Affine Spheres: Discretization via Duality Relations

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CONTENTS

Affine spheres with definite and indefinite Blaschke metric are discretized in a purely geometric manner. The technique is based on simple relations between affine spheres and their duals which possess natural discrete analogues. The geometry of these duality relations is discussed in detail. Cauchy problems are posed and shown to admit unique solutions. Particular discrete definite affine spheres are shown to include regular polyhedra and some of their generalizations. Connections with integrable partial difference equations and symmetric mappings are recorded.

1. INTRODUCTION

The study of affine differential geometry has a long history. According to [Nomizu and Sasaki 1994], it was initiated by Tzitzeica in the first decade of the twentieth century, with a remarkable paper [Tzitzeica 1910] on a particular class of hyperbolic surfaces and its invariance under a Bäcklund–Moutard-type transformation. Even though Tzitzeica's analysis was undertaken in the language of standard classical differential geometry, his class of surfaces turns out to be of particular importance in affine differential geometry. Indeed, the Tzitzeica property proves to be invariant under (equi)affine transformations, and his surfaces are now known as affine spheres (Affinsphären) [Blaschke 1923] because they are analogues of spheres in affine differential geometry. An elementary introduction to this topic and a novel characterization of affine spheres is given in Section 2.

The issue of canonical discretizations of geometries in the context of integrable systems has recently become a subject of extensive studies. Indeed, one may propose various discrete problems having the same continuum limit but rather different properties. Thus, is there a distinct discretization

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which one should choose? The problem of finding a proper discretization of a given geometric model belongs to experimental mathematics. Many concepts in this area of research have been developed on use of geometric and algebraic methods with an essential input of trial and error. The current problem of discretizing affine spheres turns out to be very much of this kind:

Early attempts to discretize particular coordinate systems on surfaces go back to Sauer. An account of his work and references may be found in [Sauer 1970], where he sets in correspondence differential geometric objects and difference geometric models. Such models include discrete conjugate and asymptotic nets. The latter were used in 1950 to define discrete models for surfaces of constant negative Gaußian curvature [Sauer 1950]. Independently, Wunderlich [1951] derived a discrete counterpart of the classical Bäcklund transformation for these discrete pseudospherical surfaces. A discrete analogue of the sine-Gordon equation which governs pseudospherical surfaces was not discussed in his work. Some tweny-five years later, an integrable discrete sine-Gordon equation was set down by Hirota [1977] and it was only recently that a connection between that discretization and the discrete pseudospherical surfaces was established by Bobenko and Pinkall [1996b]. Since then integrable discrete models of surfaces of constant mean curvature [Bobenko and Pinkall 1999] and isothermic surfaces [Bobenko and Pinkall 1996a] have been constructed.

The geometry of higher-dimensional integrable lattices has also been the subject of recent studies. Thus, the conjugate nets of Sauer [1970] were generalized to higher dimensions and investigated in connection with integrability in [Bogdanov and Konopelchenko 1995; Doliwa 1997; Doliwa and Santini 1997]. Multi-dimensional lattices which model curvature (conjugate and orthogonal) coordinate lines were defined in [Bobenko 1999] and their geometric and analytic integrability investigated in [Cieśliński et al. 1997; Doliwa et al. 1998]. Remarkably, two-dimensional lattices (nets) of this type (cyclic nets) have been used earlier in computer-aided surface design [Martin et al. 1986; Nutbourne 1986]. Two- and three-dimensional cyclic lattices in Euclidean spaces of arbitrary dimension were derived via suitable eigenfunction constraints in [Konopelchenko and Schief 1998], where explicit parametrizations of lattices on the line, plane and in $\mathbb{R}^3$ are given.

In [Bobenko and Schief 1999] we solved in a purely geometric manner the problem of discretizing affine spheres with indefinite Blaschke metric in such a way that integrability is preserved. We exploited the fact that asymptotic lines on indefinite affine spheres possess a property which we term 'affine Lorentz harmonicity' for lack of a better expression. Thus, by demanding that discrete indefinite affine spheres constitute both discrete asymptotic and discrete affine Lorentz harmonic nets, discrete indefinite affine spheres have been constructed. Section 3 starts with a brief review of this construction. However, an analogous route is not available in the convex case. We conclude Section 3 with an alternative but equivalent definition of discrete indefinite affine spheres in terms of simple duality relations involving a dual or conormal lattice. It is this description which may be adopted in the definite case. We note that the concept of dual surfaces may also be used in the definition of discrete isothermic surfaces and surfaces of constant mean curvature [Bobenko and Pinkall 1999; Hertrich-Jeromin et al. 1999].

Section 4 is concerned with the definition and properties of discrete definite affine spheres and their duals. It turns out that the duality relations are such that the discrete surfaces regarded as lattices may be of any type, for instance triangular, quadrilateral or honeycomb. Examples of discrete definite affine spheres include certain symmetric solids, in particular the regular polyhedra. In the case of surfaces $\mathbb{Z}^2 \to \mathbb{R}^3$, it is shown that the discrete duality relations are self-consistent in the sense that a Cauchy problem may be formulated and its well-posedness proven. These discrete surfaces are discussed in detail in Section 5. The corresponding discrete Gauß equations are formulated and it is shown that their compatibility conditions (discrete Gauß-Weingarten equations) lead to a discrete elliptic version of the classical Tzitzeica equation which admits a well-posed Cauchy problem. The paper concludes with a particular class of discrete definite affine spheres which is governed by a one-dimensional integrable mapping well known in soliton theory. This may be regarded as an indication of the integrability in the generic case.

2. CLASSICAL CASES

In this section, we present the well-known description of affine spheres with indefinite and definite Blaschke metrics (for more details see, for example, [Blaschke 1923; Simon and Wang 1993]). Here, affine differential geometry is treated in its classical setup, that is as the geometry which investigates properties of surfaces in $\mathbb{R}^3$ that are invariant under (equi)affine transformations

$$\boldsymbol{r} \mapsto A\boldsymbol{r} + \boldsymbol{a}, \quad \text{with } A \in \mathrm{SL}(3,\mathbb{R}),\ \boldsymbol{a} \in \mathbb{R}^3.$$

Thus, let

$$\boldsymbol{r} : \mathrm{M} \to \mathbb{R}^3, \qquad (x,y) \mapsto \boldsymbol{r}(x,y)$$

be an immersion with a nondegenerate second fundamental form (Gaußian curvature $\mathcal{K} \neq 0$). Introduce

$$L = |\boldsymbol{r}_y, \boldsymbol{r}_x, \boldsymbol{r}_{xx}|,$$
$$M = |\boldsymbol{r}_y, \boldsymbol{r}_x, \boldsymbol{r}_{xy}|,$$
$$N = |\boldsymbol{r}_y, \boldsymbol{r}_x, \boldsymbol{r}_{yy}|,$$

where $|\cdot,\cdot,\cdot|$ denotes the standard determinant in $\mathbb{R}^3$.

Definition 2.1. The quadratic form

$$g = \frac{L\,dx^2 + 2M\,dx\,dy + N\,dy^2}{|LN - M^2|^{1/4}} \tag{2–1}$$

is equiaffine invariant. It is called the *Blaschke metric* of the immersion.

The Blaschke metric is conformally equivalent to the second fundamental form of the immersion.

Definition 2.2. A transversal vector field $\boldsymbol{\xi}$ on a surface $\boldsymbol{r}(\mathrm{M})$ is called *affine normal* if it satisfies

$$\boldsymbol{\xi} = -\tfrac{1}{2}\Delta_g \boldsymbol{r},$$

where Δ_g is the Laplace–Beltrami operator of the Blaschke metric.

In contrast to Blaschke's definition, we choose the affine normal to point outwards a convex surface (see Figure 1). The reason is that in the convex case we prefer to 'look' at the surface from outside rather than inside. The direction of the affine normal has a simple geometrical meaning [Blaschke 1923]. In the hyperbolic case, it can be described as follows. Consider an infinitesimal quadrilateral composed of asymptotic lines. Build two planes, each being parallel to a pair of opposite edges of the quadrilateral. The affine normal is parallel to the line of intersection of these two planes. If $\boldsymbol{r}, \boldsymbol{r}_{12}$ and $\boldsymbol{r}_1, \boldsymbol{r}_2$ are two pairs of opposite vertices of the quadrilateral (see Section 3 for notations), then the affine normal is parallel to

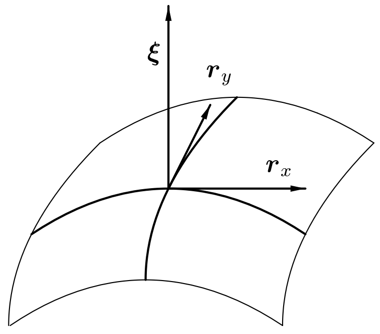


FIGURE 1. The orientation of the triad $\{\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{\xi}\}$.

$$\boldsymbol{\xi} \sim \boldsymbol{r}_{12} + \boldsymbol{r} - \boldsymbol{r}_1 - \boldsymbol{r}_2. \tag{2–2}$$

In the convex case, the description is different. Let $\boldsymbol{r}(\mathrm{M})$ be a locally convex surface in $\mathbb{R}^3$ and $p \in \mathrm{M}$. For any small t consider a plane π_t parallel to the tangent plane $\pi_0 = d\boldsymbol{r}(T_p\mathrm{M})$. The plane π_t intersects the surface in a plane curve $\Pi(t)$ which is the boundary of a planar domain $D(t)$:

$$\partial D(t) = \Pi(t).$$

Let $\gamma(t)$ be the centre of gravity of $D(t)$. The tangent vector $\gamma'(0)$ to the curve γ defines the affine normal direction at the point $\boldsymbol{r}(p)$.

Regarding the normalization of $\boldsymbol{\xi}$, we mention two properties of the affine normal:

- $d\boldsymbol{\xi} \in d\boldsymbol{r}(TM)$;
- $|d\boldsymbol{r}(\cdot), d\boldsymbol{r}(\cdot), \boldsymbol{\xi}|$ is the volume form corresponding to the Blaschke metric, that is,

$$|\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{\xi}| = |LN - M^2|^{1/4}. \tag{2–3}$$

Definition 2.3. A non-degenerate surface in $\mathbb{R}^3$ is called an *affine sphere* if all affine normals intersect at a point. If this point is not infinite it may be chosen as the origin of $\mathbb{R}^3$ so that

$$\boldsymbol{\xi} = \mathcal{H}\boldsymbol{r}, \quad \mathcal{H} : \mathrm{M} \to \mathbb{R}.$$

$\mathcal{H}$ is called the *affine mean curvature.*

One can prove that in the case of a non-degenerate Blaschke metric the affine mean curvature of the affine sphere is constant. In the following, we consider proper affine spheres; that is, we assume that $\mathcal{H} \neq 0$. Consequently, $\mathcal{H}$ may be normalized to $\mathcal{H} = \pm 1$ on use of a scaling transformation of the ambient space $\mathbb{R}^3$. It is natural to distinguish between the cases of a definite Blaschke metric (convex surfaces $\mathcal{K} > 0$) and of an indefinite Blaschke metric (hyperbolic surfaces $\mathcal{K} < 0$). In the convex case, $\mathcal{H}$ defined as above is positive and in the indefinite case, it can be set positive by a change of the orientation $\boldsymbol{r} \to -\boldsymbol{r}$. In the sequel, we assume that

$$\mathcal{H} = 1.$$

2A. Indefinite affine spheres

In the case of an indefinite Blaschke metric, it is natural to consider the asymptotic line parametrization of surfaces

$$L = N = 0,$$

which is affine invariant. Applying if necessary an orientation-preserving transformation

$$(x, y) \to (y, -x),$$

one can always achieve $M > 0$.

