On the Representations of xy + yz + zx

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CONTENTS

We show that there are at most 19 integers that are not of the form xy + yz + xz with x, y, z $\geq$ 1. Eighteen of them are small and easily found. The remaining possibility must be greater than 10^{11} and cannot occur if we assume the Generalized Riemann Hypothesis.

1. INTRODUCTION

Recently, Crandall [1999] used Andrews' identity for the cube of the Jacobian theta function θ_4,

$$\theta_4^3(q) = \left(\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\right)^3$$
$$= 1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1+q^n} - 2\sum_{\substack{n=1\\|j|<n}}^{\infty}\frac{q^{n^2-j^2}(1-q^n)(-1)^j}{1+q^n},$$

to derive new representations for several interesting numbers. He considered the three-dimensional Epstein zeta function $M(s)$, which is the analytic continuation of the series

$$\sum_{\substack{x,y,z\in\mathbb{Z}\\(x,y,z)\neq(0,0,0)}}\frac{(-1)^{x+y+z}}{(x^2+y^2+z^2)^s}.$$

The number $M(\frac{1}{2})$ is the celebrated *Madelung constant*. Using a reworking of Andrews' identity, he obtained the formula

$$M(s) = -6(1-2^{1-s})^2\zeta^2(s) - 4U(s)$$

where $\zeta(s)$ is the Riemann zeta function and

$$U(s) := \sum_{x,y,z\geq 1}\frac{(-1)^{x+y+z}}{(xy+yz+xz)^s}.$$

In view of this representation, Crandall asked what integers are of the form of $xy+yz+xz$ for positive integers x, y, z, and he made a conjecture that every odd integer greater than one can be so written. Crandall's conjecture is indeed true. In fact, we show a somewhat stronger result:

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Theorem 1.1. *There are at most* 19 *integers that are not of the form* $xy + yz + xz$ *with* $x, y, z \geq 1$. *The only such integers that are not square-free are* 4 *and* 18. *The first* 16 *square-free integers are*

$$\begin{aligned} &1,\ 2,\ 6,\ 10,\ 22,\ 30,\ 42,\ 58,\ 70,\\ &78,\ 102,\ 130,\ 190,\ 210,\ 330,\ 462. \end{aligned} \tag{1–1}$$

If the 19*th integer exists, it must be greater than* 10^{11}. *Moreover, assuming the Generalized Riemann Hypothesis (GRH), the* 19*th integer does not exist and the list* (1–1) *is then complete.*

If we consider only the even numbers in the list (1.1) and divide them by 2, we get the list

$$\begin{aligned} &1,\ 3,\ 5,\ 11,\ 15,\ 21,\ 29,\ 35,\\ &39,\ 51,\ 65,\ 95,\ 105,\ 165,\ 231. \end{aligned} \tag{1–2}$$

This is the same list as for disjoint discriminants less than 10,000 found by Dickson in 1929. A discriminant of a binary quadratic form is *disjoint* if each *genus* of this discriminant contains exactly one reduced form. So if $N = 2p_1p_2 \cdots p_r$ with distinct odd prime p_j, then $-4N$ is a disjoint discriminant if and only if the *class number* $h(-4N)$ equals to 2^r. It was Heilbronn who first showed that $\lim_{d \longrightarrow \infty} h(-d) = \infty$. Chowla [1934] improved this result by proving that

$$\lim_{d \longrightarrow \infty} \frac{h(-d)}{2^t} = \infty$$

where t is the number of distinct prime factors of d. Chowla's result immediately implies that there are only finitely many disjoint discriminants. However, in order to determine all of them, an explicit estimate is needed. Weinberger [1973] used the zero-free region of the L-function $L(s) = \sum_{n=1}^{\infty} \left(\frac{-d}{n}\right)/n^s$, where $\left(\frac{\cdot}{n}\right)$ is the Kronecker symbol, and proved that the list (1–2) contains the all disjoint discriminants less than 10^{11} and there is exactly one more possible exception, which must be greater than 10^{11}. The exception actually comes from the possible existence of the *Siegel zero* of the above L-function. Thus, if we further assume GRH (in fact, we only need to assume the Siegel zero doesn't exist), then the list (1–2) is indeed complete. In our Theorem Theorem 3.1 below, we prove that a square-free N is not of the form of $xy + zy + xz$ if and only if $-4N$ is a disjoint discriminant. Hence, using this and Weinberger's result, we derive Theorem 1.1 for the square-free case.

