Computing Arithmetic Invariants of 3-Manifolds

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Snap is a computer program for computing arithmetic invariants of hyperbolic 3-manifolds, built on Jeff Weeks's SnapPea and the number theory package Pari. Its approach is to compute the hyperbolic structure to very high precision, and use this to find an exact description of the structure. Then the correctness of the hyperbolic structure can be verified, and the arithmetic invariants of Neumann and Reid can be computed. Snap also computes high precision numerical invariants such as volume, Chern–Simons invariant, eta invariant, and the Borel regulator.

1. INTRODUCTION

This paper describes Snap, a computer program for computing arithmetic invariants of hyperbolic three-manifolds. Snap is based on the program SnapPea [Weeks 1993] and the number theory package Pari [Batut et al. 1998]. SnapPea computes the hyperbolic structure on a finite volume hyperbolic 3-manifold numerically (from its topology) and uses it to compute much geometric information about the manifold. Snap's approach is to compute the hyperbolic structure to very high precision, and use this to find an exact description of the structure. Then the correctness of the hyperbolic structure can be verified, and the arithmetic invariants of Neumann and Reid [1992a] can be computed. Snap also computes high precision numerical invariants such as volume, Chern–Simons invariant, eta invariant, and the Borel regulator. As sources of examples both Snap and SnapPea include the Hildebrand–Weeks census of all 4,815 orientable cusped manifolds triangulated by up to seven ideal simplices [Hildebrand and Weeks 1989], and the Hodgson–Weeks census of 11,031 low-volume closed orientable manifolds having no geodesic of length less than 0.3 [Hodgson and Weeks $\geq$ 2000]. (SnapPea also includes a census of nonorientable cusped manifolds.)

Snap is available electronically; see the Web page http://www.ms.unimelb.edu.au/~snap.

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2. IDEAL TRIANGULATIONS

SnapPea and Snap represent an orientable hyperbolic 3-manifold of finite volume as a set of ideal tetrahedra in $\mathbb{H}^3$ with face pairings. Identifying the sphere at infinity of $\mathbb{H}^3$ with $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the orientation preserving congruence class of a tetrahedron is given by the cross ratio of its vertices; oriented tetrahedra, whose vertices are numbered consistently with the orientation, correspond to cross ratios with positive imaginary part. After choosing orderings for the vertices of each tetrahedron, the tetrahedra are given by complex numbers

$$\{z_1, \dots, z_n\},$$

called their *shape parameters*, lying in the upper half plane. Changing the vertex ordering of a tetrahedron may replace z_j by $1 - z_j^{-1}$ or $(1 - z_j)^{-1}$.

For the result of gluing these tetrahedra to represent a hyperbolic 3-manifold, the following *gluing conditions* must be satisfied:

1. Around each edge of the complex, the sum of the dihedral angles must be 2π, and the edge must be glued to itself without translation.
2. Each cusp (neighborhood of an ideal vertex) must either (i) have a horospherical torus cross section, or (ii) admit a compactification by adding a closed geodesic around which there is an angle of 2π and no translation.

Remark 2.1. A probably more familiar situation is that of gluing the faces of a compact polytope to obtain a closed geometric manifold. In this case the translation condition is unnecessary since it is automatically satisfied.

If every cusp has a horospherical torus cross section, the glued complex is a complete hyperbolic 3-manifold. If some cusps require compactification, the result is a Dehn filling of the glued complex. Ideal triangulations are described in much more detail in [Thurston 1979].

The above conditions are equivalent to a set of equations in the z_i which we shall describe shortly. First we need to define a kind of "complex dihedral angle" for the edges of an ideal tetrahedron. For each edge of an ideal tetrahedron, there is a loxodromic transformation, having the edge as axis, and taking one of the two adjacent faces onto the other. The *logarithmic edge parameter* of the edge is $r + i\theta$, where r is the translation distance of the transformation, and θ is the angle through which it rotates. For oriented tetrahedra, with consistently numbered vertices, we can take $\theta \in (0, \pi)$. The corresponding *edge parameter* is $e^{r+i\theta}$. If the tetrahedron has shape parameter z, each edge parameter is one of

$$z, \ 1 - z^{-1}, \ \text{or} \ (1 - z)^{-1}.$$

Condition 1 is that for each edge of the 3–complex the sum of the logarithmic edge parameters is $2\pi i$. Condition 2 can be similarly expressed; the exact set of terms which are added depends on whether the cusp is complete or filled. We call these the *logarithmic gluing equations* of the ideal triangulation.

When SnapPea is given a 3-manifold topologically, as a set of face pairings for ideal tetrahedra, and perhaps also Dehn fillings for some of the cusps, it attempts to solve the logarithmic gluing equations numerically. A solution is called *geometric* if all the z_i lie in the upper half plane. Corresponding ideal tetrahedra can then be glued together, along some of the faces, to give an ideal fundamental polyhedron for the manifold; the remaining face pairings give a faithful representation of its fundamental group into $\mathrm{PSL}(2, \mathbb{C})$.

If not all of the z_i lie in the upper half plane the solution may still have a meaningful interpretation. Regard *any* quadruple of points in $\tilde{\mathbb{C}}$ as a tetrahedron. Call it *geometric* if the cross ratio lies in the upper half plane, *flat* if it is real and not equal to 0 or 1, *degenerate* if it is $0, 1, \infty$ or undefined (i.e., if two or more vertices coincide), and *negatively oriented* if it is in the lower half plane. A solution without degenerate tetrahedra certainly gives a representation of the fundamental group of the manifold into $\mathrm{PSL}(2, \mathbb{C})$. However, the representation need not be faithful and may not have a discrete image.

It follows from the existence of canonical ideal cell decompositions of finite volume hyperbolic manifolds [Epstein and Penner 1988] that every such 3-manifold can be represented using only geometric and flat tetrahedra: decompose each cell into tetrahedra, then match differing face triangulations using flat tetrahedra (if necessary). It is conjectured that in fact only geometric tetrahedra are needed in this case.

For closed hyperbolic 3-manifolds the situation is less clear. Certainly every such manifold can be obtained topologically by Dehn filling a suitable hyperbolic link complement. This means that any solution of the gluing equations will give a representation of the fundamental group into $\mathrm{PSL}(2,\mathbb{C})$. Unless, however, the solution is geometric, it cannot be guaranteed that the representation is faithful or discrete.

What is important, for present purposes, is that the gluing conditions can also be given as a set of polynomial equations, with rational coefficients, in the z_i. The *gluing equations* are obtained from the logarithmic gluing equations by exponentiation. These equations state that certain products of edge parameters (of the form z_i, $1-z_i^{-1}$, and $(1-z_i)^{-1}$) equal 1. Multiplying through by suitable powers of z_i and $(1-z_i)$ we obtain polynomial equations. Note that the gluing equations only specify that the angle sum, around each edge or filled cusp, is a multiple of 2π. In terms of numerical computation however, it is straightforward to check if a solution actually gives an angle sum of precisely 2π.

Mostow–Prasad rigidity (see [Mostow 1973]) implies that the solution set of the gluing equations is 0-dimensional. It follows that the z_i in any solution are algebraic numbers: compare [Macbeath 1983, proof of Theorem 4.1]. For example, the complement in S^3 of the figure 8 knot has an ideal triangulation by two tetrahedra with shape parameter

$$z_1 = z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2} i.$$

This is actually the shape parameter of a *regular* ideal tetrahedron.

We can also assume that the entries of a set of $\mathrm{PSL}(2,\mathbb{C})$ matrices for the fundamental group are algebraic: position the fundamental polyhedron so that one tetrahedron has three of its vertices at 0, 1, and ∞. The remaining vertices will be algebraic, as will entries of the face pairing transformation matrices. The other matrices, being products of these, will also have algebraic entries.

3. COMPUTATION WITH ALGEBRAIC NUMBERS

In order to give an exact representation of a 3-manifold we clearly need a way to represent algebraic numbers. We give a brief discussion, referring to [Cohen 1993; Pohst 1993] for more details. The most obvious way to represent an algebraic number is to give a polynomial with rational coefficients, whose roots include the number in question, and somehow specify which root is intended. The latter can be done by giving the root numerically to sufficient precision. The roots can also be sorted and given by number.

Carrying out the field operations with algebraic numbers given in this way is slightly non-trivial: a "resultant trick" enables us, given two numbers, to compute a polynomial whose roots include the sum of the two numbers. We must then determine which root is the sum, perhaps by computing the latter numerically to sufficient precision. Differences, products and quotients can be similarly computed.

In fact we do not use quite this approach. We specify one number, τ say, in the above manner, then represent other numbers as $\mathbb{Q}$-polynomials in τ. Let f be the minimum polynomial of τ and let n be the degree of f. Then the field $\mathbb{Q}(\tau)$ is a degree n extension of $\mathbb{Q}$, and each element of $\mathbb{Q}(\tau)$ has a unique representation as a $\mathbb{Q}$-polynomial in τ of degree at most $n-1$.

Field operations in $\mathbb{Q}(\tau)$ are now very easy: sum and difference computations are obvious; a product can be computed directly then reduced to a polynomial of degree at most $n-1$ by subtracting a suitable multiple of $f(\tau)$. A quotient $g_1(\tau)/g_2(\tau)$ is computed by using the Euclidean algorithm to find polynomials a, b such that $af + bg_2 = 1$, whence

$$b(\tau) = g_2(\tau)^{-1}.$$

Pari [Batut et al. 1998] implements this kind of arithmetic: the expression `Mod(g,f)`, called in Pari a *polymod*, represents $g(\tau)$ where τ is a root of f. Note that it is not necessary to specify *which* root is chosen to do arithmetic with polymods, since a change of root is a field isomorphism.

Of course if we want to add α, β belonging to different number fields we must either fall back on the first approach, or find a new primitive element, σ such that

$$\mathbb{Q}(\sigma) \supseteq \mathbb{Q}(\alpha, \beta),$$

and re-express both α and β in terms of σ. For the most part, however, our approach is to first find a number field which contains all the numbers we are interested in and then carry out the required computations inside this field.

Our aim then, given a 3-manifold with shape parameters $\{z_1, \dots, z_n\}$, is to find an irreducible polynomial $f \in \mathbb{Z}[x]$ with root τ such that

$$z_1, \dots, z_n \in \mathbb{Q}(\tau).$$

In outline what we do is this:

1. compute each z_i to high precision (typically some 50 decimal places);
2. use the LLL algorithm [Lenstra et al. 1982] to guess a polynomial in $\mathbb{Z}[x]$ vanishing on each z_i;
3. check if all the z_i belong to the field generated by one of them, also using the LLL algorithm.

Step 1 is done by Newton's method, using the solution provided by SnapPea as a starting point. Usually the check in Step 3 is successful. When it is not we try small rational linear combinations of the z_i to find a primitive element for $\mathbb{Q}(z_1, \dots, z_n)$. A side effect of Step 3 is that we obtain an expression for each of the z_i in terms of the primitive element.

Since the LLL algorithm is fundamental we describe a little further what it is and how it is applied in Steps 2 and 3 above. Most of what follows is described much more precisely in [Cohen 1993; Pohst 1993].

The LLL algorithm is an algorithm which finds a "good" basis for an integer lattice with respect to a given inner product. A good basis is one which consists of short and approximately orthogonal elements. Roughly, how it does this is to apply Gram–Schmidt "orthogonalization" to the starting basis, but modified so that only nearest integer multiples of basis elements are added or subtracted. Whenever an element is obtained which is significantly shorter than the preceeding ones, it is moved in front of them, and Gram–Schmidt is started again from there. The resulting basis always contains elements not too far from being shortest in the lattice. We emphasize that the result is dependent on the *inner product*: the lattice in our case is always simply the integer lattice $\mathbb{Z}^n$; it is by varying the inner product that we obtain useful results.

Now suppose that z approximates an algebraic number τ. To find an integer polynomial, of degree at most m, vanishing on τ we look for one which is small on z. In fact we use LLL to find a short vector in $\mathbb{Z}^{m+1}$ with respect to the inner product given by the quadratic form[1]

$$(a_0, \dots, a_m) \mapsto$$
$$a_0^2 + \dots + a_m^2 + N|a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m|^2,$$

where N is a large number, around $10^{1.5d}$ if z is given to d decimal places. If $a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$ is not zero, to approximately the precision to which z is known, the term $N|a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m|^2$ will make $(a_0, \dots, a_n)$ long. Thus if LLL finds any short vectors, it has most likely found $(a_0, \dots, a_m)$ such that $a_0 + a_1\tau + a_2\tau^2 + \dots + a_m\tau^m = 0$. By factoring this polynomial, and identifying which irreducible factor has τ as a root, we can determine τ's minimum polynomial. Of course, if τ's minimum polynomial has degree greater than m, this whole process is doomed to failure. Assuming however that we have chosen m sufficiently large, this application of LLL completes Step 2 above.