For affine spheres we set $\boldsymbol{\xi} = \boldsymbol{r}$ as indicated. On introduction of the function

$$h = |\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{r}| \tag{2–4}$$

and the cubic differentials $a\,dx^3$, bdy^3, where

$$a = |\boldsymbol{r}_x, \boldsymbol{r}_{xx}, \boldsymbol{r}|, \quad b = -|\boldsymbol{r}_y, \boldsymbol{r}_{yy}, \boldsymbol{r}|, \tag{2–5}$$

one obtains

$$|\boldsymbol{r}_x, \boldsymbol{r}_y, \boldsymbol{r}_{xy}| = h^2 \tag{2–6}$$

and the following linear system for the immersion $\boldsymbol{r}$ (*Gauß equations*):

$$\begin{aligned} \boldsymbol{r}_{xx} &= \frac{h_x}{h}\boldsymbol{r}_x + \frac{a}{h}\boldsymbol{r}_y, \\ \boldsymbol{r}_{xy} &= h\boldsymbol{r}, \\ \boldsymbol{r}_{yy} &= \frac{h_y}{h}\boldsymbol{r}_y + \frac{b}{h}\boldsymbol{r}_x. \end{aligned} \tag{2–7}$$

Using $\boldsymbol{r}_{xx}, \boldsymbol{r}_{yy} \in \operatorname{span}\{\boldsymbol{r}_x, \boldsymbol{r}_y\}$, one can easily prove the following fact:

Lemma 2.4. *An asymptotically parametrized surface $(x, y) \to \boldsymbol{r}(x, y)$ is an affine sphere if and only if*

$$\boldsymbol{r}_{xy} \parallel \boldsymbol{r}.$$

The compatibility conditions for (2–7) yield

$$(\log h)_{xy} = h - abh^{-2}, \quad a_y = 0, \quad b_x = 0. \tag{2–8}$$

The above system is invariant with respect to the transformation

$$a \to \lambda a, \quad b \to \lambda^{-1} b \tag{2–9}$$

with arbitrary $\lambda \in \mathbb{R}$. This gives rise to the following result:

Theorem 2.5. *Every indefinite affine sphere possesses a one-parameter ($\lambda \in \mathbb{R}$) family of deformations preserving the Blaschke metric and the differential $ab\,dx^3\,dy^3$. These deformations are described by the transformation* (2–9). *The system*

$$\begin{aligned} \begin{pmatrix} \boldsymbol{r}_x \\ \boldsymbol{r}_y \\ \boldsymbol{r} \end{pmatrix}_x &= \begin{pmatrix} h_x h^{-1} & \lambda a h^{-1} & 0 \\ 0 & 0 & h \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_x \\ \boldsymbol{r}_y \\ \boldsymbol{r} \end{pmatrix}, \\ \begin{pmatrix} \boldsymbol{r}_x \\ \boldsymbol{r}_y \\ \boldsymbol{r} \end{pmatrix}_y &= \begin{pmatrix} 0 & 0 & h \\ \lambda^{-1} b h^{-1} & h_y h^{-1} & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_x \\ \boldsymbol{r}_y \\ \boldsymbol{r} \end{pmatrix} \end{aligned} \tag{2–10}$$

determines the corresponding family of immersions.

In the terminology of modern soliton theory, the linear system (2–10) is the *Lax representation* for the system (2–8). In fact, for $a \neq 0$ and $b \neq 0$, one may re-parametrize the asymptotic coordinates according to

$$x \to \tilde{x}(x), \quad y \to \tilde{y}(y)$$

in such a way that $a = 1$, $b = \varepsilon = \pm 1$ and the orientation is preserved. Thus, we obtain

$$(\log h)_{xy} = h - \varepsilon h^{-2}. \tag{2–11}$$

Finally, the parameter ε may be absorbed if one allows negative valued solutions h. Indeed, if $h(x, y)$ is a solution to the integrable *Tzitzeica equation* [Tzitzeica 1910]

$$(\log h)_{xy} = h - h^{-2}$$

and $\boldsymbol{r}(x, y)$ is the corresponding immersion, then $\tilde{h}(x, y) = \varepsilon h(x, \varepsilon y)$ is a solution to (2–11) with associated immersion $\tilde{\boldsymbol{r}}(x, y) = \boldsymbol{r}(x, \varepsilon y)$.

It is readily verified that the quantity

$$\boldsymbol{\nu} = \frac{1}{h}\boldsymbol{r}_x \times \boldsymbol{r}_y \tag{2–12}$$

is a solution of the *adjoint* or *dual* system represented by $\lambda = -1$. The geometric significance of

$\boldsymbol{\nu}$ is as follows. By definition, the vector $\boldsymbol{\nu}$ is perpendicular to the tangent plane spanned by $\boldsymbol{r}_x, \boldsymbol{r}_y$. Furthermore, the normalization (2–3) shows that

$$\boldsymbol{r} \cdot \boldsymbol{\nu} = 1.$$

Hence $\boldsymbol{\nu}$ is called a *conormal*. The conormal defines discrete affine spheres which may be considered dual to those given by $\boldsymbol{r}$. This is made precise in the following

Theorem 2.6 (Duality relations for indefinite affine spheres). *Indefinite affine spheres and their duals are equivalently described by the* Lelieuvre formulae

$$\begin{aligned} \boldsymbol{r}_x &= \boldsymbol{\nu} \times \boldsymbol{\nu}_x, \quad & \boldsymbol{\nu}_x &= \boldsymbol{r}_x \times \boldsymbol{r}, \\ \boldsymbol{r}_y &= \boldsymbol{\nu}_y \times \boldsymbol{\nu}, \quad & \boldsymbol{\nu}_y &= \boldsymbol{r} \times \boldsymbol{r}_y, \end{aligned} \tag{2–13}$$

which imply that $\boldsymbol{r} \cdot \boldsymbol{\nu} = 1$.

One can directly verify that $\boldsymbol{\nu}$ as given by (2–12) obeys the duality relations (2–13). Conversely, two vector-valued functions $\boldsymbol{r}$ and $\boldsymbol{\nu}$ which satisfy the duality relations define two affine spheres with conormals $\boldsymbol{\nu}$ and $\boldsymbol{r}$ respectively. For symmetry reasons, it is sufficient to focus only on $\boldsymbol{r}$, say. Thus, the Lelieuvre formulae yield

$$\boldsymbol{r}_{xx} \perp \boldsymbol{\nu}, \quad \boldsymbol{r}_{yy} \perp \boldsymbol{\nu}, \quad \boldsymbol{r}_x \perp \boldsymbol{\nu}, \quad \boldsymbol{r}_y \perp \boldsymbol{\nu},$$

which implies that $\boldsymbol{r}$ forms an asymptotic net and $\boldsymbol{\nu}$ is a corresponding normal. Secondly, the compatibility condition $\boldsymbol{\nu}_{xy} = \boldsymbol{\nu}_{yx}$ for $(2–13)_{2,4}$ reads

$$\boldsymbol{r}_{xy} \times \boldsymbol{r} = 0 \quad \Longrightarrow \quad \boldsymbol{r}_{xy} \parallel \boldsymbol{r},$$

which means that $\boldsymbol{r}$ is an affine sphere by virtue of Lemma 2.4. Finally, if we insert $\boldsymbol{\nu}_x$ as given by $(2–13)_2$ into $(2–13)_1$, we obtain $\boldsymbol{r} \cdot \boldsymbol{\nu} = 1$.

2B. Definite affine spheres

In the convex case, it is natural to consider the conformal Blaschke metric

$$g = 2h\, dz\, d\bar{z}, \quad \text{with } h > 0,\ z = x + iy,$$

i.e.,

$$\frac{L}{4} = \frac{N}{4} = h^2 = i\,|\boldsymbol{r}_z, \boldsymbol{r}_{\bar{z}}, \boldsymbol{r}_{z\bar{z}}|, \quad M = 0$$

in (2–1). For affine spheres ($\boldsymbol{\xi} = \boldsymbol{r}$), the relation (2–3) implies that we have another representation for h, namely

$$ih = |\boldsymbol{r}_z, \boldsymbol{r}_{\bar{z}}, \boldsymbol{r}|.$$

The cubic differentials $a\, dz^3, b\, d\bar{z}^3$, where

$$a = i\,|\boldsymbol{r}_z, \boldsymbol{r}_{zz}, \boldsymbol{r}|, \quad b = -i\,|\boldsymbol{r}_{\bar{z}}, \boldsymbol{r}_{\bar{z}\bar{z}}, \boldsymbol{r}|,$$

are complex conjugates. The Gauß equations of definite affine spheres read

$$\begin{aligned} \boldsymbol{r}_{zz} &= \frac{h_z}{h}\boldsymbol{r}_z - \frac{a}{h}\boldsymbol{r}_{\bar{z}}, \\ \boldsymbol{r}_{z\bar{z}} &= -h\boldsymbol{r}, \\ \boldsymbol{r}_{\bar{z}\bar{z}} &= \frac{h_{\bar{z}}}{h}\boldsymbol{r}_{\bar{z}} - \frac{b}{h}\boldsymbol{r}_z. \end{aligned} \tag{2–14}$$

Using

$$\boldsymbol{r}_{xx} - \boldsymbol{r}_{yy}, \boldsymbol{r}_{xy} \in \operatorname{span}\{\boldsymbol{r}_x, \boldsymbol{r}_y\},$$

one can easily prove the following fact:

Lemma 2.7. *A convex immersion* $(z, \bar{z}) \mapsto \boldsymbol{r}(z, \bar{z})$ *with conformal Blaschke metric* (*conformal second fundamental form*) *is an affine sphere if and only if*

$$\boldsymbol{r}_{z\bar{z}} \parallel \boldsymbol{r}.$$

The compatibility conditions for (2–14) yield

$$(\log h)_{z\bar{z}} + h + |a|^2 h^{-2} = 0, \quad a_{\bar{z}} = 0. \tag{2–15}$$

The above system is invariant with respect to the transformation

$$a \to \lambda a, \quad \bar{a} \to \frac{1}{\lambda}\bar{a} \tag{2–16}$$

with arbitrary $\lambda \in \mathbb{C}, |\lambda| = 1$. This fact gives rise to the following Lax representation for the system (2–15):

Theorem 2.8. *Every definite affine sphere possesses a one-parameter* ($|\lambda| = 1$) *family of deformations which preserve the Blaschke metric and the differential* $|a|^2\, dz^3\, d\bar{z}^3$. *These are described by the transformation* (2–16). *The system*

$$\begin{aligned} \begin{pmatrix} \boldsymbol{r}_z \\ \boldsymbol{r}_{\bar{z}} \\ \boldsymbol{r} \end{pmatrix}_z &= \begin{pmatrix} h_z h^{-1} & -\lambda a h^{-1} & 0 \\ 0 & 0 & -h \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_z \\ \boldsymbol{r}_{\bar{z}} \\ \boldsymbol{r} \end{pmatrix}, \\ \begin{pmatrix} \boldsymbol{r}_z \\ \boldsymbol{r}_{\bar{z}} \\ \boldsymbol{r} \end{pmatrix}_{\bar{z}} &= \begin{pmatrix} 0 & 0 & -h \\ -\lambda^{-1}\bar{a}h^{-1} & h_{\bar{z}}h^{-1} & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{r}_z \\ \boldsymbol{r}_{\bar{z}} \\ \boldsymbol{r} \end{pmatrix} \end{aligned} \tag{2–17}$$

determines the corresponding family of immersions.

For $a \neq 0$, by a conformal reparametrization $z \to \tilde{z}(z)$, one can normalize $a = 1$ and obtain the Tzitzeica equation for definite affine spheres

$$(\log h)_{z\bar{z}} + h + h^{-2} = 0.$$

It is easy to check that the conormal ($\boldsymbol{\nu}\cdot\boldsymbol{r}=1$)

$$\boldsymbol{\nu} = -\frac{i}{h}\boldsymbol{r}_z \times \boldsymbol{r}_{\bar z}$$

is descriptive of an affine sphere represented by the system (2–17) with $\lambda = -1$. We call this affine sphere the *dual* of $\boldsymbol{r}$. We conclude this section by a complete analogue of Theorem 2.6.

Theorem 2.9 (Duality relations for definite affine spheres). *Definite affine spheres and their duals are equivalently described by the symmetric (with respect to $\boldsymbol{r}$ and $\boldsymbol{\nu}$) Lelieuvre formulae*

$$\boldsymbol{r}_z = i\boldsymbol{\nu}_z \times \boldsymbol{\nu}, \quad \boldsymbol{\nu}_z = i\boldsymbol{r}_z \times \boldsymbol{r},$$
$$\boldsymbol{r}_{\bar z} = i\boldsymbol{\nu} \times \boldsymbol{\nu}_{\bar z}, \quad \boldsymbol{\nu}_{\bar z} = i\boldsymbol{r} \times \boldsymbol{r}_{\bar z},$$

which imply that $\boldsymbol{r}\cdot\boldsymbol{\nu}=1$. In the real coordinates x, y (with $z = x+iy$), the Lelieuvre formulae take the form

$$\begin{aligned}\boldsymbol{r}_x &= \boldsymbol{\nu}_y \times \boldsymbol{\nu}, & \boldsymbol{\nu}_x &= \boldsymbol{r}_y \times \boldsymbol{r},\\ \boldsymbol{r}_y &= \boldsymbol{\nu} \times \boldsymbol{\nu}_x, & \boldsymbol{\nu}_y &= \boldsymbol{r} \times \boldsymbol{r}_x.\end{aligned} \tag{2–18}$$

3. A DISCRETIZATION OF INDEFINITE AFFINE SPHERES

In [Bobenko and Schief 1999], affine spheres with indefinite Blaschke metric have been discretized in a purely geometric manner. The corresponding discrete Gauß and Gauß–Codazzi equations have been set down and used to derive a discrete analogue of the classical Tzitzeica transformation for affine spheres. An interpretation of the discrete Gauß equations in terms of loop groups has also been given. Here, we recall the elementary properties of these discrete affine spheres and show that they may be equivalently defined via simple duality relations involving the conormal. Duality relations of similar kind will prove key to the discretization of convex affine spheres discussed in the following section.