2. THE NON-SQUARE-FREE CASE

Let f be the *ternary quadratic form* $xy+xz+zy$. An integer N is representable by f if there are positive integers x, y and z such that

$$N = f(x, y, z) = xy + xz + yz.$$

Lemma 2.1. *An integer* N *is representable by* f *if and only if there exist integers* k, d_1 *and* d_2 *with* $1 \leq k < d_1, d_2$ *such that*

$$N + k^2 = d_1 d_2. \tag{2–1}$$

Furthermore, if N *is representable by* f, *we can choose* $d_1, d_2 \geq 2k \geq 1$.

Proof. Suppose there are positive integers x, y and z such that $N = xy + xz + zy$. Without loss of generality, we assume that $x \geq y \geq z$. Let $d_1 = y + z$. Then

$$N \equiv yz \equiv z(d_1 - z) \equiv -z^2 \pmod{d_1}$$

and thus we can find a positive integer d_2 such that $N + z^2 = d_1 d_2$. Then $d_1 \geq 2z$. Also

$$d_2 = \frac{N + z^2}{d_1} = \frac{N + z^2}{y + z} \geq 2z,$$

as $x \geq z$. Hence condition (2–1) is satisfied with $k = z$ and $2k \leq d_1, d_2$. Conversely, if $N + k^2 = d_1 d_2$ for some k satisfying the ostensibly weaker condition $1 \leq k < d_1, d_2$, then

$$\begin{aligned} N &= d_1 d_2 - k^2 = d_1(d_2 - k) + (d_1 - k)k \\ &= (d_2 - k) \cdot (d_1 - k) + (d_2 - k) \cdot k + (d_1 - k) \cdot k, \end{aligned}$$

and hence N is representable by f. □

Lemma 2.2. *If* N *is a positive integer satisfying one of the following conditions, then* N *is representable by* f:

(i) N *is odd*;
(ii) $N \equiv 0 \pmod 4$, *and* $N > 4$;
(iii) $N + 1$ *is not a prime.*

Proof. The lemma follows easily from Lemma 2.1 with $k = 1$ and 2. □

We first prove that the numbers 4 and 18 are the only non-square-free integers which are not representable by f. It is easy to check 4 is not representable by f. So we suppose $N \geq 5$.

Lemma 2.3. *If N is not representable by f, then either N is square-free or $N = Mp^2$ with prime p, square-free integer M and $M < p$.*

Proof. Suppose N is not square-free. Let p^2 divides N and write $N = p^2M$. If $M \geq p$, then

$$N + p^2 = p^2M + p^2 = p^2(M + 1).$$

So the condition (2–1) is satisfied with $k = p$ and this contradicts our assumption by Lemma 2.1. Thus M must be less than p. If M is not square-free, say $q^2 \mid M$, then we may assume $q < p$, and $Mp^2/q^2+1 > q$. Thus,

$$N + q^2 = q^2\left(\frac{Mp^2}{q^2} + 1\right)$$

shows that the condition (2–1) is satisfied with $k = q$ and from Lemma 2.1, N is representable by f. This completes our proof. □

Lemma 2.4. *If $N = 2p^2$ is not representable, then $p = 3$ and hence $N = 18$.*

