Galois Representations, Hecke Operators, and the mod-p Cohomology of $\mathrm{GL}(3, \mathbb{Z})$ with Twisted Coefficients

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CONTENTS

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We compute the degree 3 homology of $\mathrm{GL}(3, \mathbb{Z})$ with coefficients in the module of homogeneous polynomials in three variables of degree g over $\mathbb{F}_p$, for $g \leq 200$ and $p \leq 541$. The homology has a "boundary part" and a "quasicuspidal" part which we determine.

By conjecture a Hecke eigenclass in the homology has an attached Galois representation into $\mathrm{GL}(3, \bar{\mathbb{F}}_p)$. The conjecture is proved for the boundary part and explored experimentally for the quasicuspidal part.

1. INTRODUCTION

The aim of this project is to study a conjecture on the existence of "reciprocity" between n-dimensional mod p Galois representations and the mod p cohomology of subgroups of finite index in the linear group $\mathrm{GL}(n, \mathbb{Z})$. Before recalling the exact statement of the conjecture, which appears as conjecture B in [Ash 1992], it may be helpful to explain some of the background and motivation for it.

Let $G_{\mathbb{Q}}$ denote the Galois group of an algebraic closure of $\mathbb{Q}$ over $\mathbb{Q}$. If we are given a ring R and a homomorphism $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, R)$ we call ρ an n-dimensional Galois representation over R. We assume that ρ is unramified outside a finite set S of primes. This means that the fixed field of the kernel K of ρ is unramified outside S . Thus for any l not in S, a Frobenius element $\mathrm{Frob}_l \in G_{\mathbb{Q}} \mid K$ is defined up to conjugacy. We thus obtain a set of characteristic polynomials, $\{\det(I - \rho(\mathrm{Frob}_l)^{-1}X)\}$, indexed by the primes $l \notin S$. In the cases we treat

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in the body of the paper, this indexed set of polynomials determines ρ up to semisimplification.

Understanding the set of Galois representations that can arise in a given situation, say as the Tate modules of elliptic curves, can give strong control over that situation. (Wiles' proof of Fermat's Last Theorem is a case in point.) For this to be useful, we need to know about the action of Frobenius at the unramified primes and about the action of the inertia subgroup of the Galois group at the ramified primes. The behavior at the ramified primes is of great interest, but unfortunately it is beyond the methods of this paper.

Ideally we would like to be able to parametrize in some way all Galois representations and give the characteristic polynomials of Frobenius in terms of the parameters. That seems to be too hard. Instead we ask about certain families of Galois representations, and whether they and their characteristic polynomials of Frobenius can be "predicted" from some other objects, like automorphic forms. Such a set-up can loosely be called a "reciprocity law". That is because the classical law of quadratic reciprocity can be interpreted as a correspondence between Galois representations ρ where the image has order 2 (i.e., Galois groups of quadratic extensions of $\mathbb{Q}$) and Dirichlet characters χ of order 2. The reciprocity law says that $\rho(\mathrm{Frob}_l) = \chi(l)$.

Suppose we have a set T of mathematical objects such as Dirichlet characters, or classical modular newforms, or (in the case of this paper) certain group cohomology classes. For each element t of T suppose that we are given a method to compute a sequence of polynomials P_l of degree n, one for each prime l outside a finite set $S(t)$. Finally, suppose that for each t there exists an n-dimensional Galois representation ρ_t unramified outside $S(t)$ such that the characteristic polynomial of $\rho_t(\mathrm{Frob}_l)$ equals P_l, for each l outside $S(t)$. We can call this situation a reciprocity law, and say that ρ_t is "associated to" t.

We can think of T as controlling the Galois representations, or the other way around, depending on which we have more information about.

Now let $T(n)$ be the set of Hecke eigenclasses in the (co)homology of a subgroup of finite index in $\mathrm{GL}(n,\mathbb{Z})$ with mod p coefficients. When $n = 1$, there is a reciprocity law for $T(1)$ in the sense above given by class field theory for $\mathbb{Q}$. When $n = 2$, through the intermediary of classical modular forms and theorems of Eichler–Shimura and Deligne, one gets a reciprocity law for $T(2)$. These two cases are dealt with in [Ash 1997].

This paper explores the conjecture that $T(3)$ has a reciprocity law. We wish to emphasize that unlike the cases mentioned in the previous paragraph, all of the mod p homology classes computed below lift to *torsion* homology in the corresponding integral homology group and do not correspond in any immediate fashion to automorphic homology classes. Compare Proposition 3.5.1 (2) in [Ash and Stevens 1986a] and its proof, where we verify this assertion for certain types of symmetric square lifts. For the classes we compute here, the assertion follows immediately from the computations.