For Step 3 we need to check if α, algebraic, approximated by w, belongs to $\mathbb{Q}(\tau)$. We use the LLL algorithm to find a small vector in $\mathbb{Z}^{n+1}$ with respect to the inner product given by the quadratic form

$$(a, a_0, \dots, a_{n-1}) \mapsto$$
$$a^2 + a_0^2 + \dots + a_{n-1}^2 + N|aw + a_0 + a_1 z + \dots + a_{n-1} z^{n-1}|^2,$$

where N is as before and n is the degree of τ's minimum polynomial. As before, if LLL finds a short vector, it most likely has found $(a, a_0, \dots, a_{n-1})$ such that $a\alpha + a_0 + a_1\tau + a_2\tau^2 + \dots + a_{n-1}\tau^{n-1} = 0$. Since n is the degree of τ's minimum polynomial, a should not be zero; so we obtain an expression for α in terms of τ. On the other hand, if $aw + a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1}$ is not zero, to approximately the precision to which z and w are known, it is likely that $\alpha \notin \mathbb{Q}(\tau)$. Refinements of these procedures can be found in [Cohen 1993; Pohst 1993].

We can give a rough analysis of the above use of the LLL algorithm. Denote by $b(a_0, \dots, a_m)$ the above quadratic form that is reduced by the LLL algorithm to find a good integer polynomial for z. One

[1] A slightly different quadratic form is actually used, namely the a_0^2 term is omitted if z is real and the a_0^2 and a_1^2 terms are omitted if z is nonreal. The reason is pragmatic: LLL initializes by doing a true Gram–Schmidt reduction of the form, and the resulting basis-change is the same to within machine precision for the modified form, but is given by a much simpler formula.

can easily check that this bilinear form has determinant approximately equal to N or N^2 according as z is real or non-real. (In this discussion, "approximately equal to" will mean "equal to a bounded multiple of". The actual determinants are very close to N and $(\operatorname{Im} zN)^2$ respectively.) In our applications z is complex, but we will analyze the algorithm without this assumption, so let $k = 1$ or 2 according as z is real or non-real. Putting $N = 10^P$, we can write

$$\det(b) \sim 10^{kP}.$$

Now crude estimates suggest that a "random" quadratic form of determinant D on $\mathbb{Z}^{m+1}$ will have minimum on $\mathbb{Z}^{m+1} - \{0\}$ of approximately $D^{1/(m+1)}$. In our case,

$$\min\{b(a_0, \dots, a_m) : (a_0, \dots, a_m) \in \mathbb{Z}^{m+1} - \{0\}\} \sim 10^{kP/(m+1)}.$$

Since the coefficients a_i contribute their squares to this minimal $b(a_0, \dots, a_m)$, they will be bounded by approximately $10^{kP/(2m+2)}$. Thus if we expect coefficients bounded by 10^c we need c less than $kP/(2m+2)$ and hence

$$P \approx 2(m+1)c/k$$

or larger. Conversely, once P is chosen, c is bounded by about $Pk/2(m+1)$.

The minimal $b(a_0, \dots, a_m)$ also includes a contribution $10^P l^2$ with

$$l = a_0 + a_1 z + \cdots + a_m z^m,$$

so we also have $10^P |l|^2 \sim 10^{kP/(m+1)}$ so

$$|l| \sim 10^{P(k-m-1)/2(m+1)}.$$

This is expected even if the original τ that z approximates does not satisfy an integer polynomial in degree m. Thus to detect that the polynomial that we find is a good one, we should use somewhat more than $P(m+1-k)/2(m+1) \approx P/2$ digits of precision.

Snap adjusts P so that it uses $d = 2P/3$ digits of precision. Since $k = 2$ in Snap's applications, this means we can hope Snap will find polynomials with coefficients up to about $10^{3d/2(m+1)}$. Snap's default (which can be changed at any time) is to work with degree 16 and precision $d = 50$, so we can hope to find polynomials with coefficients bounded by about $10^{4.5}$, and expect to find them if the coefficients are significantly smaller than this.

We can roughly quantify the likelihood of finding "false positives" in these applications of LLL. Given n random complex numbers $\zeta_1, \dots, \zeta_n$ in the unit disk, the number of complex numbers of the form

$$a_1\zeta_1 + \cdots + a_n\zeta_n$$

in the unit disk with $|a_i| \le 10^c$ is approximately $10^{(n-2)c}$, so the total area covered by a disk of radius 10^{-p} around each will be approximately $10^{(n-2)c-2p}\pi$ if p is significantly larger than $(n-2)c$. Thus the probability of one of these linear combinations $a_1\zeta_1 + \cdots + a_n\zeta_n$ being "accidentally" within 10^{-p} of 0 is about $10^{(n-2)c-2p}$. With a machine precision of 10^{-d} and coefficients up to 10^c we should take $p = d - c$, so the likelihood of a false positive becomes about 10^{nc-2d}. With Snap's defaults ($n = 17$, $d = 50$) described above and $c = 4.5$, this is about 10^{-23}.

As we increase both precision and degree, the running time of the algorithm goes up. We were unable to find any estimates of the expected running time of the LLL algorithm in the literature, but experiment suggests that typical running times using Pari 2.03's implementation on a Sparc 5 machine are given approximately, in seconds, by

$$3.7 \times 10^{-7} \text{precision}^{2.6} \text{degree}^{2.7},$$

for degrees between 10 and 20 and precisions between 80 and 180.

Finally, note that whatever choice we make for n in Step 2, it is the degree of the minimum polynomial found which governs n in Step 3: often this will be smaller and the LLL computations in Step 3 will run correspondingly faster.

Snap follows the procedure outlined above to find a number field containing all the shape parameters of a given 3-manifold, and an exact expression for each shape in terms of a primitive element for that field. Snap's "verify" function then substitutes the exact shapes back into the gluing equations to check that they are satisfied. Here is sample output for the figure 8 knot complement.

```
Shapes (Numeric)
shape(1) = 0.5000000000000000000000 +
           0.8660254037844386467637*i
shape(2) = 0.5000000000000000000000 +
           0.8660254037844386467637*i
```

```
Shape Field
min poly: x^2 - x + 1
root number: 1
numeric value of root:
           0.5000000000000000000000 +
           0.8660254037844386467637*i

Shapes (Exact)
shape(1) = x 1.33737 E-67
shape(2) = x 5.24561 E-68

Gluing Equations
Meridians:
1, 0; 0, 1; 0 -> 1 : 9.27301 E-68
Longitudes:
0, -2; 0, 4; 2 -> 1 :  0.E-57
Edges:
2, -1; -1, 2; 0 -> 1 : 1.29822 E-67
-2, 1; 1, -2; 0 -> 1 : 1.29822 E-67
```

The root number says which root of the minimum polynomial is used as a primitive element for the field. The numbering scheme used will be described when we discuss canonical representations of number fields in $\mathbb{C}$. The small number following each exact shape (e.g., `1.33737 E-67`) gives the accuracy of the originally computed numerical shape. It is included only as a sanity check.

Finally we have the gluing equations. As we have already noted, the gluing equations come down to the requirement that certain products of terms of the form z_i, $1-z_i^{-1}$ and $(1-z_i)^{-1}$ give 1. This is equivalent to certain products of powers of z_i, $1-z_i$ and -1, giving 1: see for example [Neumann and Zagier 1985]. Reading each gluing equation horizontally we have powers of $z_1, \dots, z_n$, powers of $1-z_1$, $\dots$, $1-z_n$, and the power of -1, followed by their product in exact arithmetic. This is followed after a colon by the precision to which the logarithmic gluing equation has been verified. Since the gluing equation is exactly correct, the logarithmic gluing equation is known to be correct up to an integer multiple of $2\pi i$, so it would suffice to verify it to much lower precision than is actually done.

Since this output shows that the logarithmic gluing equations have been verified exactly, and since the shape parameters were in the upper half plane, signifying correctly oriented simplices, it proves the existence of a hyperbolic structure with an ideal triangulation with the given simplex shapes.

The meridian and longitude referred to in the printout are curves, in a cross section of the cusp, which give a basis for the first homology group of that cross section. Typically SnapPea uses shortest curve and next shortest independent curve, in the Euclidean structure on a horospherical cusp cross section, as the meridian and longitude respectively. (For knot and link complements, SnapPea uses the conventional terminology: where a meridian means a curve bounding a disk transverse to the knot or link, while a longitude means a curve that runs parallel to the knot or to a component of the link and is null-homologous in S^3 minus the knot or link component.) Corresponding to each meridian or longitude is a gluing equation for the cusped hyperbolic structure. The gluing equations for a Dehn filled manifold include one equation for each filled cusp, corresponding to the filling curves.

4. COMMENSURABILITY INVARIANTS

Two finite volume, orientable, hyperbolic 3-manifolds are said to be *commensurable* if they have a common finite-sheeted cover. Subgroups $\Gamma, \Gamma' \subset \mathrm{PSL}(2,\mathbb{C})$ are commensurable if there exists $g \in \mathrm{PSL}(2,\mathbb{C})$ such that $g^{-1}\Gamma g \cap \Gamma'$ is a finite index subgroup of both $g^{-1}\Gamma g$ and Γ'. Therefore, by Mostow rigidity orientable hyperbolic 3-manifolds of finite volume are commensurable if and only if their fundamental groups are commensurable as subgroups of $\mathrm{PSL}(2,\mathbb{C})$.

4A. The Invariant Trace Field

Let Γ be the group of covering transformations of such a manifold, and let $\tilde\Gamma$ denote the preimage of Γ in $\mathrm{SL}(2,\mathbb{C})$. The traces of elements of $\tilde\Gamma$ generate a number field $\mathbb{Q}(\operatorname{tr}\Gamma)$ called the *trace field* of Γ. That $\mathbb{Q}(\operatorname{tr}\Gamma)$ is a number field follows from the observation that Γ is finitely generated and, by conjugating suitably (as described at the end of Section 2) we can assume that the generators have algebraic entries. The trace field $\mathbb{Q}(\operatorname{tr}\Gamma)$ is almost, but not quite, a commensurability invariant of Γ: see [Reid 1990].

The *invariant trace field* $k(\Gamma)$ of Γ may be defined as the intersection of all the fields $\mathbb{Q}(\operatorname{tr}\Gamma')$, as Γ' varies over all finite index subgroups of Γ. Defined in this way it is clear that $k(\Gamma)$ is a commensurability invariant of Γ. What is less clear is that it is ever non-trivial. We have, however, the following.

Theorem 4.1 [Reid 1990].

$$k(\Gamma) = \mathbb{Q}(\{\mathrm{tr}^2(\gamma) \mid \gamma \in \Gamma\}) = \mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)}),$$

where $\Gamma^{(2)}$ is the finite index subgroup of Γ generated by squares $\{\gamma^2 \mid \gamma \in \Gamma\}$.

We have seen how it is possible, given a set of generators for a field, to guess a primitive element for that field along with its corresponding minimum polynomial. In order to compute the trace and invariant trace fields of Γ we must find *finite* sets of generators for the two fields.

Theorem 4.2. *Let $\tilde{\Gamma} \subset \mathrm{SL}(2,\mathbb{C})$ be finitely generated by $\{g_1,\dots,g_n\}$. The trace, $\mathrm{tr}(g_{i_1}\cdots g_{i_k})$, of an element of $\tilde{\Gamma}$ can be expressed as a polynomial with rational coefficients in the traces: $\mathrm{tr}(g_i), 1 \le i \le n$, $\mathrm{tr}(g_ig_j), 1 \le i < j \le n$, and (if $n > 2$) the trace of one triple product of generators, e.g. $\mathrm{tr}(g_1g_2g_3)$. Also, $\mathrm{tr}(g_{i_1}\cdots g_{i_k})$ is an algebraic integer if each $\mathrm{tr}(g_i)$, for $1 \le i \le n$, and each $\mathrm{tr}(g_ig_j)$, for $1 \le i < j \le n$, is an algebraic integer.*

Proof. For the trace relations used in the following, see [Magnus 1980]. Let K be the field generated over $\mathbb{Q}$ by the traces $\mathrm{tr}(g_i), 1 \le i \le n$, and $\mathrm{tr}(g_ig_j), 1 \le i < j \le n$. Let $P_{ijk} = \mathrm{tr}(g_ig_jg_k) + \mathrm{tr}(g_ig_kg_j)$ and $Q_{ijk} = \mathrm{tr}(g_ig_jg_k).\,\mathrm{tr}(g_ig_kg_j)$. Then $\mathrm{tr}(g_ig_jg_k)$ and $\mathrm{tr}(g_ig_kg_j)$ are the roots of $z^2 - P_{ijk}z + Q_{ijk} = 0$. Fricke's Lemma [Magnus 1980] implies that P_{ijk} and Q_{ijk} are integer polynomials in the $\mathrm{tr}(g_i)$ and $\mathrm{tr}(g_ig_j)$, hence they are in K. Writing $\Delta(g_i,g_j,g_k)$ for the discriminant $P_{ijk}^2 - 4Q_{ijk}$ it is clear that for any extension K_1 of K, $\mathrm{tr}(g_ig_jg_k) \in K_1$, if and only if both $\mathrm{tr}(g_ig_jg_k)$ and $\mathrm{tr}(g_ig_kg_j) \in K_1$, if and only if $\sqrt{\Delta(g_i,g_j,g_k)} \in K_1$.

By [Magnus 1980, Lemma 2.3], for any i,j,k and i',j',k' in $\{1,\dots,n\}$,

$$\sqrt{\Delta(g_i,g_j,g_k)} \cdot \sqrt{\Delta(g_{i'},g_{j'},g_{k'})} \in K.$$

Therefore

$$\sqrt{\Delta(g_i,g_j,g_k)} \in K_1 \iff \sqrt{\Delta(g_{i'},g_{j'},g_{k'})} \in K_1.$$

If we now put $K_1 = K(\mathrm{tr}(g_1g_2g_3))$ it follows from the above observations that $\mathrm{tr}(g_ig_jg_k) \in K_1$ for all i,j,k in $1,\dots,n$.

