On the Volume of a Certain Polytope

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Let $n \geq 2$ be an integer and consider the set T_n of $n \times n$ permutation matrices π for which $\pi_{ij} = 0$ for $j \geq i+2$.

We study the convex hull P_n of T_n, a polytope of dimension $\binom{n}{2}$. We provide evidence for several conjectures involving P_n, including Conjecture 1: Let v_n denote the minimum volume of a simplex with vertices in the affine lattice spanned by T_n. Then the volume of P_n is v_n times the product

$$\prod_{i=0}^{n-2} \frac{1}{i+1}\binom{2i}{i}$$

of the first $n-1$ Catalan numbers.

We also give a related result on the Ehrhart polynomial of P_n.

Editor's note: After this paper was circulated, Doron Zeilberger [1998] proved Conjecture 1, using the authors' reduction of the original problem to a conjectural combinatorial identity, and sketched the proofs of two others. The problems and methodology presented here gain even further interest thereby.

1. INTRODUCTION

Let $n \geq 2$ be an integer and consider the set T_n of $n \times n$ permutation matrices π for which $\pi_{ij} = 0$ for $j \geq i+2$ and P_n the convex hull of T_n.

Let V_n be the relative volume of P_n. That is, the volume of P_n expressed in units of the minimum volume v_n of a simplex with vertices in the affine lattice spanned by T_n. Our main purpose in this paper is to provide evidence for the following conjecture.

Conjecture 1. The relative volume V_n of P_n is

$$V_n = \prod_{i=0}^{n-2} \frac{1}{i+1}\binom{2i}{i},$$

the product of the first $n-1$ Catalan numbers.

This conjecture arose from the study [Chan and Robbins 1999] of the polytope B_n of all doubly stochastic matrices, which is the convex hull of the set of all $n \times n$ permutation matrices. It is easily shown

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that P_n is a face of B_n of dimension $\binom{n}{2}$ with 2^{n-1} vertices. In [Chan and Robbins 1999] we discuss two methods for finding the volume of B_n and its faces. We assume some familiarity with these methods, which apply to the calculation of the volume of P_n. The reader may also wish to consult the references [Billera and Sarangarajan 1996; Diaconis and Gangolli 1995, Hibi 1992, Chapter 9; Stanley 1980], which provide background for the work in [Chan and Robbins 1999].

The first method discussed in that earlier paper consists of decomposing the polytope into simplices, each of volume v_n, and counting the simplices. By adapting the method slightly we were able easily to find the relative volumes of P_n and its faces provided that $n \leq 10$. This provided the first evidence for Conjecture 1.

The second method discussed in [Chan and Robbins 1999] computes the Ehrhart polynomial of the polytope [Ehrhart 1977]. In general the Ehrhart polynomial of a d-dimensional polytope P with integer vertices is a degree d polynomial (in t) denoted $e(P,t)$, with the property that the number of integer points in the polytope $t\cdot P$ is $e(P,t)$ when $t \geq 0$. A basic property of the Ehrhart polynomial is that the relative volume of the polytope is given by $d!$ times its leading coefficient. A common method for computing the Ehrhart polynomial is to count the numbers of lattice points in $t\cdot P$ for small t and then to find the polynomial by interpolation. For a typical face of B_n the Ehrhart polynomial method seems to be more expensive than the simplicial decomposition method. However for B_n itself the Ehrhart polynomial method is less expensive because it is possible to exploit the symmetries of B_n. These symmetries do not help with the calculation of the Ehrhart polynomial of P_n. However, different simplifications in the case of P_n allow us to compute the Ehrhart polynomial and thus verify Conjecture 1 for $n \leq 12$, as described in Section 2 of this paper.

In Section 3 of this paper we give a proof of a bijection between the simplices in a decomposition of P_n and a set of easily described integer arrays, which suggest that a combinatorial proof of Conjecture 1 may exist. We also discuss a generalization of the conjecture which arises from the bijection.

In Section 4 we discuss formulas for the relative volumes of some of the facets of P_n, which we pursued as an alternative path toward proving Conjecture 1. The formulas were discovered by using the simplicial decomposition method mentioned above.