In this section, discrete surfaces constitute two-dimensional lattices in Euclidean space $\mathbb{R}^3$, that is maps

$$\boldsymbol{r} : \mathbb{Z}^2 \supset \Gamma \to \mathbb{R}^3, \quad (n_1, n_2) \mapsto \boldsymbol{r}(n_1, n_2). \tag{3–1}$$

It is convenient to suppress the arguments of functions of n_1 and n_2 since we only deal with autonomous difference equations. Thus, increments of discrete variables are denoted by subscripts, for instance

$$\begin{aligned}\boldsymbol{r} &= \boldsymbol{r}(n_1, n_2),\\ \boldsymbol{r}_1 &= \boldsymbol{r}(n_1{+}1, n_2),\\ \boldsymbol{r}_2 &= \boldsymbol{r}(n_1, n_2{+}1),\\ \boldsymbol{r}_{11} &= \boldsymbol{r}(n_1{+}2, n_2),\\ \boldsymbol{r}_{12} &= \boldsymbol{r}(n_1{+}1, n_2{+}1),\\ \boldsymbol{r}_{22} &= \boldsymbol{r}(n_1, n_2{+}2),\end{aligned}$$

and decrements are indicated by overbars, that is,

$$\boldsymbol{r}_{\bar 1} = \boldsymbol{r}(n_1-1, n_2), \quad \boldsymbol{r}_{\bar 2} = \boldsymbol{r}(n_1, n_2-1).$$

We also adopt the following notation of the usual difference operators:

$$\Delta_i\boldsymbol{r} = \boldsymbol{r}_i - \boldsymbol{r}, \quad \Delta_{12}\boldsymbol{r} = \boldsymbol{r}_{12} - \boldsymbol{r}_1 - \boldsymbol{r}_2 + \boldsymbol{r}.$$

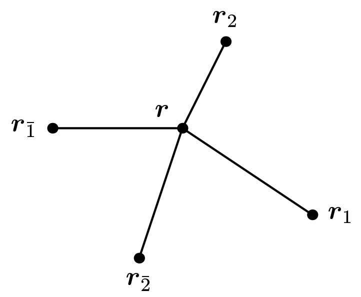
3A. Definition and properties

Definition 3.1. A two-dimensional lattice (*discrete net*) in three-dimensional Euclidean space

$$\boldsymbol{r} : \mathbb{Z}^2 \supset \Gamma \to \mathbb{R}^3$$

is called a *discrete indefinite affine sphere* if it has the following properties:

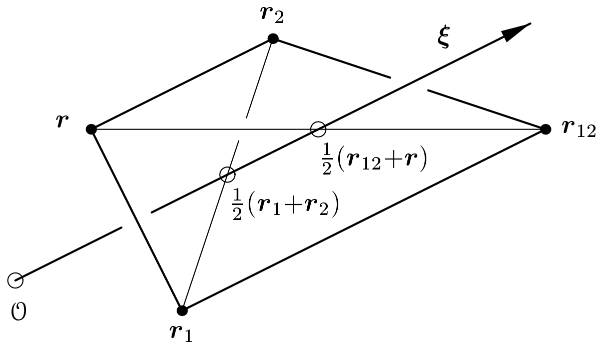
[A] Any point $\boldsymbol{r}(n_1, n_2)$ and its neighbours $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_{\bar 1}$, $\boldsymbol{r}_{\bar 2}$ lie on a plane.



[H] All *discrete affine normals* $\boldsymbol{\xi}$, whose directions are defined by

$$\boldsymbol{\xi} \sim \Delta_{12}\boldsymbol{r},$$

intersect at a point $\mathcal{O}$.



Nets possessing property [A] are called *asymptotic*. They model asymptotic lines on smooth surfaces [Sauer 1970]. In analytic terms, [A] translates into

$$|\boldsymbol{r}_1-\boldsymbol{r},\boldsymbol{r}_2-\boldsymbol{r},\boldsymbol{r}-\boldsymbol{r}_{\bar{1}}|=0,\quad |\boldsymbol{r}_1-\boldsymbol{r},\boldsymbol{r}_2-\boldsymbol{r},\boldsymbol{r}-\boldsymbol{r}_{\bar{2}}|=0.$$

Nets of type [H] may be termed *affine Lorentz harmonic*, i.e., there exists a function ρ such that

$$\Delta_{12}\boldsymbol{r}=\rho(\boldsymbol{r}_{12}+\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r});$$

equivalently,

$$\boldsymbol{r}_{12}+\boldsymbol{r}\parallel \boldsymbol{r}_1+\boldsymbol{r}_2.$$

For these conditions to hold, $\mathcal{O}$ has to be chosen as the origin of the ambient space $\mathbb{R}^3$. Remarkably, the definition of the discrete affine normal coincides exactly with the classical one (2–2) for hyperbolic surfaces if one regards the edges of elementary quadrilaterals as infinitesimal asymptotic line segments on a surface.

A straightforward analysis of the conditions [A] and [H] now leads to the following theorem [Bobenko and Schief 1999]:

Theorem 3.2 (The Gauß equations for discrete indefinite affine spheres). *Discrete indefinite affine spheres are governed by the discrete Gauß equations*

$$\begin{aligned}\boldsymbol{r}_{11}-\boldsymbol{r}_1&=\frac{H_1-1}{H_1(H-1)}(\boldsymbol{r}_1-\boldsymbol{r})+\frac{A_1}{H-1}(\boldsymbol{r}_{12}-\boldsymbol{r}_1),\\ \boldsymbol{r}_{12}+\boldsymbol{r}&=H(\boldsymbol{r}_1+\boldsymbol{r}_2),\\ \boldsymbol{r}_{22}-\boldsymbol{r}_2&=\frac{H_2-1}{H_2(H-1)}(\boldsymbol{r}_2-\boldsymbol{r})+\frac{B_2}{H-1}(\boldsymbol{r}_{12}-\boldsymbol{r}_2).\end{aligned}\tag{3–2}$$

They are compatible modulo

$$\begin{aligned}&A_2=(H/H_{\bar{1}})A,\quad B_1=(H/H_{\bar{2}})B,\\ &H_{12}H(H_1+H_2-H_1H_2+A_{12}B_{12})\\ &\qquad\qquad=H_{12}+H-1.\end{aligned}\tag{3–3}$$

The discrete analogues of the formulae (2–4)–(2–6) *derived in the previous section read*

$$\begin{aligned}|\boldsymbol{r},\Delta_1\boldsymbol{r},\Delta_2\boldsymbol{r}|&=c(H-1)/H,\\ |\Delta_1\boldsymbol{r},\Delta_2\boldsymbol{r},\Delta_{12}\boldsymbol{r}|&=2c(H-1)^2/H,\\ |\boldsymbol{r},\Delta_1\boldsymbol{r},\Delta_{11}\boldsymbol{r}|&=cA_1,\\ |\boldsymbol{r},\Delta_2\boldsymbol{r},\Delta_{22}\boldsymbol{r}|&=-cB_2.\end{aligned}\tag{3–4}$$

Here c is a constant of 'integration'.

The *discrete Gauß–Codazzi equations* (3–3) may be visualized as follows: The affine Lorentz harmonicity condition $(3–2)_2$ suggests associating the function H with the elementary quadrilateral

$$(\boldsymbol{r},\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_{12}).$$

The functions A and B naturally 'live' on the vertex $\boldsymbol{r}$. Thus, the linear equation $(3–3)_1$ may be regarded as a relation for any quadrupel consisting of two H-functions and two A-functions which enclose a 'vertical' edge $[\boldsymbol{r},\boldsymbol{r}_2]$ of the lattice. Similarly, any 'horizontal' edge $[\boldsymbol{r},\boldsymbol{r}_1]$ is enclosed by two H-functions and two B-functions. They are linked by $(3–3)_2$. The situation is illustrated in Figure 2.

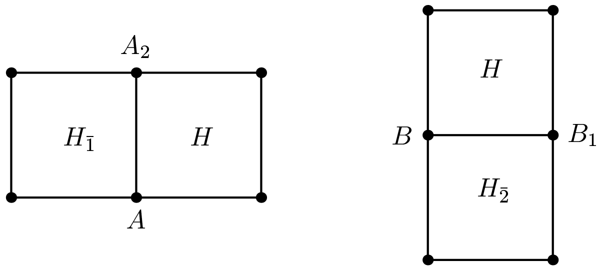


FIGURE 2. The A- and B-equations.

The remaining equation $(3–3)_3$ relates a product of A- and B-functions which is defined on a vertex to the four adjacent H-functions; see Figure 3.

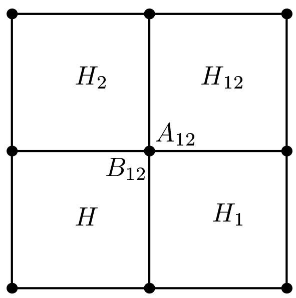


FIGURE 3. The H-equation.

In the natural continuum limit one regards a discrete function $f : \mathbb{Z}^2 \to \mathbb{R}$ as an approximation of a smooth function $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$, that is

$$f(n_1, n_2) = \tilde{f}(x, y)\big|_{(x,y)=(\varepsilon_1 n_1, \varepsilon_2 n_2)}$$

for small ε_i. Then, on dropping the tilde, the Taylor expansions

$$\begin{aligned} f_1 &= f + \varepsilon_1 f_x + O(\varepsilon_1^2),\\ f_2 &= f + \varepsilon_2 f_y + O(\varepsilon_2^2) \end{aligned}$$

apply. Now, the form of the discrete Gauß equations (3–2) suggests the natural expansion

$$H = 1 + \tfrac{1}{2}\varepsilon_1\varepsilon_2 h, \quad A = \tfrac{1}{2}\varepsilon_1^3 a, \quad B = \tfrac{1}{2}\varepsilon_2^3 b,$$

so that the discrete Gauß–Codazzi equations reduce to

$$(\log h)_{xy} = h - abh^{-2}, \quad a = a(x), \quad b = b(y)$$

in the limit $\varepsilon_i \to 0$. Similarly, the discrete Gauß equations become the continuous ones (2–7) in this limit. It is therefore justified to term (3–3) a *discrete Tzitzeica system.*

The linear equations $(3\text{–}3)_{1,2}$ for A and B may be solved identically by introducing a potential τ which parametrizes A, B and H according to

$$A = \tilde{c}\frac{\tau^2}{\tau_{\bar{1}}\tau_1}, \quad H = \frac{\tau_1\tau_2}{\tau\tau_{12}}, \quad B = \hat{c}\frac{\tau^2}{\tau_{\bar{2}}\tau_2},$$

where $\tilde{c}$ and $\hat{c}$ are arbitrary constants. This, inserted into the nonlinear equation $(3\text{–}3)_3$ for H, results in the *discrete Tzitzeica equation*

$$\begin{vmatrix} \tau_{22} & \tau_{122} & \tau_{1122}\\ \tau_2 & \tau_{12} & \tau_{112}\\ \tau & \tau_1 & \tau_{11} \end{vmatrix} = \tilde{c}\hat{c}\tau_{12}^3. \tag{3–5}$$

Note that the Tzitzeica equation

$$(\log h)_{xy} = h - h^{-2}$$

may be brought into the form

$$\begin{vmatrix} \tau_{yy} & \tau_{xyy} & \tau_{xxyy}\\ \tau_y & \tau_{xy} & \tau_{xxy}\\ \tau & \tau_x & \tau_{xx} \end{vmatrix} = \tfrac{1}{4}\tau^3$$

in terms of a τ-function defined via

$$h = -2(\log\tau)_{xy}.$$

This underlines the analogy between the classical continuous case and the discrete formalism presented here.