We show next, by induction on $k \ge 3$, that K_1 contains the traces of all k–fold products of the generators g_i. We have just shown that this is so for $k = 3$. Suppose then that $k > 3$ and K_1 contains the traces of all $(k-1)$–fold products of generators. Then for each product $g' = g_{i_1}\cdots g_{i_{k-2}}$, K_1 contains all traces, and all traces of products of pairs, of elements in the set $\{g_1,\dots,g_n,g'\}$. Moreover it contains at least one triple product, namely $\mathrm{tr}(g_1g_2g_3)$. By the above argument it follows that K_1 contains the traces of all triple products of elements of this set. In particular, K_1 contains the trace of $g'g_ig_j$ for each g' as above, and i,j in $\{1,\dots,n\}$. Since these are all the k–fold products of the g_i, this proves the first statement.

Finally, if the $\mathrm{tr}(g_i)$ and $\mathrm{tr}(g_ig_j)$ are all algebraic integers, P_{ijk} and Q_{ijk} are also. Therefore $\mathrm{tr}(g_ig_jg_k)$ and $\mathrm{tr}(g_ig_kg_j)$, being roots of a monic polynomial with algebraic integer coefficients, are again integral. The same induction argument then shows that all traces of k–fold products of the g_i are in the ring of integers of K_1. $\square$

Theorem 4.2 enables us to compute the trace field of $\Gamma = \langle g_1,\dots,g_n\rangle$. There is no particularly obvious set of generators for $\Gamma^{(2)}$ which we can use to compute the invariant trace field of Γ. Fortunately, Corollary 3.2 of [Hilden et al. 1992] tells us that $\mathbb{Q}(\mathrm{tr}\,\Gamma^{(2)}) = \mathbb{Q}(\mathrm{tr}\,\Gamma^{SQ})$ where $\Gamma^{SQ} = \langle g_1^2,\dots,g_n^2\rangle$, as long as $\mathrm{tr}(g_i) \neq 0$, for $i = 1,\dots,n$.

Snap computes trace fields and invariant trace fields in much the same way that it computes a field containing all the shape parameters. It first computes high precision numeric expressions for a set of generators for the group of covering transformations of a manifold. Then it uses LLL to find a primitive element in terms of which the appropriate set of traces can be expressed.

For example: $(6,1)$–Dehn filling on the figure 8 knot complement yields a closed hyperbolic 3-manifold with volume $1.284485300468\dots$. Its group of covering transformations is $\langle g_1,g_2\rangle$, where g_1 and g_2 are given respectively by

$$\begin{pmatrix} -1.135368 + 0.572291i & 0.0 \\ 0.702328 + 0.354014i & -0.702328 - 0.354014i \end{pmatrix},$$

$$\begin{pmatrix} -1.226699 + 1.467712i & 2.689343 + 1.705870i \\ -0.265154 + 0.168189i & 0.0 \end{pmatrix}.$$

Snap prints the invariant trace field as follows:

```
Invariant trace field
minumum polynomial: x^3 + 2*x - 1
root number: 2
numeric value of root:
          -0.2266988257582018 +
           1.467711508710224*i
signature: [1, 1]
discriminant: -59
...
```

It also gives exact expressions for the traces used to generate this field:

```
Invariant trace field generators
tr(g1^2) = Mod(-x^2 - x - 1, x^3 + 2*x - 1)
tr(g2^2) = Mod(x^2 - 2*x - 1, x^3 + 2*x - 1)
tr(g1^2g2^2) = Mod(-x + 2, x^3 + 2*x - 1)
```

This is all very well but there is not much point in computing invariants, like the invariant trace field, if we cannot compare two and decide whether they are the same. Simple invariants of a number field include its degree (dimension over $\mathbb{Q}$), which is equal to the degree of the minimum polynomial of any primitive element, and its signature (r_1, r_2), where r_1 is the number of real roots, and r_2 is the number of conjugate pairs of non-real roots of a minimum polynomial.

We also have the *discriminant*, defined as follows. The algebraic integers of a number field $\mathbb{Q}(\tau)$ form a free $\mathbb{Z}$–submodule of $\mathbb{Q}(\tau)$ of rank $[\mathbb{Q}(\tau):\mathbb{Q}]$. The bilinear map $(x,y) \mapsto \operatorname{tr}(xy)$ gives a nondegenerate inner product on $\mathbb{Q}(\tau)$ as a $\mathbb{Q}$-vector space. Given any basis of the ring of integers we form the matrix of inner products of basis elements, taken pairwise. The determinant of this is in $\mathbb{Z}$ and is independent of the choice of basis. It is called the discriminant of the number field.

In fact we can construct a *canonical minimum polynomial* which is a complete isomorphism invariant for number fields. The so-called T_2 norm of a number field is given by the the inner product

$$(x,y) \mapsto \sum_{i=1}^{n} \sigma_i(x)\overline{\sigma_i(y)},$$

where $\sigma_1, \dots, \sigma_n$ are the embeddings of the number field $\mathbb{Q}(\tau)$ into $\mathbb{C}$, and the bar denotes ordinary complex conjugation. This gives a positive definite inner product on $\mathbb{Q}(\tau)$, and we can enumerate integers of $\mathbb{Q}(\tau)$ in order of their T_2 norm. The set of integers of smallest norm that generate $\mathbb{Q}(\tau)$ is canonical. Their minimum polynomials include one which is lexicographically first, and this serves as a canonical minimum polynomial. See [Cohen 1993] for further discussion.

The trace fields we compute are not just abstract number fields, they are actually subfields of $\mathbb{C}$. Complex conjugate subfields arise from complex conjugate representations in $\mathrm{PSL}(2,\mathbb{C})$ of the same fundamental group, and just correspond to reversing the orientation of a hyperbolic 3-manifold. Otherwise different subfields mean essentially different values of the invariant. Since several roots of the canonical minimum polynomial might generate the same subfield of $\mathbb{C}$, we sort the roots into some fixed order and take the first which gives the required subfield. This gives us a *canonical root number* for the subfield.[2]

For example: in [Hodgson and Weeks $\geq$ 2000] the closed hyperbolic 3-manifolds m010$(-1,3)$ and s594$(-4,3)$ have isomorphic invariant trace fields, with canonical minimum polynomial $x^4 + x^2 - x + 1$, *but* they have different canonical root numbers, namely 1 and 2 respectively. Therefore their invariant trace fields differ and they are not commensurable.

4B. The Invariant Quaternion Algebra

Let K be a field of characteristic zero. A *quaternion algebra over K* is a simple central algebra of dimension 4 over K. These are discussed in detail in [Vignéras 1980]. Let (a,b) be a pair of nonzero elements of K. Up to isomorphism, there is a unique quaternion algebra A containing elements i, j satisfying $i^2 = a$, $j^2 = b$, and $ij = -ji$, and such that $\{1, i, j, ij\}$ form a basis for A as a K-vector space. Such a pair (a,b) is called a *Hilbert symbol* for A.

[2] In fact, we make an *ordered list* of the real roots followed by the complex roots having positive imaginary part; these are arranged in increasing order of real part and increasing absolute value of imaginary part (if real parts are equal). We then try each real root in turn (if the field is real) or each complex root followed by its complex conjugate (if the field is non-real). Finally, we assign the root a number: if the root has non-negative real part we give its position in the list, otherwise we give the negative of the number for its conjugate.

Every quaternion algebra over K has a Hilbert symbol, but the symbol is far from being unique.

A is a division algebra if and only if the equation

$$aX^2 + bY^2 - Z^2 = 0$$

has no non-trivial solutions for $X, Y, Z \in K$. If A is not a division algebra, it is isomorphic with $M(2, K)$, the algebra of all 2 by 2 matrices over K (and conversely, the latter is not a division algebra for any K). Over $\mathbb{R}$ there are just two quaternion algebras: the "usual" Hamiltonian quaternion algebra, which has Hilbert symbol $(-1, -1)$ and is a division algebra, and $M(2, \mathbb{R})$. Over $\mathbb{C}$ there is just $M(2, \mathbb{C})$.

As before, let $M = \mathbb{H}^3/\Gamma$ be a finite volume, orientable hyperbolic 3-manifold. Denote by $\tilde{\Gamma}^{(2)}$ the preimage in $\mathrm{SL}(2, \mathbb{C})$ of the group generated by squares of elements of $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$. The *invariant quaternion algebra* $A(\Gamma)$ *of* Γ, is the k–subalgebra of $M(2, \mathbb{C})$ generated by $\tilde{\Gamma}^{(2)}$, where k denotes the invariant trace field of Γ.

Theorem 4.3 [Hilden et al. 1992]. *Let g, h be non-commuting elements of $\Gamma^{(2)}$ with $\mathrm{tr}(g) \neq \pm 2$. Then $A(\Gamma)$ has Hilbert symbol*

$$(\mathrm{tr}(g^2) - 2,\ \mathrm{tr}([g, h]) - 2),$$

where $[g, h]$ denotes the commutator $ghg^{-1}h^{-1}$.

Snap computes a Hilbert symbol for the invariant quaternion algebra of a 3-manifold by finding $g, h \in \Gamma^{(2)}$ as above, and computing exact expressions for $\mathrm{tr}(g^2) - 2$ and $\mathrm{tr}([g, h]) - 2$. The non-uniqueness of the Hilbert symbol means that this is not, by itself, enough to tell us whether or not two 3-manifolds have the same quaternion algebra.

The remainder of this section describes how the classification of quaternion algebras over a number field gives a complete invariant which we can compute. We fix a number field K, and quaternion algebra A over K with Hilbert symbol (a, b).

Theorem 4.4 [Vignéras 1980]. *Let K and A be as above. The isomorphism class of A is determined by the (finite) set of real and finite places of K at which A is ramified. The total number of places, real and finite, at which A ramifies, is even.*

Recall that a *place* of a number field K is an equivalence class of absolute values $|\cdot| : K \to \mathbb{R}$. A place is called *real* or *complex* if the completion of K with respect to $|\cdot|$ is isomorphic with $\mathbb{R}$ or $\mathbb{C}$, respectively. The real places of K are in one-to-one correspondence with embeddings $\sigma : K \to \mathbb{R}$; likewise the complex places of K correspond to conjugate pairs of non-real embeddings $\sigma : K \to \mathbb{C}$.

A place is called *finite* if it arises from a valuation $v : K^* = K - \{0\} \to \mathbb{Z}$, i.e., there is a real number $\lambda \in (0, 1)$ such that $|x| = \lambda^{v(x)}$ for all $x \in K^*$. These valuations, in turn, are in one-to-one correspondence with prime ideals of $\mathbb{Z}_K$, the ring of integers of K: if $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$, then for each $x \in K^*$, let $v_{\mathfrak{p}}(x) = r$ where r is the unique integer such that $x \in \mathfrak{p}^r - \mathfrak{p}^{r+1}$.

For a fixed place of K, let $\sigma : K \to \overline{K}$ denote the embedding of K into its completion. Then $A \otimes_\sigma \overline{K}$ is a quaternion algebra over $\overline{K}$. A is said to be *ramified at* σ if $A \otimes_\sigma \overline{K}$ is a division algebra. In general, over a complete field with absolute value (e.g. $\mathbb{R}$), there exists at most one quaternion division algebra.

Computing the real ramification of A is straightforward: $A \otimes_\sigma \mathbb{R}$ has Hilbert symbol $(\sigma(a), \sigma(b))$. Therefore A is ramified at σ if and only if both $\sigma(a)$ and $\sigma(b)$ are negative.

For the remainder of this section we consider the problem of computing the finite ramification of A. Slightly different notation is convenient. Let $\mathfrak{p} \subset \mathbb{Z}_K$ be a prime ideal and let $K_{\mathfrak{p}}$ denote the corresponding completion of K. We regard K as a subfield of $K_{\mathfrak{p}}$, omitting any explicit mention of an embedding. Finally, we write $A_{\mathfrak{p}}$ for $A \otimes K_{\mathfrak{p}}$.

Proposition 4.5. *Let K, A and (a, b) be as above. Let $\mathfrak{p} \not| 2$ be a prime ideal of $\mathbb{Z}_K$. Then $A_{\mathfrak{p}}$ is a division algebra if and only if none of a, b and $-ab$ are squares in $K_{\mathfrak{p}}$. If a, b and $-ab$ all have even $\mathfrak{p}$-adic valuation, at least one of them is a square.*

Proof. See [Vignéras 1980, Lemma II.1.10 and subsequent table]. (Note that Vignéras uses the notation $\{a, b\}$ for our Hilbert symbol (a, b).) □

This proposition has two useful consequences. First, that the finite ramification of A is restricted to the finite set of primes $\mathfrak{p}$ dividing $2ab$. Secondly, for primes $\mathfrak{p}$ not dividing 2, the question of whether A is ramified reduces to determining whether certain $c \in K$ are squares in $K_{\mathfrak{p}}$. Proposition 4.7 settles this question for us. The proof uses Hensel's Lemma

[Lang 1986, page 42], which is valid for any prime $\mathfrak{p} \subset \mathbb{Z}_K$, and corresponding absolute value $|x| = \lambda^{v_\mathfrak{p}(x)}$. Here, $\overline{\mathbb{Z}}_K$ refers to the closure of $\mathbb{Z}_K$ in $K_\mathfrak{p}$.