Proof. From Lemma Lemma 2.2 (ii) and (iii), $N + 1$ must be a prime and $p > 2$. If $p > 3$, then $p^2 \equiv 1 \pmod 3$ and hence

$$N + 1 = 2p^2 + 1 \equiv 0 \pmod 3.$$

This contradicts $N+1$ being prime and so $p = 3$. □

In the quadratic field $\mathbb{Q}(\sqrt{-N})$, we factor the algebraic integer $-k + \sqrt{-N}$ as

$$-k + \sqrt{-N} = \left(A_1 + \sqrt{-N}B_1\right)\left(A_2 + \sqrt{-N}B_2\right)$$

where A_i and B_i are integers. If both the norms of $A_i + \sqrt{-k}B_i$ are greater than k then the condition (2–1) will be satisfied and so N is representable by f. The following lemma tells us the above factorization is possible when $N = Mp^2$ with $M > 2$.

Lemma 2.5. *Let M be a square-free integer greater than 1 and p be any integer. Suppose there are co-prime integers $B_1 > B_2 > 0$ such that*

$$\begin{gathered}\min\left\{4MB_1^2B_2^2,\, 4MB_1^2(B_1 - B_2)^2\right\} > p^2,\\ p^2 > 4(M + 1)B_2^2(B_1 - B_2)^2.\end{gathered} \tag{2–2}$$

Then there exist integers A_1 and A_2 such that

$$-k + p\sqrt{-M} = \left(A_1 + \sqrt{-M}B_1\right)\left(A_2 + \sqrt{-M}B_2\right)$$

and $\mathcal{N}\left(A_i + \sqrt{-M}B_i\right) > k > 0$ for $i = 1, 2$, where $k = MB_1B_2 - A_1A_2$.

Proof. Given B_1 and B_2 in the lemma, we define

$$\begin{aligned} f_p(x) &:= B_2x^2 - px + MB_1^2B_2,\\ g_p(x) &:= (B_1 - B_2)x^2 + px + MB_1^2(B_1 - B_2),\\ h_p(x) &:= (B_2 - B_1)x^2 + px + MB_2^2(B_2 - B_1).\end{aligned}$$

Using (2–2), we have

$$f_p(x) \geq f_p\left(\frac{p}{2B_2}\right) = \frac{1}{4B_2}(4MB_1^2B_2^2 - p^2) > 0$$

for any x. Suppose A_1 and A_2 are any integers satisfying $A_1B_2 + A_2B_1 = p$. Thus

$$\begin{aligned} k = MB_1B_2 - A_1A_2 &= MB_1B_2 - A_1\left(\frac{p - A_1B_2}{B_1}\right)\\ &= \frac{1}{B_1}f_p(A_1) > 0.\end{aligned}$$

Similarly,

$$\begin{aligned} g_p(x) &\geq g_p\left(-\frac{p}{2(B_1 - B_2)}\right)\\ &= \frac{1}{4(B_1 - B_2)}(4MB_1^2(B_1 - B_2)^2 - p^2)\\ &> 0\end{aligned}$$

from (2–2). Hence

$$A_1^2 + MB_1^2 - MB_1B_2 + A_1A_2 = \frac{1}{B_1}g_p(A_1) > 0.$$

So $\mathcal{N}(A_1 + \sqrt{-M}B_1) > k$. We now consider $h_p(x)$ and let α and β ($\alpha < \beta$) be the positive real roots of $h_p(x) = 0$. In fact,

$$\alpha = \frac{p - \sqrt{p^2 - 4MB_2^2(B_1 - B_2)^2}}{2(B_1 - B_2)},$$

$$\beta = \frac{p + \sqrt{p^2 - 4MB_2^2(B_1 - B_2)^2}}{2(B_1 - B_2)}.$$

Again using (2–2), we know that

$$h_p\left(\frac{p}{2(B_1-B_2)}\right) = \frac{\left(p^2 - 4MB_2^2(B_1 - B_2)^2\right)}{4(B_1-B_2)} > 0.$$