2. THE EHRHART POLYNOMIAL OF P_n

One approach to calculating the volume of P_n is to calculate its Ehrhart polynomial. Denote by $e(P_n,t)$ the Ehrhart polynomial of P_n evaluated at t. Then $e(P_n,t)$ is the number of ways of filling a left-justified array of n rows of lengths $2,3,\dots,n-1,n,n$ with nonnegative integers in such a way that all row and column sums are t. Thus $e(P_3,1)=4$ since the only four suitable arrays are

$$\begin{matrix}1&0&\\0&1&0\\0&0&1\end{matrix}\qquad\begin{matrix}1&0&\\0&0&1\\0&1&0\end{matrix}\qquad\begin{matrix}0&1&\\1&0&0\\0&0&1\end{matrix}\qquad\begin{matrix}0&1&\\0&0&1\\1&0&0\end{matrix}$$

It is known that $e(P_n,t)$ is a polynomial in t whose degree as a function of t is the dimension of P_n or $\binom{n}{2}$.

We have $e(P_n,0)=1$ for all n. Also it is easily verified that $e(P_n,1)=2^{n-1}$ for all n. These are special cases of a more general principle.

Theorem 1. *For every nonnegative integer t, the sequence*

$$e(P_0,t), e(P_1,t), \dots, e(P_n,t), \dots$$

satisfies a linear recursion of degree $p(t)$ with integer coefficients, where $p(t)$ is the number of partitions of t.

It is conceivable, as far as we know, that the sequences also satisfy recursions of lower degree. However, for all cases $t=0,\dots,12$, where we have computed the coefficients of the linear recursion, the associated characteristic polynomial has been irreducible over the integers so, in these cases, no lower degree recursion exists.

Proof. Fix a nonnegative integer t. Let

$$\pi = (x_1, x_2, \dots, x_l)$$

be a partition of t of length $l \geq 1$. That is, $0 < x_1 \leq \cdots \leq x_l$ and $x_1+\cdots+x_l = t$. For integers $n \geq 2$ let $F(\pi,n)$ denote the set of arrays of n left-justified rows of nonnegative integers of lengths $l+1$, $l+2$, …, $l+n-2$, $l+n-1$, $l+n-1$, such that the first l column sums are $x_1,\dots,x_l$, the remaining column sums are t, and all row sums are t. Let $f(\pi,n)$ be the cardinality of $F(\pi,n)$, with $f(\pi,1)=1$. (If π is the one-part partition (t), we have $f(\pi,n)=e(P_n,t)$.)

Suppose that $n\geq 2$. Set $x_{l+1}=t$. Let $y_1,\ldots,y_{l+1}$ be any nonnegative integers with $y_i\leq x_i$ for $i=1,\ldots,l+1$ such that $y_1+\cdots+y_{l+1}=t$. If $y_1,\ldots,y_{l+1}$ is the first row of one of the arrays of $F(\pi,n)$, then the rest of the array has its first $l+1$ column sums equal to $z_i=x_i-y_i$. By deleting the z_i's which equal 0 and sorting the remaining z_i's, we obtain another partition σ of t, of length at most $l+1$, and the number of ways of completing the array is clearly $f(\sigma,n-1)$. (Since $f(\sigma,1)=1$, this also holds for $n=2$.) Now for every partition σ, let $M(\pi,\sigma)$ denote the number of $(l+1)$-tuples $y_1,\ldots,y_{l+1}$ for which our process of forming the z's by subtracting from the x's, deleting 0's, and sorting yields the partition σ. Then we have shown that

$$f(\pi,n)=\sum_{\sigma}M(\pi,\sigma)f(\sigma,n-1).$$

For fixed n we can regard the array of $f(\pi,n)$, as π varies over partitions of t, as a column vector of integers of length $p(t)$. When $n=1$ we have the vector of all 1's. The preceding equation shows that the n-th vector is obtained by applying the matrix M^{n-1} to the all 1's vector. Thus the sequence of vectors satisfies a linear recursion with integer coefficients given by the characteristic polynomial of the matrix M. In particular, the component of the column vector $f(\pi,n)$ corresponding to the partition (t) is $e(P_n,t)$, which proves our theorem. □