Lemma 4.6 (Hensel). *Let $f(X)$ be a polynomial in $\mathbb{Z}_K[X]$. Let x_0 be an element of $\mathbb{Z}_K$ such that*

$$|f(x_0)| < |f'(x_0)^2|,$$

where f' denotes the formal derivative of f. Then f has a root x in $\overline{\mathbb{Z}}_K$ such that $|x - x_0| < 1$.

Proposition 4.7. *For each $c \in K^*$ and prime $\mathfrak{p} \subset \mathbb{Z}_K$ there exists $w \in K^*$ such that $cw^2 \in \mathbb{Z}_K$ and $v_\mathfrak{p}(cw^2) \in \{0, 1\}$. Suppose now $\mathfrak{p} \not| \ 2$. Then c is a square in $K_\mathfrak{p}$ if and only if $v_\mathfrak{p}(cw^2) = 0$ and cw^2 projects to a square in the finite field $\mathbb{Z}_K/\mathfrak{p}$.*

Proof. Let $w_1 \in \mathbb{Z}_K$ be the denominator of c. Then $cw_1^2 \in \mathbb{Z}_K$. By the Chinese Remainder Theorem, we can find an element $u \in K^*$ such that $v_\mathfrak{p}(u) = -1$ while $v_\mathfrak{q}(u) \geq 0$ for all prime ideals $\mathfrak{q} \neq \mathfrak{p}$. Then $w = w_1 u^m$, where m is the integer part of $v_\mathfrak{p}(cw_1^2)/2$, has the required property.

Let $c' = cw^2$. Let $f(X) = X^2 - c'$. If $v_\mathfrak{p}(c') = 0$ and c' projects to a square in $\mathbb{Z}_K/\mathfrak{p}$ we can lift a square root to obtain $x_0 \in \mathbb{Z}_K$ such that $f(x_0) \in \mathfrak{p}$ while $f'(x_0) = 2x_0 \notin \mathfrak{p}$. Lemma 4.6 then implies that c' is a square in $K_\mathfrak{p}$.

Conversely, if $c' = x^2$ for some $x \in K_\mathfrak{p}$, $v_\mathfrak{p}(c') = 0$ and x projects to a square root of the projection of c' in $\mathbb{Z}_K/\mathfrak{p}$. □

We turn now to the case $\mathfrak{p} \mid 2$. Recall that $A_\mathfrak{p}$ is a division algebra if and only if the equation $aX^2 + bY^2 - Z^2 = 0$ has no non-trivial solution for $X, Y, Z \in K_\mathfrak{p}$. Denote by $e > 0$ the p-adic valuation of 2, i.e., $e = v_\mathfrak{p}(2)$, or equivalently, $|2| = \lambda^e$. (This e is in fact the ramification index of the field extension $K/\mathbb{Q}$ at $\mathfrak{p}$.) We omit the easy proof of the following lemma.

Lemma 4.8. *Let $X, X' \in K_\mathfrak{p}$ and suppose $|X| \leq 1$ and $|X - X'| \leq \lambda^k$ for some non-negative integer k. Then $|X^2 - X'^2| \leq \lambda^{\min\{k+e, 2k\}}$.*

Multiplying a and b by suitable squares if necessary, by the first part of Proposition 4.7, we can assume $a, b \in \mathbb{Z}_K$ and $v_\mathfrak{p}(a), v_\mathfrak{p}(b) \in \{0, 1\}$.

Proposition 4.9. *Let $a, b \in \mathbb{Z}_K$ be such that*

$$v_\mathfrak{p}(a), v_\mathfrak{p}(b) \in \{0, 1\}.$$

Let R be a finite set of representatives for the ring $\mathbb{Z}_K/\mathfrak{p}^{e+3}$, where $e = v_\mathfrak{p}(2)$. The equation

$$aX^2 + bY^2 - Z^2 = 0 \tag{4–1}$$

has a solution for $X, Y, Z \in K_\mathfrak{p}$, if and only if there exist elements $X', Y', Z' \in R$ such that

$$|aX'^2 + bY'^2 - Z'^2| \leq \lambda^{2e+3}$$

and $\max\{|X'|, |Y'|, |Z'|\} = 1$.

Proof. Let (X, Y, Z) be a solution of (4–1) in $K_\mathfrak{p}$. After multiplying through, if necessary, by a suitable power of a uniformizing element π with $v_\mathfrak{p}(\pi) = 1$, we can assume $X, Y, Z \in \overline{\mathbb{Z}}_K$ and

$$\max\{|X|, |Y|, |Z|\} = 1.$$

Since $\mathbb{Z}_K$ is dense in $\overline{\mathbb{Z}}_K$, and R is λ^{e+3}-dense in $\mathbb{Z}_K$, we can choose $X', Y', Z' \in R$ such that each of $|X - X'|$, $|Y - Y'|$, $|Z - Z'|$ is at most λ^{e+3}. By Lemma 4.8, $|aX'^2 + bY'^2 - Z'^2| \leq \lambda^{2e+3}$.

Conversely, let $X', Y', Z' \in R$ be such that $|aX'^2 + bY'^2 - Z'^2| \leq \lambda^{2e+3}$ and $\max\{|X'|, |Y'|, |Z'|\} = 1$. If $|X'| = 1$ then $|2aX'| \geq \lambda^{e+1}$, and therefore

$$|aX'^2 + bY'^2 - Z'^2| < |2aX'|^2.$$

Regarding $aX'^2 + bY'^2 - Z'^2$ as a polynomial in X' alone, by Lemma 4.6 there exists $X \in \overline{\mathbb{Z}}_K$ such that $aX^2 + bY'^2 - Z'^2 = 0$. The same argument applies if $|Y'| = 1$ or $|Z'| = 1$. Since at least one of the three cases must hold, the result follows. □

Propositions 4.5, 4.7 and 4.9 reduce the task of computing the finite ramification of a quaternion algebra over a number field to a finite number of steps. We remark that the details of these computations are readily handled by Pari. In particular, Pari has functions for factoring algebraic numbers and ideals into primes, and for computing valuations. The uniformizing element and the element u, invoked in the proof of Proposition 4.7, are constituent parts of Pari's way of representing a prime ideal (and are thus readily available).

Remark 4.10. For an invariant quaternion algebra $A = A(\Gamma)$, the calculation of finite ramification can sometimes be simplified by using the following observation from [Gehring et al. 1997]. Assume that all traces of elements in $\tilde{\Gamma}$ are algebraic *integers*, and let g, h be non-commuting loxodromic elements of $\tilde{\Gamma}^{(2)}$. Then any prime $\mathfrak{p}$ which ramifies the quaternion algebra A must divide $\operatorname{tr}([g,h])-2$, where $[g,h]$ denotes the commutator $ghg^{-1}h^{-1}$.

4C. Arithmeticity

Finally we describe the "arithmetic" construction of Kleinian groups of finite co-volume. Let A be a quaternion algebra over a number field K. The integers of A, i.e., elements of A which have a monic minimum polynomial with integral coefficients over K, do not in general form a subring of A. The analogous role in A, to that of $\mathbb{Z}_K$ in K, is now played by an order of A. An *order* $\mathcal{O}$ of A is a rank 4 $\mathbb{Z}_K$-submodule of the set of integers of A, containing 1_A, and closed as a subring of A. Orders always exist but are not generally unique. The units $\mathcal{O}^1$ of $\mathcal{O}$ form a multiplicative subgroup. For each real or complex place σ of K, σ induces a map of A into $\mathbb{H}$, $M(2,\mathbb{R})$ or $M(2,\mathbb{C})$. If K has precisely one complex place, and every real place is ramified (i.e., maps A into $\mathbb{H}$), then the image of $\mathcal{O}^1$ in $M(2,\mathbb{C})$ is a discrete subgroup of $SL(2,\mathbb{C})$ of finite co-volume. This group is said to be *derived from a quaternion algebra.* A subgroup Γ of $SL(2,\mathbb{C})$ is *arithmetic* if it is commensurable with one derived from a quaternion algebra. The K and A of the construction can be recovered as the invariant trace field, and invariant quaternion algebra respectively, of Γ.

Remark 4.11. This is not really *the* definition of arithmeticity; there is a much more general definition in the context of lattices in semi-simple Lie groups. It is a result of Borel that the above construction yields all the arithmetic subgroups of $SL(2,\mathbb{C})$.

A result of Reid [1987] (see also [Takeuchi 1975; Hilden et al. 1992]), shows that a discrete subgroup Γ of $SL(2,\mathbb{C})$ is arithmetic if and only if the following conditions are satisfied:

1. The invariant trace field $k = \mathbb{Q}(\operatorname{tr}\Gamma^{(2)})$, has exactly one complex place.
2. $A(\Gamma)$ is ramified at every real place of k.
3. Γ has integer traces (equivalently, $\operatorname{tr}\Gamma^{(2)} \subseteq \mathbb{Z}_k$).

This enables us to determine whether or not hyperbolic 3-manifolds are arithmetic. See Tables 2 and 3 for some examples.

Arithmetic subgroups of $\mathrm{SL}(2,\mathbb{C})$ are commensurable if and only if they have the same invariant quaternion algebra. Therefore the arithmetic manifolds grouped together in Table 3 are commensurable. Non-arithmetic manifolds with the same invariant trace field, quaternion algebra and integrality or otherwise of traces, may still be incommensurable. It is work in progress to find a *computable,* complete commensurability invariant for the non-arithmetic case.

Example 4.12. Betley, Przytycki and Zukowski describe in [Betley et al. 1987] an interesting family of hyperbolic "twins" — pairs of non-homeomorphic closed hyperbolic 3-manifolds with the same volume. These examples are obtained by Dehn filling on the manifold denoted m009 in SnapPea's notation; this is the once-punctured torus bundle over S^1 with monodromy given by the matrix $\left[\begin{smallmatrix}3&2\\1&1\end{smallmatrix}\right]$. We use the geometric choice of basis for homology of the boundary torus consisting of shortest geodesic and next shortest independent geodesic on a horospherical torus cross section. Then the Dehn fillings $\mathrm{m009}(p,q)$ and $\mathrm{m009}(-p,q)$ give non-homeomorphic closed manifolds of equal hyperbolic volume, for each pair of relatively prime integers (p,q), except for the 8 non-hyperbolic Dehn fillings $(\pm3,1)$, $(\pm2,1)$, $(\pm1,1)$, $(0,1)$, and $(1,0)$.

Pzrzytycki asks in [Kirby 1997, Problem 3.60(H)] if these pairs are commensurable. Using Snap, we find that these pairs of manifolds generally have the isomorphic invariant trace fields, but have different invariant quaternion algebras so are not commensurable. However, there is one pair, $\mathrm{m009}(5,1)$ and $\mathrm{m009}(-5,1)$, of arithmetic manifolds of volume $1.8319311883544380\ldots$ having the same invariant quaternion algebra, and hence commensurable. Table 1 shows some arithmetic data for the lowest volume twins. (See Section 8 on page 145 for the interpretation of the last three columns.)

5. THE CHERN–SIMONS AND ETA INVARIANTS

The eta-invariant $\eta(M)$ and the Chern–Simons invariant $\mathrm{cs}(M)$ are geometrically defined invariants of an hyperbolic 3-manifold M. These invariants often take rational values, but are conjecturally "usually" transcendental; see [Neumann and Yang 1995b] for a precise conjecture. Snap computes these invariants

Manifold	Volume	Homology	Invariant trace field	Quaternion algebra	Int/Ar
m009(4, 1)	1.4140610441653916	$\mathbb{Z}/6$	$x^3-x^2+1[1,1](2)$	$(5,x-2)[1]$	Y/Y
m009(−4, 1)	1.4140610441653916	$\mathbb{Z}/10$	$x^3-x^2+1[1,1](-2)$	$(19,x-3)[1]$	Y/Y
m009(5, 1)	1.8319311883544380	$\mathbb{Z}/2+\mathbb{Z}/4$	$x^2+1[0,1](1)$	$(2,x+1)(5,x+2)[\,]$	Y/Y
m009(−5, 1)	1.8319311883544380	$\mathbb{Z}/2+\mathbb{Z}/6$	$x^2+1[0,1](1)$	$(2,x+1)(5,x+2)[\,]$	Y/Y
m009(−1, 2)	1.8435859723266779	$\mathbb{Z}/6$	$x^5-2x^4-2x^3+4x^2-x+1[3,1](-4)$	$(2,x^2+x+1)(5,x+1)[1,2]$	Y/N
m009(1, 2)	1.8435859723266779	$\mathbb{Z}/2$	$x^5-2x^4-2x^3+4x^2-x+1[3,1](4)$	$[1,2]$	Y/N
m009(−3, 2)	1.9415030840274678	$\mathbb{Z}/10$	$x^5-x^4-2x^3-x^2+2x+2[3,1](4)$	$(2,x)(19,x+2)[2,3]$	Y/N
m009(3, 2)	1.9415030840274678	$\mathbb{Z}/2$	$x^5-x^4-2x^3-x^2+2x+2[3,1](-4)$	$(2,x)(2,x^3+x^2+1)[2,3]$	Y/N
m009(−6, 1)	2.0624516259038381	$\mathbb{Z}/14$	$x^5-x^4+x^3+2x^2-2x+1[1,2](2)$	$(2,1+x)[1]$	Y/N
m009(6, 1)	2.0624516259038381	$\mathbb{Z}/10$	$x^5-x^4+x^3+2x^2-2x+1[1,2](-2)$	$(2,x+1)(19,x+9)(2,x^3+x^2+1)[1]$	Y/N
m009(−5, 2)	2.1340163368014022	$\mathbb{Z}/14$	$x^5-3x^3-2x^2+2x+1[3,1](4)$	$(71,x-11)[1,3]$	Y/N
m009(5, 2)	2.1340163368014022	$\mathbb{Z}/6$	$x^5-3x^3-2x^2+2x+1[3,1](-4)$	$(2)(5,x-2)[1,3]$	Y/N

TABLE 1. A family of pairs of closed manifolds with equal volume. See Section 8 on page 145 for discussion.

to high precision. The Chern–Simons invariant is also computed (to lower precision) by SnapPea. In the following two subsections we say in more detail what these invariants are and how Snap computes them.