It follows that $h_p(x) > 0$ for any $\alpha < x < \beta$. Consider the linear diophantine equation

$$B_1x + B_2y = p.$$

Since $d := \gcd(B_1, B_2) = 1$ (in fact we only need $\gcd(B_1, B_2) \mid p$), so all the solutions of the above diophantine equation is given by

$$x = x_0 + \frac{kB_2}{d} \quad \text{and} \quad y = y_0 - \frac{kB_1}{d}$$

for any integer k and some integers x_0 and y_0. Now since

$$\beta-\alpha=\frac{\sqrt{p^2-4MB_2^2(B_1-B_2)^2}}{(B_1-B_2)}>B_2$$

from (2–2), we can always choose A_2 between α and β and some A_1 such that

$$B_1A_2+B_2A_1=p$$

but for this particular A_2, we have $h_p(A_2)>0$ and hence

$$A_2^2+MB_2^2-MB_1B_2+A_1A_2=\frac{1}{B_2}h_p(A_2)>0.$$

Therefore,

$$-k+p\sqrt{-M}=\big(A_1+\sqrt{-M}B_1\big)\big(A_2+\sqrt{-M}B_2\big)$$

and $\mathcal{N}\big(A_i+\sqrt{-M}B_i\big)>k>0$ for $i=1,2$. This proves the lemma. □

Theorem 2.6. *The numbers* 4 *and* 18 *are the only non-square-free integers not representable by* f.

Proof. We suppose $N>4$. Then from Lemma 2.3, $N=Mp^2$ with $M<p$ where p is a prime and M is square-free. From Lemma 2.2 (i), M must be even. If $M=2$ then by Lemma 2.4, $N=18$. Suppose $M\geq 4$. By direct checking, we can assume that $p\geq 2305$. For any integer $L\geq 5$, we let $B_1=2L$ and $B_2=L+1$ so that $\gcd(B_1,B_2)=1$. The condition (2–2) in Lemma 2.5 now becomes

$$16ML^2(L-1)^2>p^2>4(M+1)(L^2-1)^2. \tag{2–3}$$

Note that for $M\geq 4$ and $L\geq 5$ we have

$$16ML^2(L-1)^2>4(M+1)((L+1)^2-1)^2,$$

and hence

$$\bigcup_{L\geq 5}\{(4(M+1)(L^2-1)^2,\ 16ML^2(L-1)^2)\}=(2304(M+1),\infty).$$

If $p^2\leq 2304(M+1)$, then

$$M<p<\sqrt{2304(M+1)}.$$

It follows that $M\leq 2305$ and thus $p\leq 2304$. Therefore, $p^2>2304(M+1)$ and there is $L\geq 5$ such that condition (2–3) is satisfied. From Lemma 2.5, we now have

$$-k+p\sqrt{-M}=\big(A_1+B_1\sqrt{-M}\,\big)\big(A_2+B_2\sqrt{-M}\,\big)$$

and $k<\mathcal{N}\big(A_i+B_i\sqrt{-M}\,\big)$ for $i=1,2$, where $k=MB_1B_2-A_1A_2$. Thus

$$\begin{aligned}N+k^2&=Mp^2+k^2\\&=\mathcal{N}\big(A_1+B_1\sqrt{-M}\,\big)\mathcal{N}\big(A_2+B_2\sqrt{-M}\,\big),\end{aligned}$$

and this shows that condition (1.1) is satisfied for this k. This proves our theorem. □

3. THE SQUARE-FREE CASE

In this section, we assume N is even and square-free. The main result in this section is to show that N is not representable by f if and only if $-4N$ is a disjoint discriminant. Crandall [1999] gave a new representation for the Madelung constant based on Andrews' identity for the cube of the Jacobi theta function θ_4. Crandall observed that

$$\theta_4^3(q)=1-6\sum_{x,y\geq 1}(-1)^{x+y}q^{xy}-4\sum_{x,y,z\geq 1}(-1)^{x+y+z}q^{xy+xz+yz}.$$