Example: It is easy to compute the matrix M for small values of t. For example, let $t=2$. If $\pi=(2)$, then $(x_1,x_2)=(2,2)$, so $(y_1,y_2)=(0,2)$, $(1,1)$ and $(2,0)$, which yield $\sigma=(2)$, $(1,1)$ and (2) respectively. If $\pi=(1,1)$, then $(x_1,x_2,x_3)=(1,1,2)$, so $(y_1,y_2,y_3)=(0,0,2)$, $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$, which yield $\sigma=(1,1)$, $(1,1)$, $(1,1)$, and (2), respectively. Thus we have

$$M=\begin{pmatrix}2&1\\1&3\end{pmatrix},$$

with the rows and columns of M indexed by the partitions (2) and (11) in that order. The characteristic polynomial of M is $\lambda^2-5\lambda+5$. Thus we have $e(P_n,2)=5e(P_{n-1},2)-5e(P_{n-2},2)$. Initial values are $e(P_1,2)=1$ (by definition) and $e(P_2,2)=3$ (by applying M to the all ones vector).

Theorem 1 and its example contain the essential ideas behind our method for evaluating $e(P_n,t)$ for small values of t. If we wish to calculate a value of $e(P_n,t)$ for which n is also small, we can simplify a little more by computing and using only the submatrix of M corresponding to partitions of length not exceeding n.

Denote the characteristic polynomial of the matrix associated to the nonnegative integer t by $f_t(\lambda)$. The first 6 polynomials are

$$\begin{aligned}
f_0&=\lambda-1,\\
f_1&=\lambda-2,\\
f_2&=\lambda^2-5\lambda+5,\\
f_3&=\lambda^3-10\lambda^2+27\lambda-20,\\
f_4&=\lambda^5-20\lambda^4+135\lambda^3-396\lambda^2+518\lambda-245,\\
f_5&=\lambda^7-36\lambda^6+480\lambda^5-3140\lambda^4\\
&\qquad+11059\lambda^3-21180\lambda^2+20560\lambda-7840.
\end{aligned}$$

As far as we have computed, all the roots of these polynomials are positive real numbers.

We can also use Ehrhart's reciprocity principle to simplify the computation of the Ehrhart polynomial. Recall that, for a d-dimensional polytope P with integer vertices, and $t>0$, Ehrhart's reciprocity principle states that

$$e^*(P,t)=(-1)^d e(P,-t)$$

where $e^*(P,t)$ is the the number of lattice points in the interior of $t\cdot P$.

An interior lattice point of $t\cdot P_n$ is an array of positive integers consisting of left-justified rows of length $2,3,\ldots,n,n$ with all row and column sums equal to t. For such an array, if $k<n$, the first k rows have lengths $2,3,\ldots,k,k+1$ and the sum of all their entries taken together is tk. On the other hand, the sum of all entries in the first $(k+1)$ columns is $(k+1)t$, and this includes all entries in the first k rows. Thus the sum of all entries in the first $k+1$ columns of the last $n-k$ rows must be t. Since all entries are positive, it follows that $t\geq(k+1)(n-k)$ and this inequality must hold for $k=0,\ldots,n-1$. Thus if an interior point exists for a given t, we

must have t at least equal to the maximum over k of $(k+1)(n-k)$. Thus for odd $n = 2m+1$, we have $e(P_n, -t) = 0$ for $t = 1, \dots, (m+1)^2 - 1$, while for even $n = 2m$, we have $e(P_n, -t) = 0$ for $t = 1, \dots, m(m+1) - 1$.

