The Tensor Product of Polynomials

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CONTENTS

Using Gröbner basis algorithms in MAGMA we find necessary and sufficient conditions for a polynomial of degree 6 over any field to be the tensor product of two polynomials, one of degree 3 and one of degree 2.

1. INTRODUCTION

In order to determine whether there exists a tensor decomposition of the natural module for a matrix group G over a field K it proved to be useful to decide whether or not there exists a tensor decomposition of the characteristic polynomial of $g \in G$ [Leedham-Green and O'Brien 1997]. This latter problem was the motivation for the present work.

Let h be a univariate polynomial of degree d over an algebraically closed field K. If $d = m + n$ then clearly h is the product of two polynomials over K of degrees m and n. But if $d = mn$, with $m, n > 1$, then h is the tensor product (as defined below) of two polynomials, one of degree m and the other of degree n, if and only if the coefficients $c_1, \dots, c_d$ of h define an element $(c_1, \dots, c_d)$ in some $(m+n-1)$-dimensional variety $V \subset K^d$. This variety is determined by a prime ideal I_{mn} in the ring $K[c_1, \dots, c_d]$. The ideal I_{22} is easily computed by hand and the ideal I_{32} is just within the range of machine computation.

2. THE TENSOR PRODUCT

Given two monic polynomials

$$f(x) = x^m - a_1 x^{m-1} + \cdots + (-1)^m a_m$$

with zeros $\alpha_1, \dots, \alpha_m$ and

$$g(x) = x^n - b_1 x^{n-1} + \cdots + (-1)^n b_n$$

with zeros $\beta_1, \dots, \beta_n$ in $K[x]$, the *tensor product* of $f(x)$ and $g(x)$ is the monic polynomial $h(x)$ of degree

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mn with roots $\alpha_j\beta_k$ for $1 \le j \le m$, $1 \le k \le n$; that is,

$$h(x) = x^{mn} - c_1 x^{mn-1} + \cdots + (-1)^{mn} c_{mn},$$

with c_i the i-th elementary symmetric function in $\alpha_j\beta_k$, for $1 \le j \le m$ and $1 \le k \le n$.

Let

$$\begin{aligned} p_i(f) &= \sum_{j=1}^{m} \alpha_j^i, \\ p_i(g) &= \sum_{k=1}^{n} \beta_k^i, \\ p_i(f \otimes g) &= \sum_{j,k} (\alpha_j\beta_k)^i \\ &= \Bigl(\sum_{j=1}^{m} \alpha_j^i\Bigr)\Bigl(\sum_{k=1}^{n} \beta_k^i\Bigr) = p_i(f)p_i(g) \end{aligned}$$

be the i-th power sums of α_j, β_k and $\alpha_j\beta_k$, where $1 \le j \le m$ and $1 \le k \le n$, respectively.

We can compute the i-th power sum p_i in terms of $\{e_1, \dots, e_i\}$ by using Newton's Formula [Macdonald 1995, p. 23],

$$ne_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r},$$

where e_j is the j-th elementary symmetric function. Then by a simple algorithm we can compute the c_i's in terms of $\{a_j : 1 \le j \le m\}$ and $\{b_k : 1 \le k \le n\}$.

The *weight* in the x's of a monomial $x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m}$ is defined by $w = \sum_{i=1}^{m} i \cdot \varepsilon_i$. Each c_i is then a homogeneous polynomial of weight i in both the a_j's and the b_k's.

In general, the condition that the polynomial h should have a tensor factorisation with factors of degrees m and n is the condition that the coefficients of h define an element $(c_1, \dots, c_{mn})$ in the variety $V \subset K^{mn}$ determined by an homogeneous ideal $I_{mn} \subset K[c_1, \dots, c_{mn}]$. I_{mn} is the kernel of the homomorphism from $K[c_1, \dots, c_{mn}]$ into

$$K[a_1, \dots, a_m, b_1, \dots, b_n]$$

taking each c_i to the corresponding polynomial in the a_j's and b_k's. Being the kernel of an homomorphism into a domain, I_{mn} is a prime ideal, hence the variety V is irreducible.

To determine the dimension of V we consider the factorisation

$$h(x) = f(x) \otimes g(x) = \prod_{j,k} (x - \alpha_j\beta_k)$$

giving the polynomial functions $\varphi_{jk} : K^{m+n} \to K$ defined by

$$\varphi_{jk}(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) = \alpha_j\beta_k.$$

It is easy to see that the $m+n-1$ elements φ_{11}, $\dots$, φ_{m1}, φ_{12}, $\dots$, φ_{1n} form a maximal set of algebraically independent elements over K, hence the dimension of V is $m+n-1$. For more details on the theory of varieties see [Cox et al. 1997, Chapters 4, 5, 9].