A well-posed Cauchy problem for the τ-equation (3–5) is associated with the Cauchy data

$$\tau(n,\, i{-}n), \quad \text{for } i = 0, \ldots, 3, \tag{3–6}$$

that is, τ is prescribed arbitrarily on four adjacent diagonal chains of lattice points (Figure 4). (In fact, τ may be prescribed on an arbitrary stairway-shaped strip.)

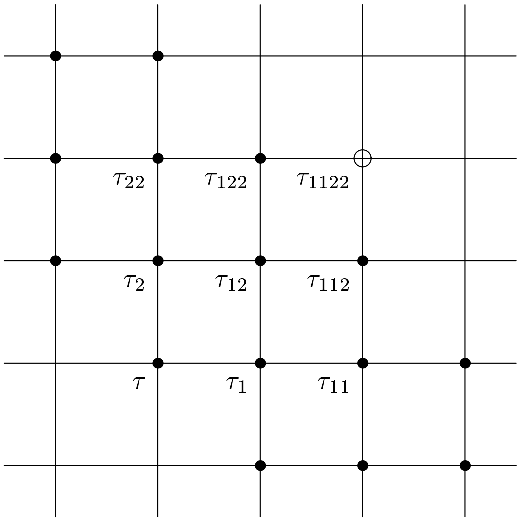


FIGURE 4. A well-posed Cauchy problem for the τ-equation (3–5).

Now, if one chooses the Cauchy data in such a way that the determinant $\tau\tau_{12} - \tau_1\tau_2$ does not vanish, then the τ-equation may be solved for τ_{1122}. Similarly, all other values of τ on the diagonal $(n, 4-n)$ may be calculated provided the relevant determinants are non-zero. This process may be iterated ad infinitum so that τ is known on the upper-right half-plane. Alternatively, the τ-equation may be regarded as an equation for the unknown values of τ on the lower-left half-plane. Thus, the Cauchy data (3–6) determine τ uniquely.

3B. The conormal and duality relations

It is evident that the discrete Tzitzeica system is invariant under $A \to \lambda A$, $B \to \lambda^{-1}B$, where λ is an arbitrary constant. This observation may be exploited to inject a parameter into the discrete Gauß equations:

Theorem 3.3 (A Lax representation for the discrete Tzitzeica system). *Every solution of the discrete Tzitzeica system* (3–3) *corresponds to a one-parameter* *family of discrete indefinite affine spheres governed by*

$$\begin{aligned} \boldsymbol{r}_{11}-\boldsymbol{r}_1 &= \frac{H_1-1}{H_1(H-1)}(\boldsymbol{r}_1-\boldsymbol{r})+\lambda\frac{A_1}{H-1}(\boldsymbol{r}_{12}-\boldsymbol{r}_1),\\ \boldsymbol{r}_{12}+\boldsymbol{r} &= H(\boldsymbol{r}_1+\boldsymbol{r}_2), \\ \boldsymbol{r}_{22}-\boldsymbol{r}_2 &= \frac{H_2-1}{H_2(H-1)}(\boldsymbol{r}_2-\boldsymbol{r})+\frac{1}{\lambda}\frac{B_2}{H-1}(\boldsymbol{r}_{12}-\boldsymbol{r}_2). \end{aligned} \tag{3–7}$$

For $\lambda = 1$, we recover the discrete Gauß equations (3–2). Moreover, it is readily verified that the quantity

$$\boldsymbol{\nu} = \frac{1}{2}\frac{H}{H-1}(\boldsymbol{r}_1-\boldsymbol{r})\times(\boldsymbol{r}_2-\boldsymbol{r}) \tag{3–8}$$

is a solution of the *adjoint* or *dual* system represented by $\lambda = -1$. The geometric significance of $\boldsymbol{\nu}$ is as follows. Since $\boldsymbol{r}$ constitutes an asymptotic net, the lattice points $\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_{\bar{1}}$ and $\boldsymbol{r}_{\bar{2}}$ lie on a plane. By definition, the vector $\boldsymbol{\nu}$ is perpendicular to this plane. Furthermore, the first integral $(3–4)_1$ shows that we may normalize $\boldsymbol{r}$ in such a way that

$$\boldsymbol{r}\cdot\boldsymbol{\nu} = 1.$$

In analogy with the continuous case, we call $\boldsymbol{\nu}$ a *conormal* and conclude that the conormal defines discrete indefinite affine spheres which may be considered dual to those given by $\boldsymbol{r}$. This is made precise in the following statement:

Theorem 3.4 (Duality relations for discrete indefinite affine spheres). *Discrete indefinite affine spheres and their duals are equivalently described by the discrete Lelieuvre formulae*

$$\begin{aligned} \boldsymbol{r}_1-\boldsymbol{r} &= \boldsymbol{\nu}\times\boldsymbol{\nu}_1, & \boldsymbol{\nu}_1-\boldsymbol{\nu} &= \boldsymbol{r}_1\times\boldsymbol{r},\\ \boldsymbol{r}_2-\boldsymbol{r} &= \boldsymbol{\nu}_2\times\boldsymbol{\nu}, & \boldsymbol{\nu}_2-\boldsymbol{\nu} &= \boldsymbol{r}\times\boldsymbol{r}_2, \end{aligned} \tag{3–9}$$

which imply that $\boldsymbol{r}\cdot\boldsymbol{\nu} = 1$.

One can directly verify that $\boldsymbol{\nu}$ as given by (3–8) obeys the duality relations (3–9). Conversely, two vector-valued functions $\boldsymbol{r} : \mathbb{Z}^2 \to \mathbb{R}^3$ and $\boldsymbol{\nu} : \mathbb{Z}^2 \to \mathbb{R}^3$ which satisfy the duality relations define two discrete indefinite affine spheres with conormals $\boldsymbol{\nu}$ and $\boldsymbol{r}$ respectively. For symmetry reasons, it is sufficient to focus only on $\boldsymbol{r}$, say. Thus, the discrete Lelieuvre formulae yield

$$\Delta_1\boldsymbol{r}\perp\boldsymbol{\nu},\quad \Delta_2\boldsymbol{r}\perp\boldsymbol{\nu},\quad \Delta_{\bar{1}}\boldsymbol{r}\perp\boldsymbol{\nu},\quad \Delta_{\bar{2}}\boldsymbol{r}\perp\boldsymbol{\nu},$$

which implies that $\boldsymbol{r}$ forms a discrete asymptotic net and $\boldsymbol{\nu}$ is a corresponding normal. Secondly, the compatibility condition $\boldsymbol{\nu}_{12} = \boldsymbol{\nu}_{21}$ for $(3–9)_{2,4}$ reads

$$(\boldsymbol{r}_{12}+\boldsymbol{r})\times(\boldsymbol{r}_1+\boldsymbol{r}_2) = 0 \quad\Longrightarrow\quad \boldsymbol{r}_{12}+\boldsymbol{r}\parallel\boldsymbol{r}_1+\boldsymbol{r}_2$$

and hence $\boldsymbol{r}$ is affine Lorentz harmonic. Finally, if we insert $\boldsymbol{\nu}_1$ as given by $(3–9)_2$ into $(3–9)_1$, we obtain

$$\Delta_1\boldsymbol{r} = (\boldsymbol{r}\cdot\boldsymbol{\nu})\Delta_1\boldsymbol{r},$$

which implies that $\boldsymbol{r}\cdot\boldsymbol{\nu} = 1$ in the generic case.

It is interesting to note that a well-posed Cauchy problem may also be formulated at the surface level using the duality relations. Thus, consider the admissible data

$$\boldsymbol{r}(n,\,-n),\qquad \boldsymbol{\nu}(n,\,1-n),\qquad \boldsymbol{\nu}(0,\,0) \tag{3–10}$$

subject to the constraints

$$\begin{aligned} \boldsymbol{\nu}(n,\,1-n)\cdot\boldsymbol{r}(n,\,-n) &= 1,\\ \boldsymbol{\nu}(n,\,1-n)\cdot\boldsymbol{r}(n-1,\,1-n) &= 1,\\ \boldsymbol{\nu}(0,\,0)\cdot\boldsymbol{r}(0,\,0) &= 1, \end{aligned} \tag{3–11}$$

as displayed in Figure 5.

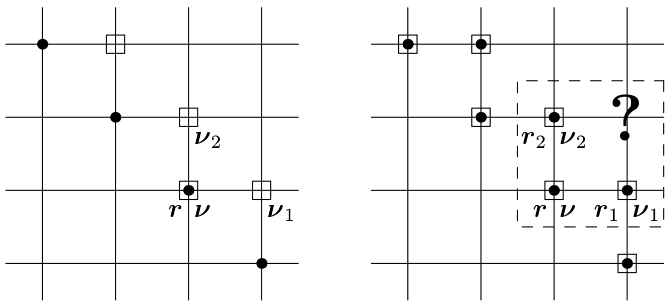


FIGURE 5. A well-posed Cauchy problem for discrete indefinite affine spheres and their duals.

Since $\boldsymbol{\nu}, \boldsymbol{\nu}_2$ and $\boldsymbol{r}$ are known, the vertex $\boldsymbol{r}_2$ may be calculated by means of the duality relations (3–9) which, in turn, enables us to determine $\boldsymbol{\nu}_{\bar{1}2}$. This process may be iterated so that $\boldsymbol{r}$ and $\boldsymbol{\nu}$ are uniquely determined on the diagonals $(n,-n)$ and $(n, 1-n)$. Note that all duality relations are satisfied due to the constraints (3–11) on the Cauchy data.

It is evident that the existence of the discrete affine sphere and its dual now depends on the solution to the following problem: Suppose the pairs of vectors $(\boldsymbol{r}_1, \boldsymbol{\nu}_1), (\boldsymbol{r}_2, \boldsymbol{\nu}_2)$ and $(\boldsymbol{r}, \boldsymbol{\nu})$ (which live on the vertices of an elementary quadrilateral) satisfy the corresponding duality relations (3–9). Is it possible to construct (uniquely) vectors $\boldsymbol{r}_{12}$ and $\boldsymbol{\nu}_{12}$ which obey the duality relations

$$\begin{aligned} \boldsymbol{r}_{12}-\boldsymbol{r}_2 &= \boldsymbol{\nu}_2\times\boldsymbol{\nu}_{12}, \quad & \boldsymbol{\nu}_{12}-\boldsymbol{\nu}_2 &= \boldsymbol{r}_{12}\times\boldsymbol{r}_2 \\ \boldsymbol{r}_{12}-\boldsymbol{r}_1 &= \boldsymbol{\nu}_{12}\times\boldsymbol{\nu}_1, \quad & \boldsymbol{\nu}_{12}-\boldsymbol{\nu}_1 &= \boldsymbol{r}_1\times\boldsymbol{r}_{12} \end{aligned} \tag{3–12}$$

(Figure 5)? The following constructive proof gives a positive answer to this question.