We have calculated the Ehrhart polynomials of P_n for $n = 2, \dots, 12$. The first few are

$$e(P_2,t) = t+1,$$

$$e(P_3,t) = \frac{1}{6}\prod_{i=1}^{3}(t+i),$$

$$e(P_4,t) = \frac{t+3}{360}\prod_{i=1}^{5}(t+i),$$

$$e(P_5,t) = \frac{(t+3)^2}{362880}\prod_{i=1}^{8}(t+i),$$

$$e(P_6,t) = \frac{(t+3)^2(t^2+12t+26)}{9340531200}\prod_{i=1}^{11}(t+i),$$

$$e(P_7,t) = \frac{(t+3)^2(14t^4+353t^3+2985t^2+9568t+10336)}{121645100408832000} \times \prod_{i=1}^{15}(t+i).$$

The factor $(t+3)^2$, which appears in $e(P_5,t)$, persists through $n = 12$, but we have no proof that it persists forever.

To check Conjecture 1, one multiplies the leading coefficient of $e(P_n,t)$ by $\binom{n}{2}!$, to get the predicted relative volume. This works through $n = 12$.

3. EXPLICIT DECOMPOSITION INTO SIMPLICES

In this section we show that the polytope P_n can be decomposed into minimal volume simplices which are in bijection with an easily described set of integer arrays. Thus the relative volume of P_n is simply the number of such integer arrays. This suggests an avenue for proving Conjecture 1, although we have not been successful thus far. Postnikov and Stanley [Postnikov and Stanley 1998] have found a bijection very much like ours, and also observed that therefore Conjecture 1 is equivalent to

$$K\left(a_1 + 3a_2 + 6a_3 + \dots + \binom{n}{2}a_{n-1}\right) = \prod_{i=0}^{n-1}\frac{1}{i+1}\binom{2i}{i}$$

where $a_1, \dots, a_{n-1}$ is a choice of simple roots and K is the Kostant partition function for the root system A_{n-1}.

To describe our decomposition we first need some notation.

The polytope P_n consists of doubly stochastic matrices $Y = (y_{ij})$ where $y_{ij} = 0$ for $j > i+1$.

However the entries y_{ij}, $j \le i \le n-1$, determine the remaining $2n-1$ entries $y_{12}, y_{23}, \dots, y_{n-1,n}$ and $y_{n1}, \dots, y_{nn}$. Thus we may view a point Y in P_n as a triangular array

$$\begin{array}{ccccc} y_{11} & & & & \\ y_{21} & y_{22} & & & \\ \vdots & \vdots & & \ddots & \\ y_{n-1,1} & y_{n-1,2} & y_{n-1,3} & \cdots & y_{n-1,n-1} \end{array}$$

where the nonnegative y_{ij}'s satisfy the conditions

$$\sum_{i=k}^{n-1} y_{ik} \le \sum_{j=1}^{k-1} y_{k-1,j} \le 1 \qquad (1)$$

for $k = 2, \dots, n-1$ and

$$\sum_{i=1}^{n-1} y_{i1} \le 1 \qquad (2)$$

so that the first column has sum ≤ 1.

Let $\mathcal{A}_n$ be the set of triangular arrays of nonnegative integers

$$\begin{array}{cccc} a_{22} & & & \\ a_{32} & a_{33} & & \\ \vdots & \vdots & \ddots & \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{array}$$

where the a's are subject to the constraints

$$\begin{aligned} &a_{22} + \dots + a_{n-1,2} \le 0, \\ &a_{33} + \dots + a_{n-1,3} \le 1 + a_{22}, \\ &a_{44} + \dots + a_{n-1,4} \le 2 + a_{32} + a_{33}, \\ &a_{55} + \dots + a_{n-1,5} \le 3 + a_{42} + a_{43} + a_{44}, \\ &\quad\vdots \\ &a_{n-1,n-1} \le n-3 + a_{n-2,2} + a_{n-2,3} + \dots + a_{n-2,n-2} \end{aligned} \qquad (3)$$

Note that this condition implies that the leftmost column in the array is all zeros.