There are two commonly used normalizations of Chern–Simons invariant in the literature, related by $\operatorname{cs}(M) = \frac{1}{2\pi^2}\operatorname{CS}(M)$. (Although the invariants are usually defined for compact M, we allow cusps; see below.) In the versions we consider, the eta-invariant $\eta(M)$ is a real invariant while the Chern–Simons invariant $\operatorname{cs}(M)$ is defined modulo $\frac{1}{2}$. Moreover, the Chern–Simons invariant is determined by the eta-invariant: $\operatorname{cs}(M)$ is simply $\frac{3}{2}\eta(M) \pmod{\frac{1}{2}}$.

Why do we bother with $\operatorname{cs}(M)$, given that it is immediately determined by $\eta(M)$? A first reason is that $\operatorname{cs}(M)$ is somewhat easier to compute. Secondly, $\operatorname{cs}(M)$ also has algebraic significance; it is closely tied to the Bloch invariant, an algebraic and number-theoretic invariant which we describe in the next section.

A less significant reason is that $\operatorname{cs}(M)$ multiplies by degree in coverings, so it is a tool for commensurability questions. However, the behaviour of $\eta(M)$ for coverings is also well understood (and related to other interesting invariants; see [Atiyah et al. 1975; Neumann 1978], for example).

5A. The Chern–Simons Invariant

The Chern–Simons invariant $\operatorname{cs}(M)$ is defined for any compact $(4k-1)$-dimensional Riemannian manifold M and is an obstruction to conformal immersion of M in Euclidean space [Chern and Simons 1971]. It is the integral of a certain $(4k-1)$-form that is defined in terms of curvature. (More generally, the Chern–Simons invariant is an invariant of a connection on a manifold and our $\operatorname{cs}(M)$ is the Chern–Simons invariant for the Riemannian connection on the tangent bundle of M).

For hyperbolic 3-manifolds Meyerhoff [1986] extended the definition of $\operatorname{cs}(M)$ to allow M to have cusps. The point is that if M' is a compact manifold obtained by Dehn filling M then $\operatorname{cs}(M')$ is naturally the sum of a term that varies analytically on hyperbolic Dehn filling space and a discontinuous summand ($-\frac{1}{2\pi}$ times the sum of torsions of the geodesics added by Dehn filling); see [Neumann and Zagier 1985; Yoshida 1985]. So one defines $\operatorname{cs}(M)$ as the value of the analytic term at the complete hyperbolic structure on M.

This leads to an invariant $\operatorname{cs}(M)$ of a hyperbolic 3-manifold M in $\mathbb{R}/\frac{1}{2}\mathbb{Z}$. If M is closed the Chern–Simons invariant is well defined modulo 1, but Snap and SnapPea still only compute modulo $\frac{1}{2}$. This is no real loss, since the Chern–Simons invariant of a closed manifold M modulo 1 can also be computed from the first homology of M together with the eta-invariant $\eta(M)$, both of which Snap can also compute.

Another significance of $\operatorname{cs}(M)$ for a hyperbolic 3-manifold is that it has natural analytic relation to $\operatorname{vol}(M)$. In fact $\operatorname{vol}(M) + 2\pi^2 i \operatorname{cs}(M)$ is a natural complexification of $\mathrm{vol}(M)$ and the formulae one uses to compute $\mathrm{cs}(M)$ give $\mathrm{vol}(M)$ as well.

The method of computation used by Snap and SnapPea is as follows. Recall that these programs compute using ideal triangulations. Let M be a cusped hyperbolic 3-manifold with ideal triangulation and $M(p,q)$ the result of hyperbolic Dehn surgery on some chosen cusp of M, triangulated by deformed versions of the original tetrahedra. Neumann [1992] gave a formula for $\mathrm{cs}(M(p,q)) + \alpha$, where α is a constant, in terms of the simplex parameters of these deformed ideal tetrahedra. The constant α is unknown, but it does not depend on p or q. Thus if the exact Chern–Simons invariant is known for just one of the manifolds $M(p,q)$ then α can be deduced, so $\mathrm{cs}(M(p,q))$ can be computed for all the $M(p,q)$. As one computes $\mathrm{cs}(M)$ for more manifolds one has more reference points to compute new families of values. Using this "bootstrapping" procedure Weeks and Hodgson computed $\mathrm{cs}(M)$ for the data-bases of manifolds in SnapPea. The computed values are included in SnapPea so that they are available for further Dehn surgeries.

In fact the constant α is always an integer multiple of $1/24$ in the version of the formula that Snap uses (this was conjectured in [Neumann 1992] but has since been proved; see [Neumann $\geq$ 2000] or the announcement in [Neumann 1998]). Thus Snap can compute the high precision value of $\mathrm{cs}(M)$ up to a multiple of $1/24$ and this multiple can then be determined from SnapPea's lower precision value. An improved formula that computes $\mathrm{cs}(M)$ exactly is now known (loc. cit.). This avoids the need of the bootstrapping procedure and will eventually be implemented in Snap.

5B. The Eta-Invariant

The eta-invariant $\eta(M)$ is also defined for any closed oriented Riemannian $(4k-1)$-manifold. It was initially defined by Atiyah, Patodi and Singer as a measure of the "asymmetry" of the spectrum of the Laplacian on M, but they proved [Atiyah et al. 1975] that it can also be given by the formula

$$\eta(M) := \int_X L - \mathrm{sign}(X),$$

where

- X is any Riemannian $4k$-manifold with $\partial X = M$ such that the metric on some collar neighbourhood of ∂X is isometric to the product metric on $M \times [0,\varepsilon)$, and
- L is the Hirzebruch L-class as a $4k$-form on X, defined in terms of curvature as in [Milnor and Stasheff 1974, Appendix], for example.

The Hirzebruch index theorem tells one that the above formula gives zero for a closed manifold X and it is then a standard argument to see that it gives an invariant of M that does not depend on the choice of X when X has boundary M as above. If $k > 1$ then M may not be the boundary of any X, but the disjoint union $2M$ of 2 copies of M is a boundary, so this formula can be used to define $\eta(2M)$, and hence define $\eta(M)$ as $\frac{1}{2}\eta(2M)$.

The relation of $\eta(M)$ to $\mathrm{cs}(M)$ for a compact 3-manifold M is

$$3\eta(M) \equiv 2\,\mathrm{cs}(M) + \tau \pmod 2$$

(see [Atiyah et al. 1975]), where τ is the number of 2-primary summands of $H_1(M;\mathbb{Z})$. Thus $\eta(M)$ completely determine $\mathrm{cs}(M)$ if M has known homology. There is also a cusped version of this: Meyerhoff and Ouyang [1997] extended the definition of $\eta(M)$ to cusped M for which one has chosen a basis of homology at each cusp.

A formula for $\eta(M(p,q))$ in terms of ideal triangulations for manifolds $M(p,q)$ as described above was given in [Meyerhoff and Neumann 1992], where it was proved "locally" (i.e., in a neighbourhood of the complete structure M in analytic Dehn filling space). It was proved globally in [Ouyang 1997]. The formula is a modification of Neumann's Chern–Simons formula by the addition of certain arithmetic terms. Again, there is an undetermined constant that is independent of p and q. Thus the above bootstrapping procedure, which will no longer be needed for computing Chern–Simons invariant, is still needed to compute $\eta(M)$ through the tables maintained by Snap and SnapPea. For a manifold M that has not yet been linked by a sequence of hyperbolic Dehn fillings and drillings (removal of closed curves) to a manifold with known eta-invariant, Snap cannot compute $\eta(M)$. This still applies to most of the knot and link complements in the standard knot and link tables, for example.

It is conjectured that the bootstrapping procedure will always work. That is:

Conjecture 5.1. *Any two hyperbolic 3-manifolds are related by a sequence of hyperbolic drillings and fillings.*

Snap and SnapPea provide good facilities for searching for such sequences, so there is much experimental evidence for the conjecture. The emphasis here is *hyperbolic* drilling and filling: that is, each drilling or filling should move between points in the appropriate analytic Dehn filling space. Without this restriction the conjecture is easy, since every 3-manifold is obtainable by Dehn surgery on some link in the 3-sphere.

Remark 5.2. The formula mentioned earlier for $\mathrm{cs}(M)$ actually computes the Chern–Simons invariant for the natural flat connection on the associated principal $\mathrm{PSL}(2,\mathbb{C})$-bundle over M rather than the Riemannian connection. It was shown by Dupont and Kamber [1993] that these are the same in $\mathbb{R}/\mathbb{Z}[\frac{1}{6}]$. In that paper they were considering a more general situation and not aiming for best denominators, and Dupont informs us that their proof works without introducing denominators in the 3 dimensional case that we are interested in.

The equality of the Riemannian and flat Chern–Simons invariants also follows if one assumes the conjecture above. Indeed, in [Yoshida 1985] the formula we use to compute Chern–Simons is proved in the context of the Riemannian Chern–Simons invariant and in [Neumann $\geq$ 2000] it is proved for the flat Chern–Simons invariant. Thus we have two formulae that differ at most by the unknown constant they contain, valid over the analytic Dehn filling space for M. Thus the difference of Riemannian and flat Chern–Simons is constant on any analytic Dehn filling space. It is zero for some examples, so if the bootstrapping conjecture is true, the bootstrapping procedure shows the difference is always zero.

6. BLOCH INVARIANT AND PSL-FUNDAMENTAL CLASS

For details on what we discuss here see [Neumann $\geq$ 2000; Neumann and Yang 1995a; 1999] or the expository article [Neumann 1998].

6A. The PSL-Fundamental Class

We first describe the "PSL-fundamental class" of an hyperbolic 3-manifold M. This is a homology class $[M]_{\mathrm{PSL}}$ in the homology group $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$, where we are taking homology of $\mathrm{PSL}(2,\mathbb{C})$ as a discrete group. If M has cusps, $[M]_{\mathrm{PSL}}$ is only defined up to an element of order 2 in $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$. We describe how we compute this invariant numerically later.

Let $M=\mathbb{H}^3/\Gamma$ be a compact hyperbolic 3-manifold. Then $H_*(\Gamma;\mathbb{Z}) = H_*(M;\mathbb{Z})$, since M is a $K(\Gamma,1)$-space. Thus $H_3(\Gamma;\mathbb{Z}) \simeq \mathbb{Z}$ with a natural generator given by the fundamental class of M. The inclusion $\Gamma\to\mathrm{PSL}(2,\mathbb{C})$ induces a map $H_3(\Gamma;\mathbb{Z})\to H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$.

Definition 6.1. The *PSL-fundamental class* $[M]_{\mathrm{PSL}}\in H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ is the image of the natural generator of $H_3(\Gamma;\mathbb{Z})$ under the above map.

If M is non-compact the PSL-fundamental class is harder to define, and we postpone it. It lies in $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})/C_2$, where C_2 is a cyclic subgroup of $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ of order 2. This cyclic subgroup exists and is unique by the next theorem. In our notation we will ignore this C_2 ambiguity and speak of $[M]_{\mathrm{PSL}}\in H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$.

Note that we can conjugate Γ to lie in a subgroup $\mathrm{PSL}(2,K)$ of $\mathrm{PSL}(2,\mathbb{C})$, where K is a number field, and $[M]_{\mathrm{PSL}}$ is then defined in $H_3(\mathrm{PSL}(2,K);\mathbb{Z})$ (this has only been proved modulo torsion in the cusped case). Usually, the smallest K for which one can do this will be a quadratic extension of the trace field of Γ (and there are infinitely many such fields which work). The following theorem tells us that if we work modulo torsion then we can actually use the invariant trace field.

This theorem summarises results of various people; see [Neumann and Yang 1999] for more details.

Theorem 6.2. 1. $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ *is the direct sum of its torsion subgroup, which is isomorphic to* $\mathbb{Q}/\mathbb{Z}$, *with an infinite dimensional* $\mathbb{Q}$ *vector space* (*conjectured to be countable*).

2. *If* $k\subset\mathbb{C}$ *is a number field then* $H_3(\mathrm{PSL}(2,k);\mathbb{Z})$ *is the direct sum of its torsion subgroup with* $\mathbb{Z}^{r_2}$, *where* r_2 *is the number of conjugate pairs of complex embeddings of* k. *Moreover, the map*

$$H_3(\mathrm{PSL}(2,k);\mathbb{Z})\to H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$$

is injective modulo torsion.