It is an open question given one of our torsion Hecke eigenclasses x in the homology of $\mathrm{GL}(3,\mathbb{Z})$ whether there always exists a *nontorsion* integral homology eigenclass y on a proper congruence subgroup of $\mathrm{GL}(3,\mathbb{Z})$ which has almost all its Hecke eigenvalues congruent mod p to those of x. If that were so, one could tensor y with $\mathbb{C}$ and obtain an autormorphic cohomology class that would have the same Galois representation attached as x does. If this always happens, we could say that all of the Galois representations predicted by the conjecture below are reductions mod p of representations over a ring of characteristic 0. Even so, our methods give the only way known at present for studying them computationally. On the other hand, if it doesn't always happen, we would have some interesting Galois representations that were somehow "intrinsically" characteristic p objects.

From the perspective of the preceding paragraph, one can say that our conjecture is a mod p version of the corresponding part of the so-called "Langlands' Program".

We now introduce some notation and formulate explicitly the conjecture we are testing in this paper. Let p be a prime number and $\mathbb{F}$ a finite field of characteristic p. Let n be a positive integer. Let Γ be the group $\mathrm{GL}(n,\mathbb{Z})$ and S the semigroup of the rational integral matrices in $\mathrm{GL}(n,\mathbb{Z}_p)$.

The Hecke algebra of double cosets $\Gamma S\Gamma$ will be denoted by $\mathbb{H}$. Note that $\mathbb{H}$ depends on p, which we suppress from the notation. In particular, it contains all double cosets of the form $\Gamma D(l,k)\Gamma$, where $D(l,k)$ is the diagonal matrix with k l's followed by $(n-k)$ 1's, where l is any prime other than p. We let $T(l,k)$ denote the corresponding double coset. In fact $\mathbb{H}$ is generated by the $T(l,k)$ for all $k=0,\dots,n$ and all $l\neq p$.

Suppose V is a finite dimensional right $M_n(\mathbb{F})$-module on which we let the elements of S with positive determinant act via their reductions mod p. Then, in the terminology of [Ash 1992], V is an admissible module. There is an action of $\mathbb{H}$ on the cohomology and homology of Γ with coefficients in V. When we view a double coset as acting on (co)homology, we call it a Hecke operator. This action is defined for example in [Ash and Stevens 1986a], where some of its basic functorial properties are also established. Below we give an explict formula for this action in Section 9, which deals with the computation of the Hecke operators.

Conjecture B in [Ash 1992] was stated in terms of the group cohomology of Γ. It is easier for us to compute homology. When Γ possesses torsion elements of order p, our computational methods yield not the homology of Γ but the Γ-invariants in the homology of a torsion-free subgroup of Γ. Therefore, we make the following definition:

Definition 1.1. $H_*^\dagger(\Gamma,V) = H_*(\Gamma(p^2),V)^\Gamma$, where $\Gamma(p^2)$ denotes the principal congruence subgroup of level p^2 (which is torsion-free) and the super Γ denotes invariants. The right hand side is invariant under $\mathbb{H}$ and we consider the left hand side as a Hecke module under this action.

If we keep p away from a finite set of primes (those which can be orders of elements in Γ) we could state our results directly in terms of the usual (non-daggered) homology of Γ. In general, if v is the virtual cohomological dimension of Γ, then we shall see that $H_v^\dagger(\Gamma,V)\approx H_v(\Delta,V)^\Gamma$ for any torsion-free normal subgroup Δ of finite index in Γ.

We now state the conjecture we study in this paper. Since homology is the dual of cohomology as Hecke modules, the conjecture below is easily implied by Conjecture B in [Ash 1992].

Conjecture 1.2. *Suppose $\beta\in H_*^\dagger(\Gamma,V)$ is an eigenclass for the action of $\mathbb{H}$, so that $T(l,k)\beta=a(l,k)\beta$ for some $a(l,k)\in\mathbb{F}$ for all $k=0,\dots,n$ and all l prime to p.*

Then there exists a continuous semisimple representation $\rho: G_{\mathbb{Q}}\to \mathrm{GL}(n,\mathbb{F})$ unramified outside p such that

$$\sum_k (-1)^k l^{k(k-1)/2} a(l,k) X^k = \det(I-\rho(\mathrm{Frob}_l)^{-1}X)$$

for all l not dividing p.