3. CASES I_{22} AND I_{32}

It is easy to prove that I_{22} is a principal ideal with generator of weight 6. The coefficients are

$$\begin{aligned} c_1 &= a_1 b_1, \\ c_2 &= a_2 b_1^2 + a_1^2 b_2 - 2a_2 b_2, \\ c_3 &= a_1 a_2 b_1 b_2, \\ c_4 &= a_2^2 b_2^2, \end{aligned}$$

so that the generator $c_1^2 c_4 - c_3^2$ can be easily obtained.

The problem of finding a set of generators for I_{32} proved surprisingly harder. This is a classical Gröbner basis problem. Considering the polynomial parametrization

$$\begin{aligned} c_1 &= q_1(a_1, \dots, a_m, b_1, \dots, b_n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ c_d &= q_d(a_1, \dots, a_m, b_1, \dots, b_n), \end{aligned}$$

let I be the ideal

$$I = \langle c_1 - q_1, \dots, c_d - q_d \rangle \subset K[a_1, \dots, a_m, b_1, \dots, b_n, c_1, \dots, c_d].$$

Then the ideal I_{mn} is the $(m+n)$-th elimination ideal $I_{mn} = I \cap K[c_1, \dots, c_d]$, and the Elimination Theorem [Cox et al. 1997, § 5.3, Theorem 1] proves that if B is a Gröbner basis for I with respect to lex order where $a_1 > \cdots > a_m > b_1 > \cdots > b_n > c_1 > \cdots > c_d$ then the set $B_{mn} = B \cap K[c_1, \dots, c_d]$ is a Gröbner basis for I_{mn}.

We were unable to get the calculation to complete on any Gröbner basis package. Clearly I_{mn} is defined over $\mathbb{Q}$ (equivalently over $\mathbb{Z}$). Working over

GF(2) without using Gröbner techniques it was possible, using MAGMA [Bosma and Cannon 1993], to find homogeneous elements of I_{32} that we believed to form a generating set. The conjecture was later confirmed when Allan Steel showed us how to carry out the complete calculation using the Gröbner basis in MAGMA, working over $\mathbb{Q}$. This was done by defining the polynomial ring

$$P = \mathbb{Q}[a_1, a_2, a_3, b_1, b_2, c_1, \dots, c_6]$$

with elimination order [Cox et al. 1997, p. 72], then defining the ideal $I = \langle c_1 - q_1, \dots, c_6 - q_6 \rangle$ in P and determining its Gröbner basis B. A Gröbner basis D for the elimination ideal I_{32} is obtained by taking the images of the basis elements $b \in B$ under the homomorphism $\psi : P \to K[c_1, \dots, c_6]$ defined by $\psi(a_j) = \psi(b_k) = 0$, and $\psi(c_i) = c_i$. Eliminating redundancies in D a minimal generating set for I_{32} is obtained. The conclusion is that a minimal generating set for I_{32} contains 16 homogeneous polynomials of weights 19 to 30, each being the sum of at least 28 monomials.

It is hoped that new development of MAGMA Gröbner basis code will enable us to compute a free homogeneous resolution of the subring M of

$$K[a_1, a_2, a_3, b_1, b_2]$$

generated by the images of $c_1, \dots, c_6$. Preliminary calculations suggest a resolution of length five

$$0 \to F_5 \to F_4 \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0,$$

where the F_i are free modules over $K[c_1, \dots, c_6]$ as follows: F_0 of rank 1 with a generator of weight 0, $F_1 = I_{32}$, F_2 generated by 34 polynomials of weights 24 to 35, F_3 by 29 polynomials of weights 28 to 38, F_4 by 12 polynomials of weights 33 to 40 and F_5 by two polynomials of weights 39 and 41.

The CPU time required for the calculation of the generators for I_{32} using MAGMA Version 2.3-1 on a Pentium II PC was 21 minutes. The polynomials are available from ftp://ftp.maths.qmw.ac.uk/pub/crlg/poly33.

We have been unable to produce any reasonable bound to the number of generators of I_{mn}, or to obtain any information about the weights of the elements of a minimal generating set, except for I_{22} and I_{32}, and have no theoretical explanation for the results obtained in these two particular cases. In particular it would be interesting to have some insight into the cohomological dimension of M.

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