3. *If k is the invariant trace field of M then some positive multiple of $[M]_{\mathrm{PSL}}$ is in the image of $H_3(\mathrm{PSL}(2,k);\mathbb{Z}) \to H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$.*

In fact, one can show that, after possibly adding a torsion element, $2^{b+1}[M]_{\mathrm{PSL}}$ is in the image of $H_3(\mathrm{PSL}(2,k);\mathbb{Z}) \to H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$, where $b = \operatorname{rank} H_1(\Gamma;\mathbb{Z}/2)$. Moreover, the coefficient 2^{b+1} can be replaced by 1 if M has cusps.

6B. Invariants of the PSL-Fundamental Class

There is a homomorphism

$$\hat{c} : H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \mathbb{C}/2\pi^2\mathbb{Z}$$

called the "Cheeger–Simons class" ([Cheeger and Simons 1985]) whose real and imaginary parts give Chern–Simons invariant and volume:

$$\hat{c}([M]_{\mathrm{PSL}}) = 2\pi^2 \operatorname{cs}(M) + i \operatorname{vol}(M)\,.$$

($\operatorname{cs}(M)$ is here appearing as the Chern–Simons invariant of the flat connection, as discussed in Remark 5.2). We therefore denote the homomorphisms given in the obvious way by the real and imaginary parts of $\hat{c}$ by

$$\begin{aligned}\operatorname{cs} &: H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \mathbb{R}/\mathbb{Z},\\ \operatorname{vol} &: H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z}) \to \mathbb{R}.\end{aligned}$$

Conjecture 6.3. *The Cheeger–Simons class is injective. That is, volume and Chern–Simons invariant determine elements of $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ completely.* (This is a special case of a general conjecture of Ramakrishnan in algebraic K-theory; see [Neumann 1998] for a discussion.)

If k is an algebraic number field and $\sigma_1,\dots,\sigma_{r_2} : k \to \mathbb{C}$ are its different complex embeddings up to conjugation then denote by vol_j the composition

$$\operatorname{vol}_j = \operatorname{vol} \circ (\sigma_j)_* : H_3(\mathrm{PSL}(2,k);\mathbb{Z}) \to \mathbb{R}.$$

The map

$$\operatorname{Borel} := (\operatorname{vol}_1,\dots,\operatorname{vol}_{r_2}) : H_3(\mathrm{PSL}(2,k);\mathbb{Z}) \to \mathbb{R}^{r_2}$$

is called the *Borel regulator*.

Theorem 6.4. *The Borel regulator maps*

$$H_3(\mathrm{PSL}(2,k);\mathbb{Z})/\,\text{Torsion}$$

injectively onto a full sublattice of $\mathbb{R}^{r_2}$.

It is known that cs is injective on the torsion subgroup of $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$. Thus, by Theorems 6.2 and 6.4, $\operatorname{cs}(M) \in \mathbb{R}/\mathbb{Z}$ and $\operatorname{Borel}([M]_{\mathrm{PSL}}) \in \mathbb{R}^{r_2(k)}$ determine the PSL-fundamental class

$$[M]_{\mathrm{PSL}} \in H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$$

completely, where k is the invariant trace field of M.

Snap computes

$$\operatorname{Borel}(M) := \operatorname{Borel}([M]_{\mathrm{PSL}}).$$

To describe how, it helps to introduce the "Bloch Group" $\mathcal{B}(\mathbb{C})$. In the next subsection we give this group a geometric description, but in fact, by a result of Bloch and Wigner and others, it is naturally the quotient of $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ by its torsion subgroup $\mathbb{Q}/\mathbb{Z}$.

We can now explain how a cusped 3-manifold has a PSL-fundamental class in $H_3(\mathrm{PSL}(2,\mathbb{C});\mathbb{Z})$ modulo an order 2 ambiguity. We shall see that it has a natural class in the Bloch group, which can be thought of as a PSL-fundamental class modulo torsion, and the Meyerhoff definition of Chern–Simons invariant then pins down the PSL-fundamental class up to the stated ambiguity. It would be nice to find a more direct definition that gives a fundamental class in $H_3(\mathrm{PSL}(2,K);\mathbb{Z})$ (modulo a similar ambiguity to the above) when $\Gamma \subset \mathrm{PSL}(2,K)$, but the above definition does not do this.

6C. The Bloch Group

There are several different definitions of the Bloch group in the literature. They differ at most by torsion and they agree with each other for algebraically closed fields. We shall use the following.

Definition 6.5. Let k be a field. The *pre-Bloch group* $\mathcal{P}(k)$ is the quotient of the free $\mathbb{Z}$-module $\mathbb{Z}(k - \{0,1\})$ by all instances of the relations

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0 \quad (6\text{–}1)$$

and

$$\begin{aligned}[x] = \left[1-\frac{1}{x}\right] &= \left[\frac{1}{1-x}\right]\\ &= -\left[\frac{1}{x}\right] = -\left[\frac{x-1}{x}\right] = -[1-x]. \quad (6\text{–}2)\end{aligned}$$

The first of these relations is usually called the *five term relation*. The *Bloch group* $\mathcal{B}(k)$ is the kernel of the map

$$\mathcal{P}(k) \to k^* \wedge_{\mathbb{Z}} k^*, \quad [z] \mapsto 2(z \wedge (1-z)).$$

For $k = \mathbb{C}$, relation (6–2) says that $\mathcal{P}(\mathbb{C})$ may be thought of as being generated by isometry classes of ideal hyperbolic 3-simplices. The five-term relation (6–1) says that in this group we can replace an ideal simplex on four ideal points by the cone of its boundary to a fifth ideal point. As is shown in [Neumann and Yang 1999, Appendix], the effect is that $\mathcal{P}(\mathbb{C})$ is a group generated by ideal polyhedra with ideal triangular faces modulo the relations generated by cutting and pasting along such faces.

6D. The Bloch Invariant

Suppose we have an ideal triangulation of an hyperbolic 3-manifold M using ideal hyperbolic simplices with cross ratio parameters $z_1, \dots, z_n$. This ideal triangulation can be a genuine ideal triangulation of a cusped 3-manifold, or a deformation of such a one as used by Snap and SnapPea to study Dehn filled manifolds, but it may be of much more general type; see [Neumann and Yang 1999].

Definition 6.6. The *Bloch invariant* $\beta(M)$ is the element $\sum_1^n [z_j] \in \mathcal{P}(\mathbb{C})$. If the z_j's all belong to a subfield $K \subset \mathbb{C}$, we may consider $\beta(M)$ as an element of $\mathcal{P}(K)$.

Theorem 6.7 [Neumann and Yang 1999]. *If $\beta(M)$ can be defined as above in $\mathcal{P}(K)$ then it actually lies in $\mathcal{B}(K) \subset \mathcal{P}(K)$ and is independent of triangulation.*

In these terms, the Borel regulator $\mathrm{Borel}(M)$ can also be thought of as an invariant of the Bloch invariant $\beta(M)$ and can be computed as follows. The invariant trace field k of M will always be contained in the field K generated by the simplex parameters z_i, $i = 1, \dots, n$. The j-th component $\mathrm{vol}_j([M]_{\mathrm{PSL}})$ of $\mathrm{Borel}(M)$ is

$$\mathrm{Borel}(M)_j = \sum_{i=1}^{n} D_2(\tau_j(z_i)),$$

where $\tau_j : K \to \mathbb{C}$ is any complex embedding that extends $\sigma_j : k \to \mathbb{C}$. Here D_2 is the "Wigner dilogarithm function"

$$D_2(z) = \mathrm{Im}\ln_2(z) + \log|z| \arg(1-z),$$

where z ranges over $\mathbb{C} - \{0,1\}$ and $\ln_2(z)$ is the classical dilogarithm function. $D_2(z)$ is also the volume of the ideal simplex with parameter z.

As described earlier, Snap specifies the invariant trace field k as a subfield of $\mathbb{C}$ by giving the minimal polynomial of a "canonical" primitive element together with the position of the this primitive element in a list of the roots of this polynomial. Snap numbers the roots with non-negative imaginary part using real roots first in order of size, say $c_1 < c_2 < \dots < c_{r_1}$, and then non-real roots in lexicographic order of size of real and imaginary parts, $c_{r_1+1}, \dots, c_{r_1+r_2}$. Finally, roots with negative imaginary part have negative indices: $c_{-j} = \bar{c}_j$. The "canonical element" is the first complex root in the list c_{r_1+1}, $\bar{c}_{r_1+1}$, c_{r_1+2}, $\bar{c}_{r_1+2}$, $\dots$ that generates the correct subfield of $\mathbb{C}$.

In printing $\mathrm{Borel}(M)$ Snap uses the complex embeddings given by the complex roots $c_{r_1+1}, c_{r_1+2}, \dots$ above. The effect is that, according as the canonical element is c_{r_1+j} or $c_{-(r_1+j)}$, the j-th component of the Borel regulator is $\mathrm{vol}(M)$ or $-\mathrm{vol}(M)$. In the latter case — more generally, whenever $k \neq \bar{k}$ — the Borel regulator $\mathrm{Borel}(-M)$ is simply $-\mathrm{Borel}(M)$. However, if $k = \bar{k}$ then Snap's printout of $\mathrm{Borel}(M)$ and $\mathrm{Borel}(-M)$ refer to the same embedding of k (both times given by the same canonical element), so the relation is given by the action of conjugation on $\mathcal{B}(k)$, which is a bit more subtle.

It can be shown that $\pm\mathrm{vol}(M)$ is, in fact, the component with largest absolute value in the Borel regulator; [Neumann and Yang 1999].

Some interesting examples with invariant trace field

$$\mathbb{Q}(x)/(x^4 + x^2 - x + 1)$$

are discussed in [Neumann and Yang 1999]. We list all examples with this invariant trace field from the closed and cusped censuses in Table 5.

To compare the Bloch invariants of manifolds with different trace fields we must compute in the Bloch group of a common field. We close this section with interesting examples which illustrate this.

Example 6.8. The manifold of conjecturally smallest volume is the so-called Weeks manifold Wks, called m003(−3, 1) in the closed census. Its invariant trace field is

$$[x^3 - x^2 + 1, -2],$$

by which we mean the subfield of $\mathbb{C}$ generated by the complex conjugate of the second root of the polynomial $x^3 - x^2 + 1$ (the first root is the real root). This field has one complex embedding, so the Borel regulator has just one component, which, by the preceding discussion, will be minus the volume:

$$\mathrm{Borel(Wks)} = [-0.942707362776927720921299603\mathbf{1}].$$

The manifold of conjecturally third smallest volume is called m007(3, 1) in the closed census. It is an arithmetic manifold of exactly half the volume of the figure eight knot complement, i.e., its volume is the volume 1.0149416... of a regular ideal simplex. Call this manifold Vol3 for short. Its invariant trace field is

$$[x^2 - x + 1,\ 1]$$

and its Borel regulator is thus

$$[1.01494160640965362502\mathbf{1}202554].$$

However, we can ask Snap to compute the Borel regulator in the field $k(\mathrm{Wks}) = [x^3 - x^2 + 1,\ -2]$ of the Weeks manifold instead. Snap complains that this field does not contain our invariant trace field, and then proceeds to compute the join of the two fields and gives us the answer in that field:

$$[x^6 - x^5 + x^4 - 2x^3 + x^2 + 1,\ -2],$$

$$\begin{aligned}[\ \ &1.01494160640965362502\mathbf{1}202554,\\ &-1.01494160640965362502\mathbf{1}202554,\\ &-1.01494160640965362502\mathbf{1}202554].\end{aligned}$$

From this we see that the joined field K is degree 6, as expected, and that it has three complex embeddings and they restrict on $k(\mathrm{Vol3})$ once to the given embedding and twice to its conjugate.

Computing with the Weeks manifold in this same field we get a Borel regulator:

$$\begin{aligned}[\ \ &0,\\ &-0.9427073627769277209212996031,\\ &\ \ 0.9427073627769277209212996031]\end{aligned}$$

(which tells us that the first complex embedding of our degree 6 field restricts to the real embedding of $k(\mathrm{Wks})$ and the next two complex embedding restricts to the complex embedding of $k(\mathrm{Wks})$ and its conjugate).

It has been asked if the Bloch group can be generated by Bloch invariants of 3-manifolds (a positive answer would imply the "Rigidity Conjecture"; see [Neumann and Yang 1999; Neumann 1998], for example). If so, one might guess that a "random" 3-manifold with invariant trace field equal to the above degree 6 field K is likely to have Borel regulator linearly independent of the above two Borel regulators, since the Bloch group has rank 3. There turn out to be just two manifolds in the closed census with this invariant trace field (as far as has been computed). They are called v2274(−3, 2) and −v2274(3, 2), and both have the same Borel regulator, namely

$$\begin{aligned}[\ \ &2.0298832128193072500424051\mathbf{0}8,\\ &-4.8580053011500904128063039\mathbf{1}7,\\ &\ \ 0.7982388755114759127214937007].\end{aligned}$$

It turns out that this is, at least numerically, equal to

$$3\,\mathrm{Borel(Wks)} + 2\,\mathrm{Borel(Vol3)}.$$

Other interesting examples are given by surgeries on the census manifold v3066, as discussed in [Neumann and Yang 1999]. This manifold gives some of the most interesting examples of the "twins" phenomenon discussed in Example 4.12. The four surgeries v3066($\pm p, q$) and v3066($\pm 2q, p/2$) all have the same volume for each p, q.