In [Ash 1992] Conjecture B was stated only for $n>1$. The analogous statement can be made for $n=1$ and in fact can be proved using class field theory in that case [Ash 1997]. The last cited paper also proves the conjecture when $n=2$, where it follows from known facts about classical modular forms and Galois representations.

The computations reported upon in this paper are designed to test this conjecture.

This project continues the research in [Ash and McConnell 1992] where numerical evidence for an analogous conjecture was given in the case where V is the trivial module, and Γ is replaced by a subgroup of finite index in Γ. In this paper, because Γ has level 1, the conjecturally associated Galois representations are unramified at all but one of the finite primes.

More specifically: in this paper we have gathered evidence to support Conjecture 1.2 when $n=3$ and V is the module V_g of homogeneous polynomials in three variables (x,y,z) of degree g over $\mathbb{F}_p$. We computed $H_3^\dagger(\Gamma,V_g)$ for $g\leq 200$ and $p\leq 541$, except for $p=2$, where we only computed $g\leq 100$.

The theory behind the computation of the group homology is recalled in Section 2. The computer implementation is described in Section 7. The dimensions of $H_3^\dagger(\Gamma, V_g)$ are tabulated on pages 381 to 386.

There is a filtration on the homology that takes into account the contributions from the homology of the boundary of the Borel–Serre compactification of the locally symmetric space we use to compute the group homology. We call this contribution the "boundary" homology. The rest of the homology is called the "interior" or "quasicuspidal" homology. This filtration and its computational ramifications are discussed in Sections 2, 3, and 7 and is reflected in the descriptions and tables in Section 8. We sketch out a proof that Conjecture 1.2 is true for the boundary homology: see Theorem 3.1.

We computed the action of the Hecke operators $T(l,1)$ and $T(l,2)$ for small l on some of the homology groups. (It would have taken too long to do this for all the homology that we found.) These computations are described in Sections 3 and 9 and the results appear in Table 11.

The technique we used for studying the boundary homology and the Hecke operators involves modular symbols with coefficients. These are described in Section 3.

All of our computations have built-in consistency checks described in Sections 7 and 9, making it very unlikely that there are any errors in them. There may be scribal errors in the tables, but we have striven to eliminate them.

In Section 4, we prove that when $n = 3$ and $p \geq 5$, if Conjecture 1.2 holds for all the modules

$$V_g, \quad \text{for } g \leq p^2 + p - 2,$$

then it holds for *all* V. This enables us to prove the conjecture for $n = 3$ and $p = 5$ or 7. We also have strong evidence that Conjecture 1.2 is true for $n = 3$, $p = 11$. We discuss these cases of small p in Section 6.

The mod p betti numbers

$$d_p(g) = \dim_{\mathbb{F}_p} H_3(\mathrm{GL}(3,\mathbb{Z}), V_g(\mathbb{F}_p))$$

are found to satisfy

$$d_p(g) \leq d_p(g + (p-1))$$

most of the time. We don't have a precise understanding of this, but it is clearly related to the fact that if an irreducible $\mathrm{GL}(3,\mathbb{Z}/p)$-module is a constituent of $V_g(\mathbb{F}_p)$ for some g, then in most cases it is so for$V_{g+p-1}(\mathbb{F}_p)$ as well. This is touched upon further in Section 4.

Since our conjecture is true for the boundary homology, we concentrated our testing for Galois representations on the quasicuspidal homology. For those classes for which we computed some of the Hecke operators, we attempted to test Conjecture 1.2 by finding Galois representations that appear to be predicted by the Hecke data. In most cases, we showed that if the desired Galois representation ρ exists, it must have very large image in $\mathrm{GL}(3,\mathbb{F}_p)$. In such a case, the fixed field of the kernel of ρ is so large that there is no known way of finding it. On the other hand, in the few cases where the image of ρ could be predicted to be small, we were able in each case to find a Galois representation that satisfies Conjecture 1.2 for as many Hecke-eigenvalues as we chose to compute.

There is an intermediate sort of case, where one has the analytic tool of the "symmetric square" lifting from automorphic representations on $\mathrm{GL}(2)$ to automorphic representations on $\mathrm{GL}(3)$. Here the associated Galois representation ρ into $\mathrm{GL}(3,\mathbb{F}_p)$ is the composition of a Galois representation σ into $\mathrm{GL}(2,\mathbb{F}_p)$ composed with the adjoint map from $\mathrm{GL}(2,\mathbb{F}_p)$ to $\mathrm{GL}(3,\mathbb{F}_p)$ (up to a twist). In these cases, ρ is generally irreducible, without having "big" image. Since σ is attached to a classical modular form, we can verify our conjecture in these cases empirically.