For example, $\mathcal{A}_5$ consists of these 10 triangular arrays:

$$\begin{matrix}0&&\\0&0&\\0&0&0\end{matrix}\qquad\begin{matrix}0&&\\0&0&\\0&0&1\end{matrix}\qquad\begin{matrix}0&&\\0&0&\\0&0&2\end{matrix}$$

$$\begin{matrix}0&&\\0&0&\\0&1&0\end{matrix}\qquad\begin{matrix}0&&\\0&0&\\0&1&1\end{matrix}\qquad\begin{matrix}0&&\\0&0&\\0&1&2\end{matrix}$$

$$\begin{matrix}0&&\\0&1&\\0&0&0\end{matrix}\qquad\begin{matrix}0&&\\0&1&\\0&0&1\end{matrix}\qquad\begin{matrix}0&&\\0&1&\\0&0&2\end{matrix}\qquad\begin{matrix}0&&\\0&1&\\0&0&3\end{matrix}$$

We will give a decomposition of P_n into simplices all of the same volume in such a way that the simplices in the decomposition will be in one-to-one correspondence with the set $\mathcal{A}_n$.

We start by defining a mapping which assigns to each element α of $\mathcal{A}_n$ a simplex contained in P_n.

Let V be the space of triangular arrays

$$\begin{matrix} x_{11} & & & \\ x_{21} & x_{22} & & \\ \vdots & \vdots & \ddots & \\ x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n-1} \end{matrix}$$

Let S be the unit simplex in V consisting of all nonnegative arrays of the preceding form in which the sum of all the entries is ≤ 1. The simplex associated to α will be the image of S under a certain linear transformation of determinant 1 which is associated to α. We will denote this linear transformation by $L(\alpha)$.

We can construct many such mappings from α's to simplices, each of which yields a suitable decomposition of P_n. A convenient way to specify a single one of these is to assign a linear ordering to the variables x_{ij}. It does not matter what linear ordering we use.

To form $L(\alpha)$ we start with the identity matrix, represented by the preceding triangle. Then $L(\alpha)$ is formed by a series of steps, one step for each column of α. At the beginning of each step we have a linear transformation consisting of a triangle of linear functions of the x's. The step itself consists in performing certain operations on the triangle, leading to another triangle of linear functions.

After each step all the linear functions in each triangle are of a particularly simple form:

C0. each linear function is a linear combination of the x's all of whose coefficients are 0 or 1; i.e. a sum of distinct x's. Moreover no two entries in any row involve the same variable and no two entries in any column involve the same variable. Also, after having used columns $2, \ldots, k$ of α, the triangle of linear functions has certain additional properties depending on k.

C1. In the rectangular subarray consisting of columns $1, \ldots, k$ of rows k through $n-1$, no two of the linear functions share any variables.

C2. Within this rectangle, if $j \geq 2$, entry ij is a sum of precisely $a_{ij}+1$ variables, while entry $i1$ is just x_{i1}, a sum of 1 variable.

C3. The only variables that occur in columns $1, \ldots, k$ of the triangle are those that were originally in these columns.

C4. In columns $k+1, \ldots, n-1$, the linear function is just the original variable x_{ij}.

C5. For $2 \leq j \leq k$ the variables appearing in column j of the array are a proper subset of those appearing in row $j-1$.

C6. In the first column of the triangular array every variable originally in the first k columns appears precisely once.

In view of (2) above, after having used columns $2, \ldots, k$ of α, there are precisely

$$k + a_{k2} + \cdots + a_{k,k}$$

variables in all the sums in the k-th row. Denote this number of variables by N. We now list these variables in the assigned order, denoting them as $z_1, \ldots, z_N$.

From our conditions above defining $\mathcal{A}_n$, we have

$$N > a_{k+1,k+1} + \cdots + a_{n-1,k+1}.$$

Before actually modifying the triangle of linear functions we first use column $k+1$ of α to parse the z's into "chunks", putting the first $a_{k+1,k+1}$ z's into the first chunk, the next $a_{k+2,k+1}$ z's into the next chunk, and so forth, one chunk for each entry $a_{i,k+1}$ in column $k+1$ of α. The inequality (3) guarantees that there is at least one more variable available than is needed to form all the chunks. Notice that some of the chunks can be empty and that at least one of the z's does not appear in any chunk.