Example 6.9. The manifolds M_1 = v3066(6, 1) and M_2 = v3066(−6, 1) have invariant trace fields

$$[x^9 - 2x^7 - 5x^6 + 12x^5 + 8x^4 + 15x^3 + 4x^2 + 2x - 1, -2]$$

and

$$[x^9 - 2x^7 - 5x^6 + 12x^5 + 8x^4 + 15x^3 + 4x^2 + 2x - 1, -5]$$

respectively. The join of these fields is

$$\begin{aligned} K_{18} = [x^{18} &- 6x^{16} - 4x^{15} + 8x^{14} + 6x^{13} + 19x^{12}\\ &+ 16x^{11} + 32x^{10} - 84x^9 - 104x^8 + 52x^7\\ &+ 67x^6 - 8x^5 + 30x^4 - 28x^3 + 8x^2 - 2x + 1,\\ -1],\ & \end{aligned}$$

with 9 complex embeddings, and the Borel regulators of the above two manifolds, computed in this join, are respectively

$$\begin{aligned}\beta_1 = [-2a_1 - a_2,\ -a_1,\ 2a_1 + a_2,\ a_1 + a_2,\ 0,&\\ -a_2,\ a_1 + a_2,\ -a_1,\ a_2]&\end{aligned}$$

$$\begin{aligned}\beta_2 = [-2a_1 - a_2,\ a_2,\ a_1,\ a_1 + a_2,\ -a_1 - a_2,&\\ a_1,\ 0,\ -2a_1 - a_2,\ a_2],&\end{aligned}$$

where

$$a_1 = 2.5689706009367088849206741 69,$$
$$a_2 = 0.6083226776636170914331534552.$$

The automorphism group of the field K_{18} is order 6. Each of β_1 and β_2 is fixed by an involution in this automorphism group, since they come from degree 9 subfields. Nevertheless, we can find three Galois conjugates of each of β_1 and β_2, so we might hope to generate up to a rank 6 subgroup of $\mathcal{B}(K_{18})$. But in fact, we only generate a rank 3 subgroup.

The Galois conjugates of β_1 are β_1 and

$$\beta_1' = [-a_1, a_2, -a_2, 0, -a_1-a_2, 2a_1+a_2, -a_1-a_2, -2a_1-a_2, -a_2],$$
$$\beta_1'' = [a_2, -2a_1-a_2, a_1, -a_1-a_2, a_1+a_2, a_1, 0, a_2, -2a_1-a_2]$$

and we find that

$$\beta_2 = \tfrac{1}{3}(2\beta_1 + 2\beta_1' - \beta_1'').$$

Various 3-manifolds can be found in the census with invariant trace fields contained in K_{18}. So far they all have Bloch invariant in the above rank 3 subgroup of $\mathcal{B}(K_{18})$. For example, the field $[x^3 + 2x - 1, 2]$ is the fixed field of $\mathrm{Aut}(K_{18})$. It occurs as the invariant trace field of various manifolds, for example v3066(1, 2), and they all have Borel regulator computed in K_{18} proportional to $\mathrm{Borel}(\mathrm{v3066}(1,2)) = \frac{2}{3}(\beta_1 + \beta_1' + \beta_1'')$. The field $[x^3 - x^2 + x + 1,\ i]$ occurs as a subfield of K_{18} for each of its three embeddings $i = 1, 2, -2$. The real embedding ($i = 1$) is in fact the real subfield of K_{18}. The complex embedding and its conjugate occur for many census manifolds and leads to Borel regulators in K_{18} that are integer multiples of $2\beta_1 - \beta_1' - \beta_1''$ or its Galois conjugate $2\beta_1'' - \beta_1 - \beta_1'$, depending on orientation. The third Galois conjugate $2\beta_1' - \beta_1'' - \beta_1$ must belong to the embedding $[x^3 - x^2 + x + 1,\ 1]$, i.e., to the real subfield of K_{18}. We will use this fact in the next section.

In addition to three embeddings of the degree 9 field already mentioned, the only other subfields of K_{18} are $\mathbb{Q}(\sqrt{-11})$ and two degree 6 fields (the joins of $\mathbb{Q}(\sqrt{-11})$ with the degree three subfields above; one of these degree 6 fields is Galois over $\mathbb{Q}$). None of these degree 2 and 6 fields have been found so far in the census. One must, however, be careful about making premature guesses from these data: arithmetic manifolds exist for any imaginary quadratic field — for $\mathbb{Q}(\sqrt{-11})$ they have just not been found in the census. The Bloch invariant for these arithmetic manifolds will lie outside the above rank three subgroup of $\mathcal{B}(K_{18})$.

7. SCISSORS CONGRUENCE

The *scissors congruence group* $\mathcal{P}(\mathbb{H}^3)$ is the abelian group generated by congruence classes of hyperbolic polyhedra of finite volume modulo all relations of the form: $P = P_1 + \cdots + P_n$ if the polyhedra $P_1, \ldots, P_n$ can be glued along faces to create the polyhedron P. Dupont and Sah [1982] showed that one obtains the same group whether one allows ideal polyhedra or not. (For an exposition and references for the material of this section see [Neumann 1998].)

The *Dehn invariant* is the map

$$\delta : \mathcal{P}(\mathbb{H}^3) \to \mathbb{R} \otimes \mathbb{R}/\pi$$

defined on generators of $\mathcal{P}(\mathbb{H}^3)$ as follows. If P is a compact polyhedron then $\delta(P) = \sum_E l(E) \otimes \theta(E)$ where the sum is over the edges E of P and $l(E)$ and $\theta(E)$ are length and dihedral angle. For an ideal polyhedron one first truncates the ideal vertices by horocycles and then uses the same definition, summing only over edges that do not bound one of the horocycle faces of the truncated polyhedron. The kernel of the Dehn invariant will be denoted

$$\mathcal{D}(\mathbb{H}^3) := \ker(\delta : \mathcal{P}(\mathbb{H}^3) \to \mathbb{R} \otimes \mathbb{R}/\pi).$$

If one subdivides an hyperbolic 3-manifold M into polyhedra then the sum of these polyhedra defines an element $\beta_0(M)$ in the scissors congruence group $\mathcal{P}(\mathbb{H}^3)$ and it is an easy exercise to see that in fact $\beta_0(M)$ is in $\mathcal{D}(\mathbb{H}^3)$.

This group $\mathcal{D}(\mathbb{H}^3)$ is closely related to the Bloch group. Since $\mathcal{B}(\mathbb{C})$ is a $\mathbb{Q}$-vector space, it splits as the direct sum

$$\mathcal{B}(\mathbb{C}) = \mathcal{B}_+(\mathbb{C}) \oplus \mathcal{B}_-(\mathbb{C})$$

of its $+1$ and -1 eigenspaces under the action of conjugation. Dupont and Sah [1982] showed:

Theorem 7.1. *The Dehn invariant kernel $\mathcal{D}(\mathbb{H}^3)$ is naturally isomorphic to $\mathcal{B}_-(\mathbb{C})$. In fact the natural map of the pre-Bloch group $\mathcal{P}(\mathbb{C})$ to $\mathcal{P}(\mathbb{H}^3)$, defined* *by mapping a class $[z]$ to the ideal simplex with parameter z, induces a surjection $\mathcal{B}(\mathbb{C}) \to \mathcal{D}(\mathbb{H}^3)$ with kernel $\mathcal{B}_+(\mathbb{C})$. The Bloch invariant $\beta(M)$ is taken to the scissors congruence class $\beta_0(M)$ by this map.*

In particular, this implies that the scissors congruence class $\beta_0(M)$ is orientation-insensitive. In fact, it was first pointed out by Gerling in a letter to Gauss that any polyhedron is scissors congruent to its mirror image. Neumann [1998] discusses to what extent one may think of the Bloch group as giving an orientation-sensitive version of scissors congruence, and [Neumann and Yang 1999] gives an explicit interpretation in terms of scissors congruence allowing only cut-and-paste along ideal triangles. However, the geometric interpretation of this for $\beta(M)$ needs care — for instance the manifold Vol3 discussed earlier appears to have no subdivision into ideal tetrahedra at all.

Note that if two manifolds have the same scissors congruence class, say $\beta_0(M_1) = \beta_0(M_2)$, this means *a priori* only that M_1 and M_2 are *stably* scissors congruent; that is, there is some polyhedron Q such that $M_1 + Q$ can be cut-and-pasted to form $M_2 + Q$. However, one can show that if M_1 and M_2 are either both compact or both non-compact then adding Q is unnecessary: M_1 can be cut into polyhedra that can be reassembled to form M_2.

Theorem 7.2. *Suppose M_1 and M_2 both have invariant trace field contained in the field K. The following statements are equivalent*:

1. *M_1 and M_2 are stably scissors congruent*; *that is*, $\beta_0(M_1) = \beta_0(M_2)$.
2. *Computed over a field containing K and $\overline{K}$, we have* $\mathrm{Borel}(M_1) + \mathrm{Borel}(-M_1) = \mathrm{Borel}(M_2) + \mathrm{Borel}(-M_2)$.
3. $\mathrm{Borel}(M_1) - \mathrm{Borel}(M_2)$ *is proportional to some* $\mathrm{Borel}(x)$ *with* $x \in \mathcal{B}(K \cap \mathbb{R})$.

Proof. The equivalence of the first two conditions follows because $\beta(-M) = -\overline{\beta}(M)$ and the map $x \mapsto \frac{1}{2}(x - \overline{x})$ defines the projection $\mathcal{B}(\mathbb{C}) \to \mathcal{B}_-(\mathbb{C})$.

Denote $\mathcal{B}(K)_{\mathbb{Q}}$ the image of $\mathcal{B}(K) \otimes \mathbb{Q}$ in $\mathcal{B}(\mathbb{C}) \otimes \mathbb{Q} = \mathcal{B}(\mathbb{C})$ (recall $\mathcal{B}(\mathbb{C})$ is a $\mathbb{Q}$-vector space). In [Neumann and Yang 1995a] it is shown that the $\mathcal{B}(K)_{\mathbb{Q}} \cap \mathcal{B}_+(\mathbb{C}) = \mathcal{B}(K \cap \mathbb{R})_{\mathbb{Q}}$. This is thus the kernel of the map $\mathcal{B}(K) \to \mathcal{P}(\mathbb{H}^3)$, proving equivalence of the third condition. $\square$

Example 7.3. Returning to the manifolds M_1 and M_2 of Example 6.9, we find that they are scissors congruent. Indeed,

$$\begin{aligned}
&\mathrm{Borel}(M_1) - \mathrm{Borel}(M_2)\\
&\quad = [0, -a_1 - a_2, a_1 + a_2, 0, a_1 + a_2,\\
&\qquad\qquad -a_1 - a_2, a_1 + a_2, -a_1 - a_2, 0]\\
&\quad = \tfrac{1}{3}(2\beta_1' - \beta_1'' - \beta_1),
\end{aligned}$$

and we pointed out in Example 6.9 that this Borel regulator comes from the real subfield of K_{18}.

The following conjecture has been made by many people. It is, as discussed in [Neumann 1998], also a consequence of Conjecture 6.3 and hence of the Ramakrishnan conjecture.

Conjecture 7.4. *The map* $\mathrm{vol} : \mathcal{D}(\mathbb{H}^3) \to \mathbb{R}$ *is injective.*

Snap provides many examples like the above which give evidence for this conjecture.

8. SOME TABLES

The tables in this section list some arithmetic and numerical invariants of hyperbolic 3-manifolds computed using Snap. Much more extensive tables of results are available; see [Goodman et al. 1998].

In the column "Invariant trace field" we list the canonical minimal polynomial p defining the field, the signature $[r_1, r_2]$, and the canonical root number as described in footnote 2 on page 134.

In the column "Quaternion algebra" we list the finite ramification (giving generators for the corresponding prime ideals), then real ramification of the invariant quaternion algebra (giving the root number for the corresponding real field embeddings). The Int column indicates whether all traces are integral and the Ar column whether the manifold is arithmetic. Manifolds are named using SnapPea notation; $*$ denotes the mirror image of a manifold.

Table 2 lists invariants for the first 12 closed hyperbolic 3-manifolds in the Hodgson–Weeks census [$\geq$ 2000]. These are conjectured to be the 12 hyperbolic 3-manifolds of smallest volume.