He also observed that this identity relates to the number of representations of $f(x,y,z)=xy+xz+yz$ as follows:

$$(-1)^N r_3(N)=-6\sum_{\substack{x,y\geq 1\\ xy=N}}(-1)^{x+y}-4\sum_{\substack{x,y,z\geq 1\\ xy+xz+yz=N}}(-1)^{x+y+z}, \tag{3–1}$$

where $r_3(N)$ is the number of representation of N as a sum of three squares.

Using (3–1), we establish:

Theorem 3.1. *Let* $N=2p_1p_2\cdots p_r$ *for distinct odd primes* $p_1,p_2,\cdots,p_r$. *Then* N *is not representable if and only if* $-4N$ *is a disjoint discriminant.*

Proof. Let $r_t(n)$ be the number of representations of N as $xy+yz+xz$ for positive integers x,y and z. Then from (3–1), we have

$$r_3(N)=6d(N)+4r_t(N)=12\cdot 2^r+4r_t(N)$$

because N is even. But $N\equiv 2 \pmod 4$ and if $N=xy+yz+xz$ then exactly two of x,y,z are even. It was proved by Gauss (see [Rose 1994, 170–171]) that $r_3(N)=12h(-4N)$. Hence $r_t(N)=0$ if and only if

$$h(-4N)=2^r.$$

Let m be the number of genera for discriminant $-4N$. From the formula for m in [Rose 1994, page 198], we have $m = 2^r$. Also, we have $h(-4N) = mg$ where g is the number of forms in a genus. Therefore, $h(-4N) = 2^r = m$ if and only if $g = 1$, or equivalently if and only if d is disjoint. $\square$

Using Weinberger's result and Theorems 2.6 and 3.1, we may now complete the proof of Theorem 1.1.

It is still an open problem as to whether the 19th integer exists. For computational purposes, the following result is useful.

Theorem 3.2. *A square-free integer N is representable by f if and only if there is an odd prime p with $p < \sqrt{\frac{4}{3}N}$ such that $-N$ is a quadratic residue mod p.* (See [Zhu and Shao 1988, Theorem 4].)

Proof. Suppose N is representable by f. Then from Lemma 2.1, we have

$$N + k^2 = d_1 d_2 \tag{3–2}$$

for some $1 \le 2k \le d_1, d_2$. Suppose $d_1 \ge d_2$. Then we claim that there is a prime p such that $p \mid d_2$ but $p \nmid N$. Suppose not, then d_2 must be square-free otherwise N is not square-free. It follows that every prime dividing d_2 must also divide k and hence $d_2 \le k$. This contradicts the condition $2k \le d_2$. This proves the claim. Hence from (3–2), we see that $-N$ is a quadratic residue modulo p and

$$N = d_1 d_2 - k^2 \ge d_2^2 - \tfrac{1}{4}d_2^2 \ge \tfrac{3}{4}p^2.$$

This proves that $p \le \sqrt{\frac{4}{3}N}$.

Conversely, if there is a prime $p \le \sqrt{\frac{4}{3}N}$ such that $-N$ is a quadratic residue modulo p, then we choose p to be the smallest among all such primes. Then we can find an integer k such that $1 \le k \le \frac{1}{2}(p-1)$ and $N + k^2 = pd_1$ for some d_1. Suppose $d_1 < p$. If there is a prime q such that $q \mid d_1$ but $q \nmid N$, then $q \le d_1 < p$ and $-N$ is a quadratic residue modulo q, which contradicts p being the smallest such prime. Hence every prime dividing d_1 must also divide N and k. Since N is square-free, so is d_1 and hence $d_1 \le k$. It follows that

$$N = pd_1 - k^2 \le pd_1 - d_1^2 \le \tfrac{1}{4}p^2$$

which contradicts $p \le \sqrt{\frac{4}{3}N}$. Thus we must have $d_1 \ge p$ and using Lemma 2.1, N is representable by f. This completes our proof. $\square$