For certain "symmetric squares" cases, in particular those that appeared in our computations (when $p = 29, 37, 41$), one can actually prove the conjecture. The details will appear in a paper by the second author and P.H. Tiep. We will discuss this further in Section 5, where we discuss all the Galois related issues.

2. HOMOLOGY

In this section and the next Δ is a torsion-free normal subgroup of finite index in $\Gamma = \mathrm{GL}(n, \mathbb{Z})$. We let X denote the symmetric space of positive-definite n-by-n symmetric matrices modulo homotheties, on which $\mathrm{GL}(n, \mathbb{R})$ acts on the right via $y \mapsto {}^t g y g$ for $y \in X$ and $g \in \mathrm{GL}(n, \mathbb{R})$. Let W denote the well-rounded retract [Ash 1980] and $\bar{X}$ the Borel–Serre bordification of X [Borel and Serre 1973].

Recall that W is a subspace of X of dimension $n(n-1)/2$. It is a Γ-invariant deformation retract of X (hence contractible) of minimal possible dimension, since the virtual cohomological dimension of Γ is $n(n-1)/2$. It comes equipped with the structure of cell complex, with a finite number of Γ-orbits of cells. Thus the following are naturally isomorphic (with any coefficient module): the group (co)homology $H(\Delta)$, the topological (co)homology $H(X/\Delta)$ and the cell-complex (co)homology $H(W/\Delta)$. We use the latter for our computations.

Here is a description of $\bar{X}$: For each parabolic subgroup P of $\mathrm{GL}(n)$ defined over $\mathbb{Q}$, there is a "face" $e(P)$ which is a contractible space on which $P(\mathbb{R})$ acts. For the improper parabolic subgroup $\mathrm{GL}(n)$, the face is X. Then $\bar{X}$ is the disjoint union of the faces. The action of Γ on X extends naturally to $\bar{X}$. For any subgroup Γ' of Γ, the quotient X/Γ' is the disjoint union of faces $e'(P) = e(P)/P \cap \Gamma$. More information on the structure of $e'(P)$ will be recalled in Section 3.

Since Δ is assumed to be torsion-free, X/Δ is a manifold, the interior of the manifold with boundary $M = \bar{X}/\Delta$. We will later be using Lefschetz duality for M. These manifolds have dimension $\frac{1}{2}n(n+1)-1$. Also, ∂M is a manifold satisfying Poincaré duality.

As stated above, X/Δ, W/Δ, and M are all homotopy equivalent. If V is any $\mathbb{F}_p[\Gamma]$-module and Y is any one of these spaces with local coefficients induced by V, then $H_*(\Delta, V)$ is naturally isomorphic to $H_*(Y, V)$.

For the following theorem, we specialize to $n=3$.

Theorem 2.1. *Let p be prime and let ρ be an arbitrary right representation of $\Gamma = \mathrm{GL}(3, \mathbb{Z})$ on a finite dimensional vector space V over $\mathbb{F}_p$. Then $H_3^{\dagger}(\Gamma, V)$ is isomorphic to the subspace consisting of all $v \in V$ such that the following conditions are satisfied*:

(i) $v \cdot (1-\rho(d)) = 0$ *for all diagonal matrices d in* Γ.
(ii) $v \cdot (1+\rho(z)) = v \cdot (1+\rho(w)) = 0$.
(iii) $v \cdot (1+\rho(h)+\rho(h^2)) = 0$.

Here the diagonal matrices are matrices of the form

$$d = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

The matrices z, w, and h are matrices of order 2, 2 and 3, respectively, given by

$$z = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$h = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. We use the explicit description of W as Γ-cell-complex in [Ash 1980]. The theorem easily follows from this and the following lemma:

Lemma 2.2. *Let A be a contractible cell complex of dimension d on which a group G acts on the right with finite stabilizers. Let W be a right $\mathbb{F}[G]$-module. Write (C_*, ∂) for the chain complex of oriented chains on A with coefficients in W. Assume that G possesses a torsion-free normal subgroup H of finite index. Then $H_d(H, W)^G$ is equal to the kernel of ∂ on C_d^G. In particular, it is independent of H.*

Proof. For each cell s of A, let $G_s = \mathrm{stab}_G(s)$ and let $\chi_s(g)$ denote the orientation character of G_s. Then C_s is the induced module $\mathrm{Ind}(G_s, G, W \otimes \chi_s)$ and $C_i = \bigoplus_{\dim s = i} C_s$. We have the boundary maps in A inducing $\partial : C_i \to C_{i-1}$.