Table 3 includes examples of closed manifolds chosen to illustrate phenomena such as

- manifolds with the same invariant trace field but different invariant quaterion algebras,

Manifold Volume	Eta invariant Chern–Simons (mod 1)	Invariant trace field Quaternion algebra	Int	Ar
m003(−3, 1) 0.94270736277692772092	0.04002871111915143667 0.06004306667872715501	$x^3-x^2+1[1,1](-2)$ $(5,x-2)[1]$	Y	Y
m003(−2, 3) 0.98136882889223208809	0.71802545350918014836 0.07703818026377022254	$x^4-x-1[2,1](3)$ $[1,2]$	Y	Y
m007(3, 1) 1.01494160640965362502	0.00000000000000000000 −0.50000000000000000000	$x^2-x+1[0,1](1)$ $(2)(3,x+1)[\,]$	Y	Y
m003(−4, 3) 1.26370923865804365588	0.92390622935375341671 0.38585934403063012507	$x^4-x^3+x^2+x-1[2,1](-3)$ $[1,2]$	Y	Y
m004(6, 1) 1.28448530046835444246	1.04528778231990871951 0.06793167347986307927	$x^3+2x-1[1,1](2)$ $(2,x^2+x+1)[1]$	Y	Y
m004(1, 2) 1.39850888415080664050	−0.83107150176717910541 −0.24660725265076865812	$x^7-2x^6-3x^5+3x^4+5x^3-x^2-3x+1[5,1](6)$ $[2,3,4,5]$	Y	N
m009(4, 1) 1.41406104416539158138	0.38440137776571728943 0.07660206664857593414	$x^3-x^2+1[1,1](2)$ $(5,x-2)[1]$	Y	Y
m003(−3, 4) 1.41406104416539158138	0.41217915554349506721 0.11826873331524260081	$x^3-x^2+1[1,1](2)$ $(19,x-3)[1]$	Y	Y
m003(−4, 1) 1.42361190029282524980	−0.25828989863587927861 −0.38743484795381891791	$x^5-x^3-x^2+x+1[1,2](2)$ $(13,x+5)[1]$	Y	N
m004(3, 2) 1.44069900672736487528	−0.49337380630786866586 0.25993929053819700120	$x^6-x^5-2x^4-3x^3+3x^2+3x-2[4,1](5)$ $(2,x)[1,3,4]$	Y	N
m004(7, 1) 1.46377664492723877337	1.37374457756475854543 0.06061686634713781814	$x^6-x^5+x^4+2x^3-4x^2+3x-1[2,2](3)$ $[1,2]$	Y	N
m004(5, 2) 1.52947732943002626282	−0.15641491224094610942 −0.23462236836141916413	$x^7-x^6-2x^5+5x^4-6x^2+x+1[5,1](6)$ $[2,3,4,5]$	Y	N

TABLE 2. Arithmetic invariants for the first 12 manifolds from the Hodgson–Weeks closed census.

- closed manifolds with the full matrix algebra as invariant quaternion algebra (i.e., no ramification),
- arithmetic and non-arithmetic manifolds with the same invariant quaterion algebra,
- manifolds with the same abstract invariant trace field, but different complex embeddings,
- manifolds with the same invariant quaternion algebra, but not commensurable (distingushed by integrality of traces).

For cusped manifolds, the invariant quaternion algebra is always the full matrix algebra over the invariant trace field. For non-arithmetic cusped manifolds with one cusp, we list another useful commensurability invariant: the density of a maximal embedded horoball cusp (see [Neumann and Reid 1992b]). A similar invariant can be defined for multicusped cusped non-arithmetic manifolds, provided that there is a finite sheeted covering where all cusps are equivalent under the symmetry group. In this case, we compute the cusp density by taking equal volume horoballs at all the cusps.

Table 4 includes examples of cusped manifolds chosen to ilustrate phenomena such as

- arithmetic and non-arithmetic manifolds with the same invariant quaterion algebra,
- non-arithmetic manifolds with the same invariant quaternion algebra but different cusp densities,
- manifolds with the same abstract invariant trace field, but different complex embeddings.

This table includes some familiar knot complements: m004, m015, m016, m032 are the complements of knots 4_1, 5_2, the $-2, 3, 7$-pretzel, and knot 6_1 respectively. A table of arithmetic invariants computed

Invariant trace field	Quaternion algebra	Int	Ar	Manifolds
$x^2+1[0,1](1)$	$(2,x+1)(3)[\,]$	Y	Y	m304(5,1) m336(−1,3) s942(−2,1) s960(−1,2)
	$(2,x+1)(5,x-2)[\,]$	Y	Y	m293(4,1) s297(−1,3) s572(1,2) s645(−1,2) s682(−3,1) s775(−1,2) s778(−3,1) v3213(−1,3) v3216(4,1)
	$(2,x+1)(5,x+2)[\,]$	Y	Y	m006(1,3) m009(−5,1) m009(5,1) m010(−2,3) m294(4,1) m312(−1,3) s296(5,1) s350(−4,1) s495(1,2) s595(3,1) s775(−3,1) s779(2,1) v3217(−1,3) v3412(5,1)
	$[\,]$	N	N	m239(−2,3) s254(−3,2)
$x^2-x+2[0,1](1)$	$(2,x)(7,x+3)[\,]$	Y	Y	m140(−4,1) v3110(3,1) v3147(−3,1)
		N	N	v3377(−3,1) v3378(−3,1) v3390(3,1)
$x^3+2x-1[1,1](2)$	$(2,x^2+x+1)[1]$	Y	Y	m004(6,1) m160(1,2) m306(−5,1) m307(−1,3) s554(3,1) s594(−3,2)* v2642(5,1) v2643(−2,3)
	$(2,x+3)(2,x^2+x+1)[\,]$	Y	N	m136(1,2) v2920(−1,2)* v3066(1,2) v3528(3,1)
$x^3+x-1[1,1](2)$	$[\,]$	Y	N	s772(−5,2) s772(3,2)* s775(−5,2) s775(3,2)* s778(−5,2) s778(3,2)* s779(−5,2) s779(3,2)* s787(−5,2) s787(3,2)*
$x^3-x-2[1,1](2)$	$(2,x+1)[1]$	N	N	m293(−2,3)* m390(3,1)*
		Y	Y	m307(−5,1)* m369(−1,3) m371(1,3)* s298(5,1) s594(1,2)* s594(2,1)
	$(2,x+1)(2,x)[\,]$	Y	N	s235(−4,3) s595(1,2)
$x^4-2x^3-x^2+2x+2[0,2](2)$	$(13,x+2)(13,x-3)[\,]$	N	N	v3207(5,1) v3209(4,3) v3210(5,1) v3208(4,3)
$x^4+x^2-x+1[0,2](2)$	$[\,]$	Y	N	s594(−3,4)* s594(−4,3)
		N	N	v2050(4,1)* v3404(1,3)
$x^4+x^2-x+1[0,2](1)$	$[\,]$	Y	N	m010(−1,3) m368(4,1) m369(3,1)* m370(−4,1)* s313(−2,3)* s554(1,3)
$x^5-x-1[1,2](2)$	$(2,x^3+x^2+1)[1]$	Y	N	v3221(1,2) v3228(−1,2)*
$x^5-x-1[1,2](3)$	$(2,x^3+x^2+1)[1]$	Y	N	v3100(1,3)

TABLE 3. Arithmetic invariants of some selected manifolds from the Hodgson–Weeks closed census.

using Snap for the complements of knots with up to 8 crossings is given in [Callahan and Reid 1998].

Table 5 lists Borel regulators and arithmetic invariants for all the closed and cusped census manifolds for which the invariant trace field has been computed to be $x^4 + x^2 - x + 1$. Some of these examples are discussed in [Neumann and Yang 1999]. Note that the first two Borel regulators are proportional for the field with root 2, while all three Borel regulators are proportional for the field with root 1. The table also includes examples of the following phenomena:

- manifolds with same Borel regulator but different invariant quaternion algebras,
- closed and cusped manifolds with the same Borel regulator,
- manifolds v2050(4,1) and v3404(1,3) with the same arithmetic invariants (invariant trace field, invariant quaternion algebra, non-integral traces)

Invariant trace field	Int	Ar	Cusp density	Manifolds
$x^2+1[0,1](1)$	Y	Y		m001 m124 m125 m126 m127 m128 m129 m130 m131 m132 m133 m134 m135 m136 m139 m140 s859 v1858 v1859
	N	N	0.614106035	m137 m138
$x^2-x+1[0,1](1)$	Y	Y		m000 m002 m003 m004 m025 m202 m203 m204 m205 m206 m207 m208 m405 m406 m407 m408 m409 m410 m411 m412 m413 m414 s118 s119 s594 s595 s596 s955 s956 s957 s958 s959 s960 v2873 v2874 v3551
	N	N	0.568850725	v2875
$x^2-x+2[0,1](1)$	Y	Y		m009 m010 s772 s773 s774 s775 s776 s777 s778 s779 s780 s781 s782 s784 s786 s787
	N	N	N/A	s785
	N	N	0.558071819	s783
	N	N	0.620079799	s788 s789 v1539 v1540
$x^3-x^2+1[1,1](2)$	Y	N	0.511270966	s898 v2202* v2203
	Y	N	0.524808681	v3428*
	Y	N	0.539001522	v3429*
	Y	N	0.545958189	v0769
	Y	N	0.575271908	s420*
	Y	N	0.604035858	v3426 v3427
	Y	N	0.612276793	v2204* v2205*
	Y	N	0.697799972	m015* m017* s899 s900
	Y	N	0.711685428	m016* s897
$x^3-x^2+x+1[1,1](2)$	Y	N	N/A	v3220 v3223*
	N	N	N/A	v3224*
	Y	N	0.616691512	m035 m037 m039* m040* v3218 v3222* v3225* v3227*
	Y	N	0.623017665	m376*
	Y	N	0.645539037	m036* m038 v3214 v3215* v3216 v3217*
	N	N	0.646337229	v3226
	N	N	0.652161114	s448
	Y	N	0.675735988	v3207 v3208 v3209 v3210
	Y	N	0.717278605	v3219 v3221 v3228*
	Y	N	0.726163222	v3211 v3212 v3213*
$x^4+x^2-x+1[0,2](1)$	Y	N	0.614493011	m161*
	Y	N	0.631076941	s919*
	Y	N	0.662737952	m159* m160
$x^4+x^2-x+1[0,2](2)$	Y	N	0.595110801	s235
	Y	N	0.630681177	m032* m033*
	Y	N	0.686680170	s435* s436*

TABLE 4. Arithmetic invariants of some selected manifolds from the Hildebrand–Weeks cusped census. (The notation N/A under "Cusp density" indicates that there are inequivalent cusps.)

Invariant trace field	Borel regulator	Quaternion algebra	Int	Ar	Manifolds	Chern–Simons (mod $\frac{1}{2}$)
$x^4+x^2-x+1(2)$	$\begin{bmatrix} -1.41510489726556334068 \\ 3.16396322888314398399 \end{bmatrix}$	$(2,x+3)(13,x+6)[\,]$	Y	N	m140$(5,2)^*$	0.17735631658981817209
		$(2,x+3)(2,x^3+x^2+1)[\,]$	Y	N	m136$(5,2)^*$	0.21902298325648483876
		$(2,x+3)(233,x+72)[\,]$	Y	N	m140$(-5,2)$	−0.23931035007684849456
		$[\,]$	Y	N	m032*	−0.15597701674351516123
					m033*	0.09402298325648483876
	$\begin{bmatrix} -2.12265734589834501103 \\ 4.74594484332471597598 \end{bmatrix}$	$[\,]$	Y	N	s435*	0.05770114155139392481
					s436*	−0.19229885844860607518
	$\begin{bmatrix} -0.21181355280835614147 \\ 4.39667280193249561612 \end{bmatrix}$	$(2,x+3)(19,x-5)[\,]$	Y	N	s855$(3,2)$	−0.24238579181095171467
		$[\,]$	Y	N	s594$(-3,4)^*$	0.00761420818904828532
					s594$(-4,3)$	−0.24238579181095171467
					s235	0.13261420818904828532
			N	N	v2050$(4,1)^*$	−0.20071912514428504800
	$\begin{bmatrix} 0.99147779164885105773 \\ 5.62938237498184724825 \end{bmatrix}$	$[\,]$	N	N	v3404$(1,3)$	−0.16212790021172160144
$x^4+x^2-x+1(1)$	$\begin{bmatrix} 1.91084379308998886955 \\ -0.34927204139222035986 \end{bmatrix}$	$[\,]$	Y	N	m010$(-1,3)$	−0.09574639997098769384
	$\begin{bmatrix} 3.82168758617997773911 \\ -0.69854408278444071973 \end{bmatrix}$	$(2,x+3)(29,x-14)[\,]$	Y	N	m294$(4,3)^*$	−0.02482613327530872102
					m293$(-4,1)$	0.22517386672469127897
		$[\,]$	Y	N	m368$(4,1)$ m369$(3,1)^*$	0.14184053339135794563
					m370$(-4,1)^*$ s554$(1,3)$	0.14184053339135794563
					m160 m161*	−0.19149279994197538769
					m159*	0.05850720005802461230
	$\begin{bmatrix} 5.73253137926996660866 \\ -1.04781612417666107959 \end{bmatrix}$	$[\,]$	Y	N	s919*	−0.16223919991296308154

TABLE 5. Bloch invariants of some closed and cusped manifolds with invariant trace field x^4+x^2-x+1.

but not commensurable as their Borel regulators are not proportional.

ADDENDUM

The referee raised the concern that the numerical solutions that Snap uses in proving a hyperbolic structure might be spurious — i.e., although the minimal polynomial evaluated at the given "numerical root" is vanishingly small, there may nevertheless actually be no root of the minimal polynomial near the given "numerical root". This is not an issue. To quote from the manual for the Pari libraries: "The algorithm used is a modification of A. Schönhage's remarkable root-finding algorithm, due to and implemented by X. Gourdon. Barring bugs, it is guaranteed to converge and to give the roots to the desired accuracy." See [Schönhage 1987] for a discussion.

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