After we form each chunk, we associate to it the first of the z's (in our ordering) that has not yet appeared in any chunk (including the chunk just formed) and call this the "cap" of the chunk just formed. Note that since some of the chunks are empty, it is possible for chunks associated to several consecutive entries $a_{i,k+1}$ to have the same cap.

Now we modify the triangle of linear functions in two substeps.

First, for each $i = k+1, \ldots, n-1$, we let z be the cap of the chunk associated to $a_{i,k+1}$, and then replace every occurrence of z in columns $1, \ldots, k$ of the triangle of linear functions with $z + x_{i,k+1}$. The order in which we perform the substitutions in this substep is immaterial since the variables $x_{i,k+1}$ did not previously appear in columns $1, \ldots, k$ (and hence also are never caps).

Second, for each variable $x_{i,k+1}$ in column $k+1$ of the triangle of linear functions, we replace that variable by a sum consisting of the variable itself plus the sum of all the variables in the chunk associated to $a_{i,k+1}$.

Conditions C0–C6 holds for the initial triangle and it is easy to see, inductively, that the modification rules above preserve the conditions. Thus they hold at every stage.

A somewhat deeper property of our inductive procedure is that, at each stage, the triangle of linear functions represents a linear transformation of determinant 1.

The first substep is a linear substitution of determinant 1. However, the second substep is not strictly a linear substitution since the first substep results in occurrences $x_{i,k+1}$ in the columns $1, \ldots, k$, while, in the second substep, we do not perform the substitution in these occurrences.

However, we can obtain the second substep by a sequence of pairs of linear substitutions of determinant 1. Indeed for each $x_{i,k+1}$ in column $k+1$ we substitute for its cap variable the cap variable minus the sum of the associated chunk variables, and then substitute for the $x_{i,k+1}$, $x_{i,k+1}$ plus the sum of the chunk variables. The effect of the these two substitutions is to leave columns $1, \ldots, k$ unchanged but to perform the desired substitution in column $k+1$.

Here is an illustration of the procedure described above. Suppose that $n = 5$ and that the array α is

$$\begin{array}{llll} 0 & & & \\ 0 & 1 & & \\ 0 & 0 & 2 & \\ 0 & 0 & 1 & 2 \end{array}$$

Take the array of variables (called x_{ij} above) to be

$$\begin{array}{lllll} A & & & & \\ B & F & & & \\ C & G & J & & \\ D & H & K & M & \\ E & I & L & N & O \end{array}$$

and order the variables alphabetically.

When the first column of α is used, all 4 chunks have length 0 and cap A. So the effect is that all four variables are added to A, yielding

$$\begin{array}{lllll} AFGHI & & & & \\ B & F & & & \\ C & G & J & & \\ D & H & K & M & \\ E & I & L & N & O \end{array}$$

where, for the rest of this example, we designate addition by juxtaposition, so that $AFGHI$ means $A+F+G+H+I$.

When the second column of α is used, the variables in the second row of the triangle are B, F and there are three chunks, the first is B and the last two are empty. All three have cap F. Thus we obtain

$$\begin{array}{lllll} AFGHIJKL & & & & \\ B & FJKL & & & \\ C & G & BJ & & \\ D & H & K & M & \\ E & I & L & N & O \end{array}$$

When the third column of α is used there, are four variables in the third row of our triangle, namely B, C, G, J and there are two chunks, B, C and G, with caps G and J respectively. The chunks are adjoined to M and N and, in the first three columns of the triangle, G is replaced by GM and J by JN. Thus we obtain

$$\begin{array}{lllll}
AFGHIJKLMN & & & & \\
\quad B & FJKLN & & & \\
\quad C & GM & BJN & & \\
\quad D & H & K & BCM & \\
\quad E & I & L & GN & O
\end{array}$$

Finally, when the last column of α is used, there are 6 variables in the fourth row of the triangle, B, C, D, H, K, M. We form one chunk of size 2, namely B, C, with cap D, obtaining

$$\begin{array}{lllll}
AFGHIJKLMN & & & & \\
\quad B & FJKLN & & & \\
\quad C & GM & BJN & & \\
\quad DO & H & K & BCM & \\
\quad E & I & L & GN & BCO
\end{array}$$

The unit simplex in V is the set of 15-tuples $A, \ldots, O$ of nonnegative reals whose sum is ≤ 1. Still, taking note of our juxtaposition notation for addition, we see that the triangle above defines a linear mapping from the unit simplex to P_5.