We recall the definition of the induced module: an element of C_s is a function $f : G \to W$ such that $f(bg) = \chi_s(b) f(g) b^{-1}$ for $b \in G_s$ and $g \in G$. An element $k \in G$ acts on f by $(fk)(g) = f(gk^{-1})$.

Now the invariants $C_s^H \approx \mathrm{Ind}(G_s, G/H, W \otimes \chi_s)$ and C_i^H can be identified with $C_i(A) \otimes_H W =$ the W-valued i-chains on A/H by

$$\Phi : \bigoplus_{\dim s = i} f_s \mapsto \sum_s \sum_{g \in G_s \backslash G/H} sg \otimes f(g) g.$$

Φ is well-defined, compatible with the G-action and with ∂.

Since H is torsion-free, it acts freely on A, so the homology of C_*^H is naturally isomorphic to $H_*(H, W)$. One easily checks that the G-action on C_*^H commutes with ∂ and induces the usual G-action on $H_*(H, W)$.

It follows that $H_*(H, W)^G \approx (H_*(C_*^H))^G$. Of course taking the invariants by G will not yield an exact functor if p divides $[G : H]$. But it is left exact, so in the top dimension d we obtain the statement of the lemma, since $(C_d^H)^G = C_d^G$. This proves the lemma and the theorem. □

For later use, we also quote here the GL(2) version of Theorem 2.1. See again [Ash 1980] for the description of W in this case. Of course, for GL(2) this construction has been known for a long time, and is closely related to the theory of Eichler–Shimura cohomology for classical congruence subgroups.

Theorem 2.3. *Let p be prime and let ρ be an arbitrary right representation of $\Gamma_2 = \mathrm{GL}(2, \mathbb{Z})$ on a finite dimensional vector space U over $\mathbb{F}_p$. Then $H_1^\dagger(\Gamma_2, U)$ is isomorphic to the subspace consisting of all $u \in U$ satisfying these conditions*:

(i) $v \cdot (1 - \rho(d)) = 0$ *for all diagonal matrices d.*
(ii) $v \cdot (1 + \rho(z_2)) = 0$.
(iii) $v \cdot (1 + \rho(h_2) + \rho(h_2^2)) = 0$.

Here the diagonal matrices are matrices of the form

$$d = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

The matrices z_2 and h_2 are given by

$$z_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

3. MODULAR SYMBOLS WITH COEFFICIENTS AND BOUNDARY HOMOLOGY

In this section we let M be the compactification of the locally symmetric space for $\Delta \subset \Gamma = \mathrm{GL}(n, \mathbb{Z})$, as in the previous section. Let $N = \frac{1}{2} n(n-1)$ denote the cohomological dimension of Δ. Dual to the cycles in $H_N(M, V)$ we have minimal modular symbols with coefficients in V^*, where V^* is the dual vector space of V given an S-module structure by $(v^* s)(v) = v^*(v s^{-1})$.

Recall that a minimal modular symbol is the image in M of a $\mathrm{GL}(n, \mathbb{Q})$-translate of the diagonal matrices. More precisely, take the set of diagonal matrices D in the space X of positive definite symmetric n-by-n matrices modulo homotheties, and consider Dg for any $g \in \mathrm{GL}(n, \mathbb{Q})$. We denote projection to M of the closure in $\bar{X}$ of Dg by the symbol $[g]$. We fix once for all an orientation on $[I]$ (where I is the n-by-n identity matrix) and give $[g]$ the orientation induced from the fixed one by translation by g.

Since $[g]$ is contractible, a section of the local coefficient system coming from V^* over it is just given by an element v^* of V^*. We shall use the symbol $[g, v^*]$ to stand for the fundamental class of this relative cycle with coefficient v^* attached. Thus $[g, v^*]$ is an element in $H_{n-1}(M, \partial M, V^*)$.

The boundary of $[g, v^*]$ is an $(n-1)$-sphere with a simplicial structure each of whose $n-1$ simplices has the coefficient v^* attached. Each simplex lies in one of the faces of ∂M.