Elimination of $\boldsymbol{r}_{12}$ and $\boldsymbol{\nu}_{12}$ from the left-hand sides of (3–12) results in

$$\begin{aligned} \boldsymbol{r}_1-\boldsymbol{r}_2 &= (\boldsymbol{\nu}_1+\boldsymbol{\nu}_2)\times\boldsymbol{\nu}_{12} \\ \boldsymbol{\nu}_1-\boldsymbol{\nu}_2 &= \boldsymbol{r}_{12}\times(\boldsymbol{r}_1+\boldsymbol{r}_2). \end{aligned} \tag{3–13}$$

Similarly, the duality relations (3–9) yield

$$\begin{aligned} \boldsymbol{r}_1-\boldsymbol{r}_2 &= \boldsymbol{\nu}\times(\boldsymbol{\nu}_1+\boldsymbol{\nu}_2) \\ \boldsymbol{\nu}_1-\boldsymbol{\nu}_2 &= (\boldsymbol{r}_1+\boldsymbol{r}_2)\times\boldsymbol{r} \end{aligned}$$

which implies that $-(\boldsymbol{r},\boldsymbol{\nu})$ is a particular solution of (3–13) if regarded as linear inhomogeneous equations for $(\boldsymbol{r}_{12},\boldsymbol{\nu}_{12})$. Thus, the latter take the form

$$\begin{aligned} \boldsymbol{r}_{12} &= -\boldsymbol{r}+\alpha(\boldsymbol{r}_1+\boldsymbol{r}_2) \\ \boldsymbol{\nu}_{12} &= -\boldsymbol{\nu}+\beta(\boldsymbol{\nu}_1+\boldsymbol{\nu}_2), \end{aligned} \tag{3–14}$$

where α,β are as yet unknown. However, the necessary conditions

$$\boldsymbol{r}_1\cdot\boldsymbol{\nu}_{12}=\boldsymbol{r}_{12}\cdot\boldsymbol{\nu}_2=1$$

imply that

$$\alpha=\beta=\frac{2}{1+\boldsymbol{r}_1\cdot\boldsymbol{\nu}_2}. \tag{3–15}$$

The fact that $\alpha=\beta$ is hardly surprising since both $\boldsymbol{r}$ and $\boldsymbol{\nu}$ satisfy the affine Lorentz harmonicity condition $(3–7)_2$ with the *same* coefficient H. In fact, the preceding analysis delivers the expressions

$$H=\frac{2}{1+\boldsymbol{r}_1\cdot\boldsymbol{\nu}_2}=\frac{2}{1+\boldsymbol{r}_2\cdot\boldsymbol{\nu}_1}$$

if one takes into account that $\boldsymbol{r}_1\cdot\boldsymbol{\nu}_2=\boldsymbol{r}_2\cdot\boldsymbol{\nu}_1$. It is now readily verified that $\boldsymbol{r}_{12}$ and $\boldsymbol{\nu}_{12}$ as given by (3–14), (3–15) indeed satisfy the relations (3–12). This proves that discrete indefinite affine spheres and their duals are uniquely determined by the Cauchy data (3–10)–(3–11).

4. DISCRETE DEFINITE AFFINE SPHERES

4A. Duality relations

In the previous section, it has been shown that discrete indefinite affine spheres and their conormals can be defined by the discrete Lelieuvre formulae (3–9). This observation suggests defining discrete definite affine spheres by means of a canonical discretization of the Lelieuvre formulae (2–18). Thus, in the discrete case, the partial derivatives ∂_x and ∂_y should correspond to differences defined on 'horizontal' and 'vertical' edges of a lattice respectively. Since the Lelieuvre formulae (2–18) mix these derivatives, it is necessary to distinguish between the lattices $\boldsymbol{r}$ and $\boldsymbol{\nu}$. We therefore denote by Γ^* the lattice dual to $\Gamma\subset\mathbb{Z}^2$. We may think of the field $\boldsymbol{\nu}:\Gamma^*\to\mathbb{R}^3$ as defined on the faces of a discrete surface $\boldsymbol{r}:\Gamma\to\mathbb{R}^3$ (see Figure 11). The following discrete Lelieuvre relations (for notations see Figure 11) constitute a natural geometric discretization of the Lelieuvre formulae (2–18):

$$\begin{aligned} \boldsymbol{r}_1-\boldsymbol{r} &= \boldsymbol{\nu}\times\boldsymbol{\nu}_{\bar 2}, \quad & \boldsymbol{\nu}-\boldsymbol{\nu}_{\bar 1} &= \boldsymbol{r}_2\times\boldsymbol{r}, \\ \boldsymbol{r}_2-\boldsymbol{r} &= \boldsymbol{\nu}_{\bar 1}\times\boldsymbol{\nu}, \quad & \boldsymbol{\nu}-\boldsymbol{\nu}_{\bar 2} &= \boldsymbol{r}\times\boldsymbol{r}_1. \end{aligned} \tag{4–1}$$

They constitute relations on edges. If we denote the four fields attached to an edge by **u**p, **d**own, **l**eft, **r**ight as in Figure 6, the duality relations take a symmetric form.

Definition 4.1. A discrete definite affine sphere $\boldsymbol{r}$ and its dual (discrete definite affine sphere) $\boldsymbol{\nu}$ are maps $\boldsymbol{r}:\Gamma\to\mathbb{R}^3$ and $\boldsymbol{\nu}:\Gamma^*\to\mathbb{R}^3$ satisfying the discrete Lelieuvre relations

$$\begin{aligned} \boldsymbol{r}_r-\boldsymbol{r}_l &= \boldsymbol{\nu}_u\times\boldsymbol{\nu}_d, \quad & \boldsymbol{\nu}_r-\boldsymbol{\nu}_l &= \boldsymbol{r}_u\times\boldsymbol{r}_d, \\ \boldsymbol{r}_u-\boldsymbol{r}_d &= \boldsymbol{\nu}_l\times\boldsymbol{\nu}_r, \quad & \boldsymbol{\nu}_u-\boldsymbol{\nu}_d &= \boldsymbol{r}_l\times\boldsymbol{r}_r. \end{aligned} \tag{4–2}$$

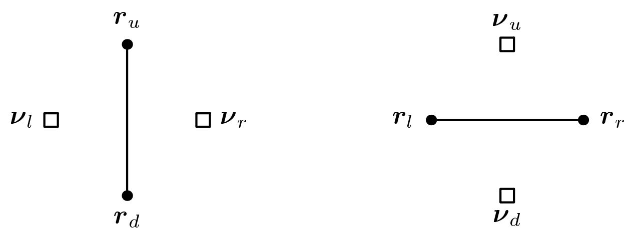


FIGURE 6. The duality relations for discrete definite affine spheres.

Since Definition 4.1 is symmetric with respect to $\boldsymbol{r}$ and $\boldsymbol{\nu}$, it is sufficient to investigate the geometrical properties of $\boldsymbol{r}$, say.

Theorem 4.2. *Discrete definite affine spheres have the following geometrical properties*:

(i) *All elementary quadrilaterals* $(\boldsymbol{r},\boldsymbol{r}_1,\boldsymbol{r}_{12},\boldsymbol{r}_2)$ *are planar*.

(ii) $\boldsymbol{\nu}$ *is the conormal of* $\boldsymbol{r}$, *i.e.*,

$$\boldsymbol{\nu}\cdot\boldsymbol{r}=\boldsymbol{\nu}\cdot\boldsymbol{r}_1=\boldsymbol{\nu}\cdot\boldsymbol{r}_2=\boldsymbol{\nu}\cdot\boldsymbol{r}_{12}=1.$$

(iii) *For each vertex and its* 4 *neighbours*

$$\boldsymbol{r}_1+\boldsymbol{r}_{\bar 1}+\boldsymbol{r}_2+\boldsymbol{r}_{\bar 2}\parallel\boldsymbol{r}. \tag{4–3}$$

Nets with this property may be termed discrete affine harmonic.

(iv) *For* $\boldsymbol{r},\boldsymbol{\nu}$ *defined by* (4–1), *we have*

$$\begin{aligned}(\boldsymbol{r}_1-\boldsymbol{r})\cdot(\boldsymbol{\nu}-\boldsymbol{\nu}_{\bar 2})&=(\boldsymbol{r}_2-\boldsymbol{r})\cdot(\boldsymbol{\nu}-\boldsymbol{\nu}_{\bar 1})=0,\\(\boldsymbol{r}_1-\boldsymbol{r})\cdot(\boldsymbol{\nu}-\boldsymbol{\nu}_{\bar 1})&=(\boldsymbol{r}_2-\boldsymbol{r})\cdot(\boldsymbol{\nu}-\boldsymbol{\nu}_{\bar 2}).\end{aligned} \tag{4–4}$$

Properties (i), (ii), (iv) follow immediately from the discrete Lelieuvre formulae. The latter are readily shown to be invariant under volume preserving equi-affine transformations. We have chosen Definition 4.1 in such a way that the normalization $\boldsymbol{r}\cdot\boldsymbol{\nu}=$ const $=1$ holds. Any other choice of this constant would result in a slightly different but equivalent (modulo appropriate scaling of $\boldsymbol{r}$ and $\boldsymbol{\nu}$) definition.

In order to derive (4–3), one should consider four neighbouring faces in Figure 11 and substitute (4–1) into the trivial identity

$$(\boldsymbol{\nu}-\boldsymbol{\nu}_{\bar 2})+(\boldsymbol{\nu}_{\bar 1}-\boldsymbol{\nu})+(\boldsymbol{\nu}_{\bar 1\bar 2}-\boldsymbol{\nu}_{\bar 1})+(\boldsymbol{\nu}_{\bar 2}-\boldsymbol{\nu}_{\bar 1\bar 2})=0.$$

Remark 4.3. In [Sauer 1970], a net (or a discrete surface) $\boldsymbol{r}:\mathbb{Z}^2\to\mathbb{R}^3$ is called *discrete conjugate* if all its elementary quadrilaterals are planar. Recently, discrete conjugate nets and their specializations have become a focus of interest in the theory of integrable systems (see Introduction). Property (i) of the theorem implies that discrete definite affine spheres are conjugate nets. This property is natural since the Blaschke metric (which is proportional to the second fundamental form) is conformal.

Remark 4.4. The *affine harmonicity* condition admits a simple geometric interpretation. We define a *discrete affine normal* $\boldsymbol{\xi}$ at the vertex $\boldsymbol{r}$ as a vector which passes through the barycenter of the vertices $\boldsymbol{r}_1$, $\boldsymbol{r}_{\bar 1}$, $\boldsymbol{r}_2$, $\boldsymbol{r}_{\bar 2}$, and $\boldsymbol{r}$, that is,

$$\boldsymbol{\xi}\sim\boldsymbol{r}_1+\boldsymbol{r}_{\bar 1}+\boldsymbol{r}_2+\boldsymbol{r}_{\bar 2}-4\boldsymbol{r},$$

and require that all affine normals meet at a point $\mathcal{O}$, say. If we identify this point with the origin of the ambient space $\mathbb{R}^3$ then the discrete affine normal is parallel to the position vector $\boldsymbol{r}$ and hence the condition (4–3) is retrieved. Note the resemblance between the definitions of the affine normal in the discrete and continuous cases.

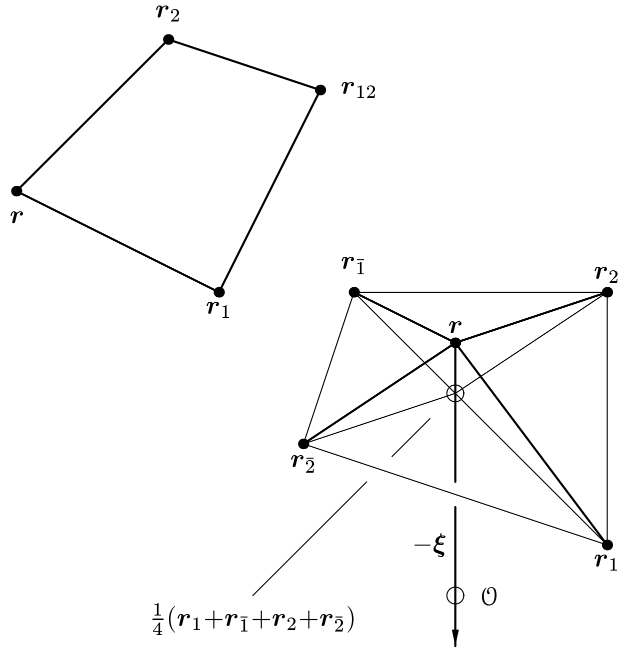


FIGURE 7. A planar quadrilateral and the discrete affine normal.

Remark 4.5. Property (iv) is a discrete analogue of the conformality of the second fundamental form

$$\boldsymbol{r}_x\cdot\boldsymbol{\nu}_y=\boldsymbol{r}_y\cdot\boldsymbol{\nu}_x=0,\qquad\boldsymbol{r}_x\cdot\boldsymbol{\nu}_x=\boldsymbol{r}_y\cdot\boldsymbol{\nu}_y,$$

where $\boldsymbol{\nu}$ is any normal field on the surface.

4B. A Cauchy problem

A canonical Cauchy problem for discrete definite affine spheres is obtained by prescribing $\boldsymbol{r}$ on a vertical chain of lattice points and the conormal $\boldsymbol{\nu}$ on adjoining quadrilaterals, that is

$$\boldsymbol{r}(0,n),\quad\boldsymbol{\nu}(0,n),$$

subject to the normalization conditions

$$\begin{aligned}\boldsymbol{r}(0,n)\cdot\boldsymbol{\nu}(0,n)&=1,\\\boldsymbol{r}(0,n)\cdot\boldsymbol{\nu}(0,n-1)&=1.\end{aligned} \tag{4–5}$$

This is schematically indicated in Figure 8. Here, bullets and boxes represent the Cauchy data $\boldsymbol{r}(0,n)$ and $\boldsymbol{\nu}(0,n)$ respectively.