It is easy to see, inductively, that this will be the case for any α in $\mathcal{A}_n$. First note the inequality (2) will always hold because of (C6). One also easily verifies that the conditions (1) always hold. The second of the inequalities is a consequence of the fact that the variables occurring in any row are always distinct. The first inequality follows from (C5).

Thus we have associated to every α in $\mathcal{A}_n$ a simplex whose volume is $1/\binom{n}{2}!$.

One needs also to show that the simplices $L(\alpha)$ cover P_n and have disjoint interiors. There is an argument, rather similar to the preceding, in which we start with a point of P_n and build up α and $L(\alpha)$ with a construction like the preceding. But we omit the details.

Thus our conjecture would be proved if we could show that the cardinality of $\mathcal{A}_n$ was given by

$$\prod_{i=0}^{n-2} \frac{1}{i+1}\binom{2i}{i}.$$

We have not been able to show this.

However, this combinatorial interpretation leads to a stronger conjecture. We can classify the elements of $\mathcal{A}_n$ according to the number of times that we have equality in (3). This can hold from 1 to $n-2$ times.

Conjecture 2. If $n \geq 2$ and D_{nk} is the number of elements of $\mathcal{A}_n$ for which equality holds for k of the inequalities (3), then D_{nk} is divisible by

$$\prod_{i=0}^{n-3} \frac{1}{i+1}\binom{2i}{i}$$

and the quotient is

$$N(n,k) = \frac{1}{n-2}\binom{n-2}{k}\binom{n-2}{k-1},$$

the *Narayana number* $N(n-2,k)$.

For example, the following two elements of $\mathcal{A}_5$ satisfy just 1 equality in (3)

$$\begin{array}{lll} 0 & & \\ 0 & 0 & \\ 0 & 0 & 0 \end{array} \qquad \begin{array}{lll} 0 & & \\ 0 & 0 & \\ 0 & 0 & 1 \end{array}$$

while the following two elements of $\mathcal{A}_5$ satisfy three equalities

$$\begin{array}{lll} 0 & & \\ 0 & 0 & \\ 0 & 1 & 2 \end{array} \qquad \begin{array}{lll} 0 & & \\ 0 & 1 & \\ 0 & 0 & 3 \end{array}$$

The remaining 6 satisfy two equalities.

To conclude this section we define a generalization of the set $\mathcal{A}_n$ and a corresponding generalization of Conjecture 1.

Let $\mathcal{A}_n^j$ denote the set of elements of $\mathcal{A}_n$ in which the first j columns consist entirely of zeros. Thus $\mathcal{A}_n^1 = \mathcal{A}_n$ and $\mathcal{A}_n^j \subset \mathcal{A}_n^{j-1}$ for all $j \geq 2$. Here is a small table of values of $\mathcal{A}_n^j$.

n	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
3	1				
4	2	1			
5	10	3	1		
6	140	28	4	1	
7	5880	840	60	5	1

Conjecture 3. The number of elements in $\mathcal{A}_n^j$ is the product

$$\prod_{i=j}^{n-3} \frac{1}{2i+1}\binom{n+i-1}{2i}.$$

With the help of Mathematica we can verify this easily for $n-j \leq 6$.

4. FACETS OF P_n AND THEIR VOLUMES

Another approach toward proving Conjecture 1 is to try to understand the relative volumes of the facets of P_n. In this section we study these facet volumes and make a conjecture concerning these volumes based on evidence obtained by the simplicial decomposition method described in [Chan and Robbins 1999].