In [Ash and Rudolph 1979] it was proved that $H_{n-1}(M, \partial M, \mathbb{Z})$ is spanned by modular symbols with trivial integer coefficients. The argument is easily adapted to show that

$$\{[g, v^*] : g \in \mathrm{GL}(n, \mathbb{Z}) \text{ and } v^* \in V^*\}$$

spans $H_{n-1}(M, \partial M, V^*)$. Since M is the quotient of $\bar{X}$ by Δ, we see that we can take as a spanning set $\{[\gamma, v^*] : \gamma \in B, v^* \in V^*\}$, where B is a set of representatives for the cosets of Δ in Γ.

From now on we set $n = 3$. Then the boundary of a modular symbol is a hexagon, and the edges lie alternately in faces conjugate to $e'(P)$ and $e'(Q)$, where P and Q are the stabilizers of a line and plane respectively.

Moreover Lefschetz duality gives a perfect pairing

$$H_3(M, V) \times H_2(M, \partial M, V^*) \to \mathbb{F}.$$

If an element of $H_3(M, V) = H_3(\Delta, V)$ is invariant under Γ, it is uniquely determined by its images under the pairing with modular symbols in the set $\{[I, v^*] : v^* \in V^*\}$, where I denotes the identity matrix.

The formula for this pairing is easily found by adapting the proof of Proposition 3.24 of [Ash et al. 1984] to the case of twisted coefficients. If $v \in V$ represents a Γ-invariant 3-cycle as in Theorem 2.1, we have

$$\langle v, [I, v^*] \rangle = v^*(v). \tag{3–1}$$

Let there be given $g \in \mathrm{GL}(3, \mathbb{Q})$ and a row vector $w \in \mathbb{Q}^3$. Denote by g_i the matrix obtained from g by replacing its i-th row with w. Then from [Ash and Rudolph 1979] we know that $[g]$ is homologous to $\sum [g_i]$, by a homology given by a contractible simplex. It follows that for any coefficient v^* we have that $[g, v^*] = \sum [g_i, v^*]$. We also record the following formula for future use:

$$[g, v^*]\gamma = [g\gamma, v^*\gamma] \qquad \text{if } g \in \mathrm{GL}(3, \mathbb{Q}) \text{ and } \gamma \in \Gamma.$$

Next we want to discuss the image of $H_r(\partial M, V)^\Gamma$ in $H_r(M, V)^\Gamma \approx H_r^\dagger(\Gamma, V)$. We will call the image the "boundary homology" and the quotient the "quasicuspidal homology", in conformity with our usage in [Ash and McConnell 1992]. The boundary homology is a Hecke equivariant subspace of the whole homology, so that the quasicuspidal homology is itself a Hecke module.

Theorem 3.1. *Let α be a Hecke-eigenclass in the boundary homology. Then Conjecture* 1.2 *holds for α and the corresponding Galois representation is reducible.*

Proof. A theorem analogous to this one for cohomology with coefficients in a finite dimensional vector space over the complex numbers is proved as Proposition 3.2.1 of [Ash and Stevens 1986a]. We shall sketch here a modification of that argument that works for homology with an admissible mod p coefficient module. Since homology is dual to cohomology, it suffices, and is more convenient, to prove that any Hecke eigenclass in the cohomology of the boundary satisfies Conjecture 1.2.

First we invoke Theorem 3.1 of [Ash 1992] which states that any system of Hecke eigenvalues occurring in cohomology of a congruence subgroup Δ of Γ of level N and with coefficients in an admissible module V also occurs in the cohomology of the principal congruence subgroup of Γ of level N with coefficients in $\mathbb{F}(\psi)$. Here $\psi : (\mathbb{Z}/N)^\times \to \mathbb{F}^\times$ is a character, and $\mathbb{F}(\psi)$ denotes the one dimensional $\mathbb{F}$-module on which S acts via $\det \circ \psi$. In Section 4 we show that Conjecture 1.2 is stable under twisting. That is, using the notation of the conjecture, if ρ is attached to $\beta \in H_*(\Delta, V)$ then $\rho \otimes \psi$ is attached to $\beta \in H_*(\Delta, V \otimes \mathbb{F}(\psi))$. Here we use ψ also to denote the corresponding character of the Galois group via class field theory for $\mathbb{Q}$.

In this way we reduce to the case of trivial coefficients. It is now convenient to use adelic methods. Let $\mathbb{A}$ denote the adeles of $\mathbb{Q}$, $\mathbb{A}_f$ the finite adeles. We will replace M by a larger space which is a finite disjoint union of spaces like M, and then take a limit as Δ shrinks. If the conjecture holds for the boundary homology of this limit, it certainly holds for the boundary homology of the original M, which injects Hecke equivariantly into the limit.