Now, comparison with the duality relations encoded in Figure 6 shows that both $\boldsymbol{r}(1,n)$ (circles) and $\boldsymbol{\nu}(-1,n)$ (dashed boxes) may be calculated from the Cauchy data. We stress that *all* duality relations are satisfied by virtue of the constraints (4–5). It is

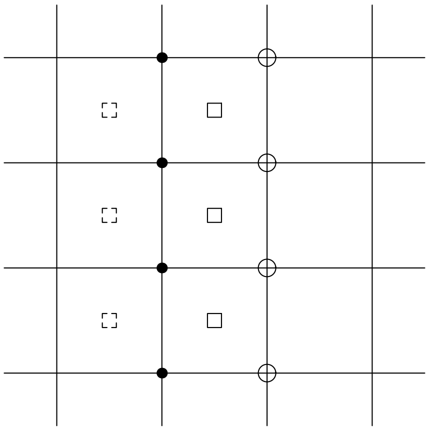


FIGURE 8. A well-posed Cauchy problem for discrete definite affine spheres.

evident that iterative application of this procedure uniquely determines the discrete affine sphere and its dual.

A particular class of discrete affine spheres is obtained if the Cauchy data $\boldsymbol{r}(0,n)$ and $\boldsymbol{\nu}(0,n)$ generate two regular polygons of order N. For given N, the affine spheres admit the discrete symmetry $\mathbb{Z}_N$ and possess one degree of freedom. However, the additional assumption of a reflection symmetry with respect to the polygon $\{\boldsymbol{r}(0,n)\}$ removes this ambiguity. In this case, it is not difficult to show that for $N=4$ the discrete affine sphere is compact as shown in Figure 9. We conjecture that for $N \neq 4$ the discrete affine spheres do *not* close in the sense that the 'discrete meridians' do not intersect at vertices.

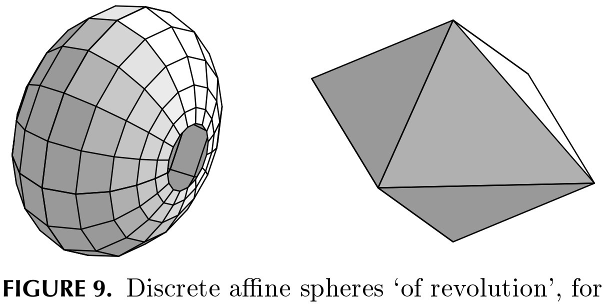


FIGURE 9. Discrete affine spheres 'of revolution', for $N = 16, 4$. For clarity, only part of the sphere corresponding to $N = 16$ is shown.

4C. General nets

Since the duality relations are defined on edges one can naturally generalize Definition 4.1 to discrete surfaces with arbitrary topology different from $\mathbb{Z}^2$. The faces may be planar polygons of any type which may meet at vertices in any number. Each internal edge must belong to exactly two faces. In this section, we are concerned with this generalized notion of the discrete definite affine spheres: Γ and Γ^* in Definition 4.1 are two arbitrary mutually dual lattices which form discrete surfaces.

Figure 9 is reminiscent of a famous result in affine differential geometry due to Blaschke [1923]:

Theorem 4.6. *All compact affine spheres are ellipsoids* (*the round sphere and its affine transforms*).

In the discrete case, it is natural to pose the same question:

Problem 4.7. *Classify compact discrete affine spheres.*

We do not have a complete solution of this problem. In the simplest cases, however, the solution is remarkably close to the above mentioned theorem of Blaschke. First of all, note that all tetrahedra are affine equivalent and the regular one is obviously a discrete affine sphere. The normalization $\boldsymbol{r} \cdot \boldsymbol{\nu} = 1$ implies that a tetrahedron affine sphere and its dual have the same volume. In our normalization, the volume of the tetrahedron affine sphere is $\frac{8}{3}$.

Theorem 4.8. *All discrete affine spheres with the topology of a cube are parallelopipeds* (*the regular cube and its affine transforms*).

To prove this theorem, denote by $\boldsymbol{\nu}_u$, $\boldsymbol{\nu}_d$, $\boldsymbol{\nu}_l$, $\boldsymbol{\nu}_r$, $\boldsymbol{\nu}_f$, $\boldsymbol{\nu}_b$ the conormals of the up, down, left, right, front and back faces of a discrete surface with the cube topology (all faces are planar quadrilaterals, three faces meet at a vertex). It is not difficult to show that the sum of all conormals

$$\boldsymbol{A} = \boldsymbol{\nu}_u + \boldsymbol{\nu}_d + \boldsymbol{\nu}_l + \boldsymbol{\nu}_r + \boldsymbol{\nu}_f + \boldsymbol{\nu}_b$$

must vanish. Indeed, consider $\boldsymbol{A} \times \boldsymbol{\nu}_u$. Since the conormals are discrete affine harmonic we have $\boldsymbol{A} \times \boldsymbol{\nu}_u = \boldsymbol{\nu}_d \times \boldsymbol{\nu}_u$, which implies that $\boldsymbol{A} = \boldsymbol{\nu}_d + \alpha \boldsymbol{\nu}_u$ with some α. For symmetry reasons $\alpha = 1$. Considering all pairs of opposite faces, we obtain

$$\boldsymbol{A} = \boldsymbol{\nu}_d + \boldsymbol{\nu}_u = \boldsymbol{\nu}_l + \boldsymbol{\nu}_r = \boldsymbol{\nu}_f + \boldsymbol{\nu}_b.$$

The sum of these three formulae ($3\boldsymbol{A} = \boldsymbol{A}$) implies that $\boldsymbol{A} = 0$ and

$$\boldsymbol{\nu}_d = -\boldsymbol{\nu}_u, \quad \boldsymbol{\nu}_l = -\boldsymbol{\nu}_r, \quad \boldsymbol{\nu}_f = -\boldsymbol{\nu}_b.$$

The opposite faces of the surface are parallel. This observation completes the proof.

In our normalization, the volume of this affine cube is 4.

Corollary 4.9. *All discrete affine spheres with the topology of an octahedron are affine transforms of the regular octahedron.*

This statement follows from the previous one by duality. The volume of the octahedron is $\frac{8}{3}$. Furthermore, the following theorem is evident:

Theorem 4.10. *Regular solids and their affine transforms are discrete affine spheres.*

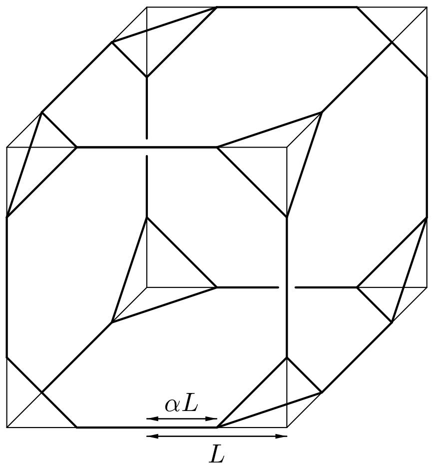


FIGURE 10. The truncated cube, shown here for $\alpha = \frac{1}{2}(\sqrt{17} - 3) = 1/(2L^3)$.

We should mention that the regular solids do not exhaust the list of discrete compact affine spheres. In particular, the truncated hexahedron as shown in Figure 10 and its dual are discrete affine spheres; they are not Archimedean solids since there are two types of edges. Another simple example is the truncated tetrahedron (also non-Archimedean) and its dual. See section on Electronic Availability below.

5. THE DISCRETE GAUSS EQUATIONS AND A DISCRETE TZITZEICA SYSTEM

The aim of this section is to derive explicitly a discrete elliptic Tzitzeica system which governs discrete definite affine spheres in the case of $\mathbb{Z}^2$-lattices. As in the continuous case, the discrete Tzitzeica system may be regarded as associated discrete Gauß–Codazzi equations. They arise as the compatibility condition of the discrete Gauß equations which are, in turn, a consequence of the defining relations (4–1) for discrete definite affine spheres and their duals. It is also recorded that the discrete Gauß equations admit an elementary interpretation in terms of the volumes

$$\begin{aligned} V &= |\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2|, \qquad V' = |\boldsymbol{r}, \boldsymbol{r}_2, \boldsymbol{r}_{\bar{1}}| \\ V'' &= |\boldsymbol{r}, \boldsymbol{r}_{\bar{2}}, \boldsymbol{r}_1|, \qquad V''' = |\boldsymbol{r}, \boldsymbol{r}_{\bar{1}}, \boldsymbol{r}_{\bar{2}}|. \end{aligned} \tag{5–1}$$

(Compare Figure 11.)

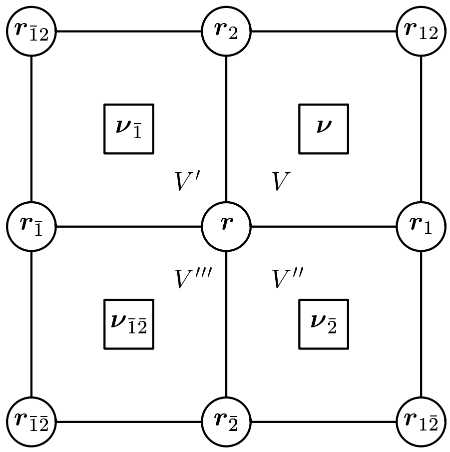


FIGURE 11. The $\mathbb{Z}^2$-lattice, its conormals and corresponding volumes.

5A. The discrete Gauß equations

The defining relations (4–1) readily imply the following result:

Theorem 5.1 (The discrete Gauß equations). *The discrete Gauß equations associated with discrete definite affine spheres take the form*

$$\begin{aligned} \boldsymbol{r}_{12} - \boldsymbol{r} &= P(\boldsymbol{r}_1 - \boldsymbol{r}) + Q(\boldsymbol{r}_2 - \boldsymbol{r}), \\ \boldsymbol{r}_1 + \boldsymbol{r}_{\bar{1}} + \boldsymbol{r}_2 + \boldsymbol{r}_{\bar{2}} &= (4 - H)\boldsymbol{r}, \\ \boldsymbol{r}_1 - \boldsymbol{r}_{\bar{1}} - \boldsymbol{r}_2 + \boldsymbol{r}_{\bar{2}} &= A\boldsymbol{r}_1 + B\boldsymbol{r}_2 + C\boldsymbol{r}, \end{aligned} \tag{5–2}$$

with constraints

$$\begin{aligned} A + B + C &= \frac{A+B}{A-B}H, \qquad S = \frac{A-B}{2H}, \\ 2\frac{S_1}{S} &= \frac{A_1}{Q}, \qquad 2\frac{S_2}{S} = -\frac{B_2}{P}. \end{aligned} \tag{5–3}$$

The conormal $\boldsymbol{\nu}$ *is given by*

$$\boldsymbol{\nu} = S(\boldsymbol{r}_1 - \boldsymbol{r}) \times (\boldsymbol{r}_2 - \boldsymbol{r}). \tag{5–4}$$

The condition $(5\text{–}2)_1$ expresses the fact that the elementary quadrilaterals are planar (compare Figure 7). Secondly the condition (4–3) implies that there exists a function H such that $(5\text{–}2)_2$ holds. The remaining relation $(5\text{–}2)_3$ is trivial if one assumes that the vectors $\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2$ are linearly independent. The additional constraints are obtained by satisfying the conditions (4–1) identically. To this end, it turns out convenient to bring the discrete Gauß equations into canonical form, by writing the vectors $\boldsymbol{r}_{11}, \boldsymbol{r}_{22}$ and $\boldsymbol{r}_{12}$ as linear combinations of $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}$. This is achieved by incrementing the sum and the difference of $(5\text{–}2)_{2,3}$, namely

$$\begin{aligned} 2\boldsymbol{r}_1+2\boldsymbol{r}_{\bar{2}} &= A\boldsymbol{r}_1+B\boldsymbol{r}_2+(4-H+C)\boldsymbol{r}, \\ 2\boldsymbol{r}_{\bar{1}}+2\boldsymbol{r}_2 &= -A\boldsymbol{r}_1-B\boldsymbol{r}_2+(4-H-C)\boldsymbol{r}, \end{aligned} \tag{5–5}$$

with respect to n_1 and n_2. We deduce that

$$\begin{aligned} A_1\boldsymbol{r}_{11} &= (4-H_1-C_1)\boldsymbol{r}_1-2\boldsymbol{r}-(2+B_1)\boldsymbol{r}_{12}, \\ \boldsymbol{r}_{12} &= \boldsymbol{r}+P(\boldsymbol{r}_1-\boldsymbol{r})+Q(\boldsymbol{r}_2-\boldsymbol{r}), \\ B_2\boldsymbol{r}_{22} &= -(4-H_2+C_2)\boldsymbol{r}_2+2\boldsymbol{r}+(2-A_2)\boldsymbol{r}_{12}, \end{aligned}$$

which is of the required form if one substitutes for $\boldsymbol{r}_{12}$.