4. ADDITIONAL COMMENTS

Our root to this development was reflective of the changing nature of mathematical research. A copy of the preprint of [Crandall 1999] was sent by the first author to Roalnd Girgensohn in Munich who reported the next morning that numerical computation showed the only square-free counterexamples less than 5,000 were the ones listed in equation (1.1). These numbers (1, 4 and 18 excluded) were familiar to the first author as corresponding to *disjoint discriminants of the second type, P* which give rise to *singular values k_{2P}* expressible as products of fundamental units [Borwein and Borwein 1987, pp. 296–300]. The most famous of these is

$$k_{210} = (\sqrt{2}-1)^2(2-\sqrt{3})(\sqrt{7}-\sqrt{6})^2(8-3\sqrt{7}) \\ \cdot (\sqrt{10}-3)^2(\sqrt{15}-\sqrt{14})(4-\sqrt{15})^2(6-\sqrt{35}),$$

given in Ramanujan's first letter to G. H. Hardy. The two larger values k_{330} and k_{462} are listed in [Borwein and Borwein 1987, (9.2.11) and (9.2.12)]. Theorem 1.1 was now irresistible as a conjecture.

After establishing Theorem 1.1, it occurred to us to consult Neil Sloane's marvelous *On-line Encyclopedia of Integer Sequences* (www.research.att.com/~njas/sequences/eisonline.html). We typed the first

```
%I A025052
%S A025052 1,2,4,6,10,18,22,30,42,58,70,78,102,130,190,210,330,462
%N A025052 Numbers not of form ab + bc + ca for 1<=a<=b<=c (probably list is complete).
%O A025052 1,2
%K A025052 nonn,fini
%A A025052 Clark Kimberling (ck6@cedar.evansville.edu)
%E A025052 Corrected by Ron Hardin (rhh@research.att.com)
```

Reply given by Sloane's *On-line Encyclopedia of Integer Sequences* to the input 1, 2, 4, 6, 10, 18, 22, 30, 42, 58, 70, 78, 102.

thirteen terms of (1.1) with 4 and 18 included, obtaining the result shown at the bottom of the preceding page.

Had we at the time asked for the square-free members, we would have drawn a blank. This is no longer the case. Had we checked Sloane's website initially, we would almost certainly not have thought further on the matter. After all, the *answer* had been found. That said, there appears to be little published literature on the subject. Some related references are to be found in [Chen and Le 1998].

REFERENCES

[Borwein and Borwein 1987] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A study in analytic number theory and computational complexity*, Canadian Math. Soc. Series of Monographs and Advanced Texts **4**, Wiley, New York, 1987.

[Chen and Le 1998] X. G. Chen and M. H. Le, "The positive integer solutions of the equation $xy + yz + zx = n$", *Acta Math. Sinica* **41**:3 (1998), 577–582. In Chinese, with English abstract.

[Chowla 1934] S. Chowla, "An extension of Heilbronn's class-number theorem", *Quart. J. Math.* **5** (1934), 304–307.

[Crandall 1999] R. E. Crandall, "New representations for the Madelung constant", *Experimental Math.* **8**:4 (1999), 367–379.

[Rose 1994] H. E. Rose, *A course in number theory*, 2nd ed., Oxford University Press, New York, 1994.

[Weinberger 1973] P. J. Weinberger, "Exponents of the class groups of complex quadratic fields", *Acta Arith.* **22** (1973), 117–124.

[Zhu and Shao 1988] F. Z. Zhu and Y. Y. Shao, "On the construction of indecomposable positive definite quadratic forms over $\mathbb{Z}$", *Chinese Ann. Math. Ser. B* **9**:1 (1988), 79–94.

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