We imitate the notation of [Ash 1988]. Let $G = \mathrm{GL}(3)$, $Z =$ the center of G, $K_\infty = SO(3)$. For any compact open subgroup K of $G(\mathbb{A}_f)$ let $\tilde{K} = Z(\mathbb{R})K_\infty K$. Set $X(K) = G(\mathbb{Q})\backslash G(\mathbb{A})/\tilde{K}$. We assume that K is chosen "deep" enough so that $X(K)$ is an orientable manifold with boundary. Denote by $Y(K)$ the Borel–Serre compactification of $X(K)$.

As we vary K we get directed systems of cohomology groups $H^*(Y(K),\mathbb{F})$ and $H^*(\partial Y(K),\mathbb{F})$. We write their limits symbolically as $H^*(Y)$ and $H^*(\partial Y)$, respectively. These are $G(\mathbb{A}_f)$-modules, and their K-invariants give back $H^*(Y(K),\mathbb{F})$ and $H^*(\partial Y(K),\mathbb{F})$. Moreover, one can compute the Hecke operator $T(l,k)$ on $H^*(Y)$ and $H^*(\partial Y)$ by averaging the action of the double coset $KD(l,k)K$. These operators restricted to the K-invariants give back the usual Hecke operators as defined on cohomology. See Section 3.1 of [Ash and Stevens 1986a] for more details.

The topological structure of the boundary $\partial Y(K)$ is recalled in Section 3.2 of [Ash and Stevens 1986a]. It is shown there that there exists a Mayer–Vietoris sequence for $H^*(\partial Y(K),\mathbb{F})$ each term of which has the form of a direct sum of $H^r(e,\mathbb{F})$'s, where e runs over the Borel–Serre faces of $Y(K)$ and r is 0, 1, 2, or 3. Each face $e = e'(P)$ is a fibration whose base is a locally symmetric space $Y_L(K_L)$ corresponding to the Levi-component L of the parabolic subgroup P of $\mathrm{GL}(3)$, and whose fiber is a nilmanifold $N_U(K_U)$ coming from the unipotent radical U of P. Here we denote $K_A = K \cap A$ for $A = L$ or U. We can take the limit of the cohomology of $Y_L(K_L)$ and $N_U(K_U)$ as K varies and denote it by $H^*(Y_L)$ and $H^*(N_U)$ resepectively.

The Mayer–Vietoris sequence and the Serre spectral sequences of the fibrations all have natural Hecke actions making the various homomorphisms and differentials Hecke equivariant. We then refer (twice) to the proof of Lemma 2.1 of [Ash and Stevens 1986a], which shows that any system of Hecke eigenvalues occurring in the middle term of an exact sequence of Hecke modules $A \to B \to C$ also occurs in one of the other two terms. It follows that any system of Hecke eigenvalues occurring in the cohomology of the boundary occurs first in one of the $H^r(e,\mathbb{F})$'s, and then in one of the terms $E_2^{p,q}$ of the E_2 page of the corresponding Serre spectral sequence.

In this way, we are reduced to showing the following: for any $\mathbb{Q}$-parabolic subgroup P of G, for any integers p and q, and for any K-invariant Hecke eigenclass in the induced module

$$\mathrm{Ind}\big(P(\mathbb{A}_f),\, G(\mathbb{A}_f),\, H^p(Y_L, H^q(N_U))\big)$$

there is an attached Galois representation as in Conjecture 1.2.

We will sketch how to do this when P is the stabilizer of a line, and leave the remaining cases to the reader. The Levi subgroup L of P can be written as $L' \times T$ where $L' \approx \mathrm{GL}(2)$ and $T \approx \mathrm{GL}(1)$. Then $H^q(N_U)$ as $T(\mathbb{A}_f)$-module decomposes into a sum of character spaces W_χ, where χ runs through $\bar{\mathbb{F}}^\times$-valued characters of $T(\mathbb{A}_f)$ of finite order. Thus our system of Hecke eigenvalues occurs on a K-invariant element in a Hecke module of the form $\mathrm{Ind}\big(P(\mathbb{A}_f), G(\mathbb{A}_f), H^p(Y'_L, W_\chi)\big)$. Here $H^p(Y'_L, W_\chi)$ denotes the limit as K varies of cohomology groups for the symmetric space of L'.