Now, the first step in the procedure is to find an explicit expression for the conormal $\boldsymbol{\nu}$. Thus, since the dual lattice is also discrete affine harmonic, there exists a function H^* such that

$$\boldsymbol{\nu}_1+\boldsymbol{\nu}_{\bar{1}}+\boldsymbol{\nu}_2+\boldsymbol{\nu}_{\bar{2}} = (4-H^*)\boldsymbol{\nu}.$$

If we increment n_1 and n_2 in $(4\text{–}1)_{2,4}$ respectively, we obtain

$$\begin{aligned} \boldsymbol{\nu}_1-\boldsymbol{\nu} &= \boldsymbol{r}_{12}\times\boldsymbol{r}_1, \\ \boldsymbol{\nu}_2-\boldsymbol{\nu} &= \boldsymbol{r}_2\times\boldsymbol{r}_{12}, \end{aligned} \tag{5–6}$$

which combined with $(4\text{–}1)_{2,4}$ yields

$$\begin{aligned} -H^*\boldsymbol{\nu} &= (\boldsymbol{\nu}_1-\boldsymbol{\nu})+(\boldsymbol{\nu}_{\bar{1}}-\boldsymbol{\nu})+(\boldsymbol{\nu}_2-\boldsymbol{\nu})+(\boldsymbol{\nu}_{\bar{2}}-\boldsymbol{\nu}) \\ &= (\boldsymbol{r}_{12}-\boldsymbol{r})\times(\boldsymbol{r}_1-\boldsymbol{r}_2). \end{aligned}$$

Hence, by virtue of the conjugacy condition $(5\text{–}2)_1$, the conormal is given by

$$\begin{aligned} \boldsymbol{\nu} &= S(\boldsymbol{r}_1-\boldsymbol{r})\times(\boldsymbol{r}_2-\boldsymbol{r}), \\ S &= \frac{P+Q}{H^*}, \end{aligned} \tag{5–7}$$

which reflects the fact that $\boldsymbol{\nu}$ is indeed orthogonal to the elementary quadrilateral $(\boldsymbol{r}, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_{12})$.

The expression for $\boldsymbol{\nu}$ may now be inserted into $(5\text{–}6)_1$. Its component in $\boldsymbol{r}_{12}$-direction is identically satisfied. The component in $\boldsymbol{r}_1$-direction produces the constraint $(5\text{–}3)_3$ while $(5\text{–}6)_1\cdot\boldsymbol{r}$ yields

$$A+B+C+H = \frac{A}{S}.$$

Similarly, evaluation of $(5\text{–}6)_2$ results in the constraint $(5\text{–}3)_4$ and the relation

$$A+B+C-H = \frac{B}{S}.$$

Thus, the relations $(5\text{–}3)_{1,2}$ are retrieved. Finally, it is readily verified that $\boldsymbol{r}\cdot\boldsymbol{\nu} = \boldsymbol{r}_1\cdot\boldsymbol{\nu}_1 = \boldsymbol{r}_2\cdot\boldsymbol{\nu}_2$ which implies that $\boldsymbol{r}$ may by chosen in such a way that $\boldsymbol{r}\cdot\boldsymbol{\nu} = 1$ and hence the defining relations $(4\text{–}1)_{1,3}$ are also satisfied.

To summarize, discrete definite affine spheres are governed by the discrete Gauß equations given in Theorem 5.1. The corresponding dual discrete definite affine spheres represented by the conormal $\boldsymbol{\nu}$ take the form (5–4).

5B. Geometric properties of the discrete Gauß equations

It has been pointed out that the first two discrete Gauß equations reflect conjugacy and affine harmonicity of the lattice. The third equation encodes the fact that, in the continuous case, the second fundamental form is not only diagonal but conformal. Furthermore, it is possible to interpret the coefficients in the discrete Gauß equations in terms of the volumes V, V', V'', V'''. Thus, consider the triple product $[(5\text{–}5)_1\times\boldsymbol{r}]\cdot\boldsymbol{r}_1$, which yields

$$2(\boldsymbol{r}_{\bar{2}}\times\boldsymbol{r})\cdot\boldsymbol{r}_1 = B(\boldsymbol{r}_2\times\boldsymbol{r})\cdot\boldsymbol{r}_1.$$

By virtue of the definitions (5–1) this becomes

$$B = -2\frac{V''}{V}.$$

Similarly, the triple product $[(5\text{–}5)_2\times\boldsymbol{r}]\cdot\boldsymbol{r}_2$ results in

$$A = 2\frac{V'}{V}.$$

Moreover, since $\boldsymbol{r}\cdot\boldsymbol{\nu} = 1$, the expression (5–7) for the conormal $\boldsymbol{\nu}$ yields

$$S = \frac{1}{V},$$

so that the constraints $(5\text{–}3)_{3,4}$ may be written as

$$P = \frac{V_2''}{V}, \quad Q = \frac{V_1'}{V}.$$

Hence, we conclude that the coefficients A, B and P, Q are essentially ratios of certain volumes associated with the lattice.

The above relations may now be used to rewrite the two expressions $(5\text{–}3)_2$ and $(5\text{–}7)_2$ for S. Indeed, it is readily verified that

$$H = V' + V'', \quad H^* = V_1' + V_2''. \tag{5–8}$$

It is evident that for symmetry reasons the conditions

$$H = V + V''', \quad H^* = V + V_{12}''' \tag{5–9}$$

must also hold. In fact, it is shown in the following subsection that these relations are a consequence of the compatibility conditions for the discrete Gauß equations. In this connection, it turns out convenient to introduce the quantities

$$s = \frac{1}{S}, \quad p = \frac{P}{S}, \quad q = \frac{Q}{S},$$

so that

$$s = V, \quad p = V_2'', \quad q = V_1'.$$

If we complete these identities by

$$t := V_{12}''',$$

the expressions (5–8)–(5–9) assume the form

$$\begin{aligned} H &= p_{\bar{2}} + q_{\bar{1}} = s + t_{\bar{1}\bar{2}}, \\ H^* &= p + q = s + t. \end{aligned} \tag{5–10}$$

These relations may be memorized in the following way:

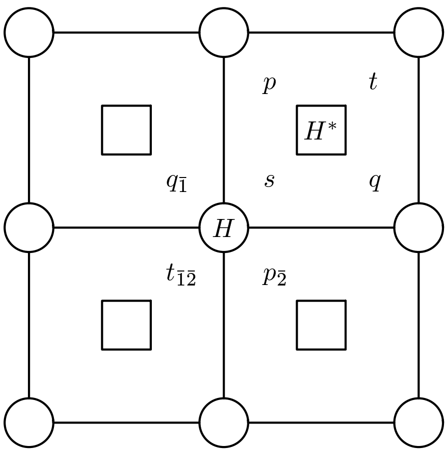


FIGURE 12. The geometric interpretation of H and H^*.

We associate the quantities H and H^* with the vertex $\boldsymbol{r}$ and the quadrilateral represented by $\boldsymbol{\nu}$ respectively. In view of the connection with the volumes V, V', V'', V''', the functions p, q, s and t may be regarded as 'living' on the quadrilaterals between H and H^* as indicated in Figure 12. The relations (5–10) then express the fact that the sum of any diagonally related pair of these functions equals the function they enclose, that is H or H^*. It is emphasized that there is a complete symmetry between the lattice $\boldsymbol{r}$ and its dual $\boldsymbol{\nu}$. This is reflected in

$$\begin{aligned} |\boldsymbol{\nu}, \boldsymbol{\nu}_{\bar{1}}, \boldsymbol{\nu}_{\bar{2}}| &= V, & |\boldsymbol{\nu}_{\bar{1}}, \boldsymbol{\nu}_{\bar{1}\bar{2}}, \boldsymbol{\nu}| &= V' \\ |\boldsymbol{\nu}_{\bar{2}}, \boldsymbol{\nu}, \boldsymbol{\nu}_{\bar{1}\bar{2}}| &= V'', & |\boldsymbol{\nu}_{\bar{1}\bar{2}}, \boldsymbol{\nu}_{\bar{2}}, \boldsymbol{\nu}_{\bar{1}}| &= V'''. \end{aligned}$$

5C. The discrete Gauß–Codazzi equations

Even though we have satisfied the defining relations (4–1) identically, there exist further constraints on the coefficients of the discrete Gauß equations due to compatibility. Thus, the consistency conditions $\boldsymbol{r}_{121} = \boldsymbol{r}_{112}$ and $\boldsymbol{r}_{122} = \boldsymbol{r}_{221}$ lead to two equations of the form

$$E^0\boldsymbol{r} + E^1\boldsymbol{r}_1 + E^2\boldsymbol{r}_2 = 0,$$

where the functions E^i depend on the coefficients A, B, P, Q and H. If we assume that the vectors $\boldsymbol{r}_1, \boldsymbol{r}_2$ and $\boldsymbol{r}$ are linearly independent, the conditions $E^i = 0$ consist of the linear system

$$H_{12} = p_1 + q_2, \quad H^* = s + t$$

and the nonlinear system

$$\begin{aligned} &\frac{H_2 - p - s_2}{ps_2} + \frac{H_{12} - t - q_2}{tq_2} \\ &\qquad + \frac{H^* - t - p}{pt} + \frac{H_2^* - s_2 - q_2}{s_2 q_2} + 1 = 0, \\ &\frac{H_1 - q - s_1}{qs_1} + \frac{H_{12} - t - p_1}{tp_1} \\ &\qquad + \frac{H^* - t - q}{qt} + \frac{H_1^* - s_1 - p_1}{s_1 p_1} + 1 = 0. \end{aligned} \tag{5–11}$$

The former system represents the additional relations (5–9) alluded to in the previous subsection. Furthermore, (5–10) may be regarded as a linear system for the functions p, q and s, t. It may be solved explicitly in terms of two potentials σ and τ according to

$$\begin{aligned} p &= \sigma + \tau_2, & q &= \sigma_1 + \tau, \\ s &= \sigma + \tau, & t &= \sigma_1 + \tau_2, \end{aligned}$$

so that

$$H = \sigma + \sigma_{\bar{2}} + \tau + \tau_{\bar{1}},$$
$$H^* = \sigma + \sigma_1 + \tau + \tau_2.$$

Insertion of these expressions into (5–11) produces a coupled system of equations for σ and τ which is encapsulated in the following statement:

Theorem 5.2 (A discrete Tzitzeica system). *The discrete Gauß–Codazzi equations associated with the discrete Gauß equations for discrete definite affine spheres may be cast into the form*

$$\begin{aligned}&\frac{\sigma-\sigma_2}{(\tau_{\bar{1}2}+\sigma)(\tau_2+\sigma)}+\frac{\sigma-\sigma_1}{(\tau_2+\sigma)(\tau+\sigma)}\\&\quad+\frac{\sigma-\sigma_{\bar{2}}}{(\tau+\sigma)(\tau_{\bar{1}}+\sigma)}+\frac{\sigma-\sigma_{\bar{1}}}{(\tau_{\bar{1}}+\sigma)(\tau_{\bar{1}2}+\sigma)}=1,\\&\frac{\tau-\tau_2}{(\sigma+\tau)(\sigma_1+\tau)}+\frac{\tau-\tau_1}{(\sigma_1+\tau)(\sigma_{1\bar{2}}+\tau)}\\&\quad+\frac{\tau-\tau_{\bar{2}}}{(\sigma_{1\bar{2}}+\tau)(\sigma_{\bar{2}}+\tau)}+\frac{\tau-\tau_{\bar{1}}}{(\sigma_{\bar{2}}+\tau)(\sigma+\tau)}=1.\end{aligned} \tag{5–12}$$

They constitute a discretized elliptic version of the classical Tzitzeica equation. On use of the reparametrization

$$\kappa = \kappa(m_1, m_2),$$
$$\kappa(n_2{+}n_1,\ n_2{-}n_1) = \sigma(n_1, n_2),$$
$$\kappa(n_2{+}n_1,\ n_2{-}n_1{-}1) = \tau(n_1, n_2),$$

the system (5–12) *may be combined to the single equation*

$$\begin{aligned}&\frac{\kappa-\kappa_{12}}{(\kappa_2+\kappa)(\kappa_1+\kappa)}+\frac{\kappa-\kappa_{1\bar{2}}}{(\kappa_1+\kappa)(\kappa_{\bar{2}}+\kappa)}\\&\quad+\frac{\kappa-\kappa_{\bar{1}\bar{2}}}{(\kappa_{\bar{2}}+\kappa)(\kappa_{\bar{1}}+\kappa)}+\frac{\kappa-\kappa_{\bar{1}2}}{(\kappa_{\bar{1}}+\kappa)(\kappa_2+\kappa)}=1.\end{aligned} \tag{5–13}$$

The potentials σ and τ may be associated with the vertical and horizontal edges of the lattice respectively. For instance, Figure 12 shows that s is enclosed by a vertical edge and a horizontal edge. On the other hand, $s = \sigma + \tau$. Hence, it is natural to label the vertical edge by σ and the horizontal edge by τ. Accordingly, the 'σ-equation' $(5\text{–}12)_1$ represents a relation between σ and its eight nearest neighbours as indicated in Figure 13. It is completely symmetric in the sense that each of the four terms consists of σ, a nearest σ-function and two adjoining τ-functions. The 'τ-equation' $(5\text{–}12)_2$ has a similar interpretation as depicted in Figure 14.