Suppose that $n \geq 2$ is an integer and that $1 \leq r, s \leq n$ and $s \leq r+1$. Consider the convex hull $P_n(r,s)$ of those permutations in T_n whose (r,s) entry is zero. Then $P_n(r,s)$ is always a face of P_n. If $n=2$ these are all facets of $P_n(r,s)$, but for $n \geq 3$, $P_n(r,s)$ is a facet of P_n precisely when $r \neq 1$ and $s \neq n$ and $s \neq r+1$.

Since the set T_n is invariant under the operation of exchanging the first two columns and the operation of exchanging the last two rows, the same symmetries apply to the volumes of the facets. Thus the volume of $P_n(r,1)$ is equal to that of $P_n(r,2)$ for all r. Also the volume of $P_n(n,s)$ equals that of $P_n(n-1,s)$. Thus, we can display the volumes of all the facets as a triangular array consisting of the volumes of $P_n(r,s)$ for $2 \leq r \leq n-1$ and $2 \leq s \leq n-1$ and $s \leq r$.

Here are the volumes of $P_n(r,s)$ for $n = 3, \ldots, 7$.

1	

1	
2	1

3		
7	4	
10	7	3

28			
70	42		
112	84	42	
140	112	70	28

840				
2180	1340			
3700	2860	1520		
5040	4200	2860	1340	
5880	5040	3700	2180	840

These arrays have some properties that are easily verified. For example, there is symmetry about the anti-diagonal. There is a slightly deeper fact. In any 2×2 submatrix of the preceding array the sum of the entries on one diagonal of the submatrix is equal to the sum of the entries on the other.

There is a slightly stronger version that can be stated a little more elegantly if we add an extra diagonal of zeroes above the main diagonal and then complete the triangle to a skew-symmetric matrix. For example, the square matrix associated to the last triangle (corresponding to $n = 7$) is

$$\begin{array}{rrrrrr}
0 & -840 & -2180 & -3700 & -5040 & -5880 \\
840 & 0 & -1340 & -2860 & -4200 & -5040 \\
2180 & 1340 & 0 & -1520 & -2860 & -3700 \\
3700 & 2860 & 1520 & 0 & -1340 & -2180 \\
5040 & 4200 & 2860 & 1340 & 0 & -840 \\
5880 & 5040 & 3700 & 2180 & 840 & 0
\end{array}$$

Each square matrix formed this way has the property that, for any of its 2×2 submatrices, the sum on the entries on one diagonal is the same as the sum of the entries on the other.

This property is easily proved. It results from the fact that the relative volume of P_n can be expressed as the sum of the relative volumes of the facets opposite any vertex. The rectangular relations arise from pairs of vertices that (when regarded as permutations) differ by a transposition.

One consequence of the rectangular relations is that all the entries in each triangle depend linearly on the main diagonal so we can describe the whole triangle, much more succinctly, in terms of the diagonal. Here are the diagonals (listed as rows) for $n = 3, \ldots, 10$. (The last four rows need to be completed to be palindromic of length $n-2$.)

1			
1	1		
3	4	3	
28	42	42	28
840	1340	1520	1340
83160	137610	167310	167310
27747720	47016970	59676120	64091020
31743391680	54669174560	71411118240	80251753120

The entries in the first two columns of this array seem to be predictable. Suppose that a_n denotes the entry in the first column and b_n the entry in the second column. Then a_n is defined for $n \geq 3$, and b_n is defined for $n \geq 4$ so that

$$a_3, a_4, a_5, a_6 \ldots = 1, 1, 3, 28 \ldots$$

and

$$b_4, b_5, b_6, b_7 \ldots = 1, 4, 42, \ldots.$$

Conjecture 4. For $n \geq 3$,

$$a_n = 3V_n \Big/ \binom{n}{2}.$$

For $n \geq 4$,

$$(n-1)\left(\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n}\right) = (n+2)\left(\frac{b_{n+2}}{a_{n+2}} - \frac{b_{n+1}}{a_{n+1}}\right).$$

The evidence for the second formula is perhaps not all that compelling since the result is known to hold only for $n = 4, \ldots, 8$. However it is not hard to check that the two formulas above, taken together with Conjecture 1, predict integral values for b_n, for all n. So this gives some additional evidence.

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