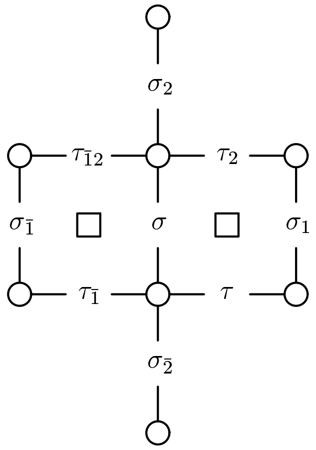


FIGURE 13. The σ-equation.

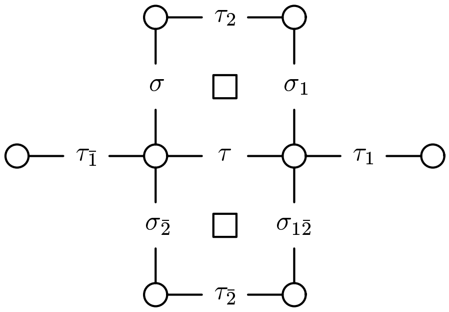


FIGURE 14. The τ-equation.

An alternative 'diagonal' labelling of the edges is displayed in Figure 15 and is associated with the 'κ-equation' (5–13).

The edges of the conormal lattice have been included in order to demonstrate the complete symmetry between the discrete affine spheres and their duals. As a consequence, the functions σ, τ and κ may also be associated with the edges of the dual lattice.

The natural Cauchy data associated with the κ-equation are given by

$$\kappa(m, m{-}1), \quad \kappa(m, m), \quad \kappa(m, m{+}1), \quad \kappa(m, m{+}2)$$

which make up four adjacent vertical chains of data, as depicted in Figure 16.

The κ-equation may then be used to calculate the values of κ on the adjoining vertical edges on the right and the horizontal edges on the left. Iteration of this procedure covers the entire lattice so that κ is indeed uniquely determined.

In order to perform a continuum limit on the discrete system (5–12), we need to introduce an arbitrary lattice step size ε. Inspection of the discrete

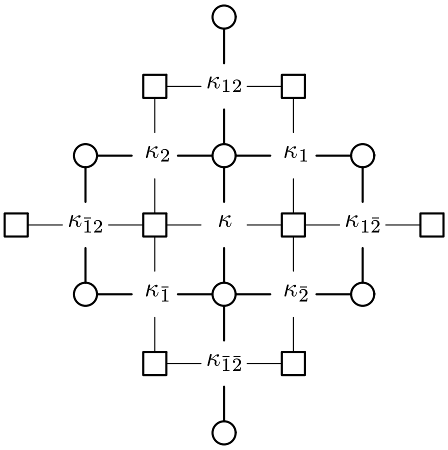


FIGURE 15. The κ-equation.

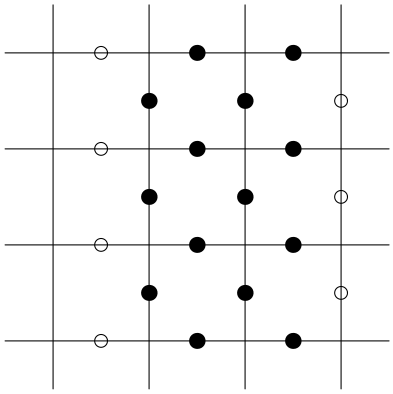


FIGURE 16. A well-posed Cauchy problem for the κ-equation.

Gauß equations reveals that the potentials σ and τ scale as

$$\sigma, \tau \sim \varepsilon^2.$$

It is therefore natural to set

$$h = \frac{\sigma + \tau}{2\varepsilon^2}, \quad \varphi = \frac{\sigma - \tau}{2\varepsilon^2}.$$

We regard the discrete variables n_1, n_2 as discretizations of some continuous variables x, y, say, that is

$$x = \varepsilon n_1, \quad y = \varepsilon n_2$$

and assume once again that any discrete function constitutes an approximation of a smooth function which admits a Taylor expansion, for instance

$$\begin{aligned} h_1 &= h + \varepsilon h_x + \tfrac{1}{2}\varepsilon^2 h_{xx} + O(\varepsilon^3), \\ h_{\bar{1}} &= h - \varepsilon h_x + \tfrac{1}{2}\varepsilon^2 h_{xx} + O(\varepsilon^3). \end{aligned}$$

In this interpretation, the discrete system (5–12) reduces to the elliptic Tzitzeica system

$$\Delta(\log h) + 4h + \frac{(\nabla\varphi)^2}{h^2} = 0, \quad \Delta\varphi = 0$$

in the limit $\varepsilon \to 0$. Here, ∇ and Δ are the usual gradient and Laplace operator respectively. Suffice it to say that the discrete Gauß equations coincide with the continuous ones discussed in Section 2 if one applies the same limit.

5D. A class of discrete affine spheres governed by elliptic functions

Here, we consider a class of discrete affine spheres which is associated with a one-dimensional reduction of the discrete Tzitzeica system (5–12). It turns out that this reduction leads to a particular member of a class of integrable mappings which is well-known in soliton theory. In order to be as general as possible, it is observed that the discrete Tzitzeica system only depends on the quantities

$$\sigma + \tau, \quad \Delta_i(\sigma + \tau), \quad \Delta_i(\sigma - \tau).$$

We assume that these depend on $n_1 + n_2$ only. Note that the special case $\sigma = \sigma(n_1 + n_2)$, $\tau = \tau(n_1 + n_2)$ reduces the number of arbitrary constants of integration by one. In terms of the variables a, b and h defined by

$$\begin{aligned} \sigma &= h + \varphi, \quad \Delta_1\varphi = a \\ \tau &= h - \varphi, \quad \Delta_2\varphi = b, \end{aligned} \tag{5–14}$$

the symmetry reduction considered here reads

$$h = h(n_1 + n_2), \quad a = a(n_1 + n_2), \quad b = b(n_1 + n_2).$$

The compatibility condition $\Delta_2 a = \Delta_1 b$ gives rise to the first integral

$$b = a + 2\mu, \qquad \mu = \text{const}.$$

Another first integral is given by

$$\frac{a + \mu}{x + x_1 - 2} = \text{const}$$

with the change of dependent variable

$$h = x + \frac{\mu}{2}, \tag{5–15}$$

so that

$$a = c(x + x_1 - 2) - \mu, \quad b = c(x + x_1 - 2) + \mu, \tag{5–16}$$

where c is another constant of integration. The discrete Tzitzeica system then reduces to the second-order difference equation

$$x_1 = \frac{f^1(x) - x_{\bar{1}} f^2(x)}{f^2(x) - x_{\bar{1}} f^3(x)}, \tag{5–17}$$

where the functions f^i are defined by

$$\begin{aligned} f^1 &= 4(1-c^2)x^4 + 8(3c^2-1)x^3 + \big((c^2-1)\mu^2 - 48c^2\big)x^2 \\ &\quad + 4\big((1-c^2)\mu^2 + 8c^2\big)x + 4c^2\mu^2 \\ f^2 &= 4(c^2-1)x^3 - 16c^2x^2 \\ &\quad + \big(\mu^2(1-c^2) + 16c^2\big)x + 2(c^2-1)\mu^2 \\ f^3 &= 4(1-c^2)x^2 + 8(1+c^2)x + (c^2-1)\mu^2. \end{aligned} \tag{5–18}$$

It has been shown in [Quispel et al. 1988; 1989] that second-order difference equations of the form (5–17) admit a first integral if the vectors

$$\boldsymbol{f} = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

obey the condition

$$\boldsymbol{f} = (\boldsymbol{A}\boldsymbol{x}) \times (\boldsymbol{B}\boldsymbol{x}), \tag{5–19}$$

where the constant matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric but otherwise arbitrary. In this case, the first integral is biquadratic and symmetric in x and x_1. It may be parametrized in terms of elliptic functions with the module being the remaining constant of integration. Proto-typical examples for integrable (differential)-difference equations which admit symmetry reductions to these integrable mappings include the (modified) Korteweg–de Vries, the nonlinear Schrödinger and the Heisenberg spin equations.

It turns out that there exist matrices $\boldsymbol{A}, \boldsymbol{B}$ such that the functions f^i given by (5–18) indeed satisfy the condition (5–19). As a consequence:

Theorem 5.3. *If the coefficients of the discrete Gauß equations for discrete definite affine spheres depend only on $n_1 + n_2$, the associated Gauß–Codazzi equations reduce to the integrable symmetric mapping*

$$K(\nu^2 + 4xx_1)(x + x_1 - 2) = c^2(x + x_1 - 2)^2 - (x + x_1)^2, \tag{5–20}$$

where K is an arbitrary constant of integration. The potentials σ and τ may be retrieved on use of the change of variables (5–14), (5–15)–(5–16).

6. PERSPECTIVES

Apart from the important geometric problem of classifying compact discrete affine spheres which may lead to an analogue of Blaschke's classical result (see Theorem 4.6), there are open questions concerning the integrability of the discrete definite affine spheres investigated in the previous section. It has been pointed out that in the case of discrete indefinite affine spheres a Lax representation of the discrete Tzitzeica system is available which may be interpreted in terms of loop groups. Furthermore, the Bäcklund transformation set down in [Bobenko and Schief 1999] obeys the usual 'tangency condition', that is the conormal $\boldsymbol{\nu}$ and its Bäcklund transform $\boldsymbol{\nu}'$ are parallel. In other words, the 'tangent planes' defined by the vertices $\boldsymbol{r}$, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_{\bar{1}}$, $\boldsymbol{r}_{\bar{2}}$ and $\boldsymbol{r}'$, $\boldsymbol{r}'_1$, $\boldsymbol{r}'_2$, $\boldsymbol{r}'_{\bar{1}}$, $\boldsymbol{r}'_{\bar{2}}$ coincide. In the continuous case, the tangency condition is satisfied for definite and indefinite affine spheres. However, in the case of discrete definite affine spheres, the conormals are associated with the planar quadrilaterals and not the vertices so that it is not immediately clear how tangent planes and an associated Bäcklund transformation should be defined.

Alternatively, one could try to inject a 'spectral' parameter into the Gauß equations for discrete definite affine spheres and find a corresponding interpretation in terms of loop groups. A Bäcklund transformation may then be derived by means of a Darboux matrix. A first step in this direction has already been taken. Thus, it turns out that in the one-dimensional case the difference equation (5–20) admits a simple Lie point symmetry which gives rise to an arbitrary parameter in the Gauß equations (5–1). Remarkably, this parameter coincides with the one introduced in (2–17) in the continuum limit. However, an analogous symmetry in the generic case is yet to be found.

Finally, we hope that future application of recently developed tests for integrability, such as the singularity confinement test of [Grammaticos et al. 1991], will inform us further about the integrable nature of the discrete Tzitzeica equation (5–13).

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ELECTRONIC AVAILABILITY

Examples of compact discrete affine spheres in Java and VRML formats, including the non-Archimedean polyhedra mentioned at the end of Section 4C, can be found at http://www-sfb288.math.tu-berlin.de/~bobenko/.

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