We can finish the argument exactly as on pages 213–214 of [Ash and Stevens 1986a] if we know that every Hecke eigenclass (with respect to the Hecke algebra for L') in $H^p(Y'_L(K'), W_\chi)$ has an attached Galois representation, where K' runs over compact open subgroups of $L'(\mathbb{A}_f)$. Now the main result of [Ash 1997] is that Conjecture 1.2 holds for $\mathrm{GL}(1)$ and $\mathrm{GL}(2)$. We apply this to $L' \approx \mathrm{GL}(2)$, noting that W_χ is an admissible module.

When doing the case of a minimal parabolic subgroup, we need to use the fact that Conjecture 1.2 holds for $\mathrm{GL}(1)$. □

It follows from this theorem that (when $n = 3$) Conjecture 1.2 will hold for all Hecke eigenclasses in $H_r^\dagger(\Gamma, V)$ for a given r and V if it can be shown to hold for any part of the homology big enough to map onto the quasicuspidal homology. Therefore, our experimental testing of Hecke eigenclasses was mostly concerned with quasicuspidal classes.

We shall now derive formulas for some boundary homology classes when $n = 3$ and $r = 3$. We believe they span the boundary homology in this dimension, but have not proved this. A proof along the lines of [Lee and Schwermer 1982] should be possible.

We note that all the morphisms of (co)homology and all the dualities used below are equivariant for the actions of the relevant Hecke algebras.

We have the long exact cohomology sequence of the pair $(M, \partial M)$ which gives us a connecting morphism $\delta : H^1(\partial M, V) \to H^2(M, \partial M, V)$. If δ is followed by the Lefschetz duality isomorphism $L : H^2(M, \partial M, V) \to H_3(M, V)$ then the image is the boundary homology. (Apply Poincaré duality to ∂M which has dimension 4.) Moreover, it's not hard to see that the image of a given class ω in $H^1(\partial M, V)^\Gamma$ can be determined by evaluating ω on the boundaries of the standard identity modular symbols $[I, v^*]$ with coefficients running through the dual module V^*.

So we need to compute $H^1(\partial M, V)^\Gamma$. Following [Lee and Schwermer 1982] and using the notation of [Borel and Serre 1973], one sees that $\partial M/\Gamma$ is the union of $e'(P)$ and $e'(Q)$ over their intersection $e'(B)$. Here P denotes the stabilizer of the line ${}^t(0, 0, *)$ in GL(3), Q denotes the stabilizer of the line $(*, 0, 0)$, and $B = P \cap Q$ is the stabilizer of a flag. For a parabolic subgroup R, $e'(R)$ denotes the corresponding face of ∂M. Taking Γ-invariants in the Meyer–Vietoris sequence gives maps

$$H^0(e'(B), V) \to H^1(\partial M, V)^\Gamma \to H^1(e'(P), V) \oplus H^1(e'(Q), V).$$

If p is prime to the index of Δ in Γ, this sequence is exact.

We can classify the boundary homology into three types, depending on whether a given class comes from $H^0(e'(B), V)$ or from $H^1(e'(P), V)$ or from $H^1(e'(Q), V)$. Set

$$L_P = P \cap {}^tP, \qquad L_Q = Q \cap {}^tQ,$$

and let U_P and U_Q be the unipotent radicals of P of Q, so $P = L_P U_P$, $Q = L_Q U_Q$. Let W be the permutation group of order 6 in Γ and

$$A = \sum_{\sigma \in W} \operatorname{sgn}(\sigma)\sigma.$$

To state the next theorem, we need one ad hoc definition: Let L denote either L_P or L_Q. Then L is isomorphic in an obvious way to $\mathrm{GL}(2) \times \mathrm{GL}(1)$. Let φ denote the obvious embedding of GL(2) into L. We shall say that an element v of an L-module satisfies equations (i)–(iii) of Theorem 2.3 under L if they hold true when the matrices d, z_2 and h_2 are replaced by their images under φ. We omit the proof of the following theorem.

Theorem 3.2. *The boundary homology in $H_3(\Gamma, V)$ contains the following subsets of V, after identifying the homology with a subspace of V in accordance with Theorem* 2.1:

Type B: $\{A(v) \mid v \in V^{B\cap\Gamma}\}$;
Type P: $\{A(v) \mid v \in V^{U_P\cap\Gamma}$ *and satisfies equations* (i), (ii), (iii) *of Theorem* 2.3 *under* $L_P\}$;
Type Q: $\{A(v) \mid v \in V^{U_Q\cap\Gamma}$ *and satisfies equations* (i), (ii), (iii) *of Theorem* 2.3 *under* $L_Q\}$;

We now apply these results when $V = V_g$. We need the invariants for the action of the unipotent radicals of P, Q, and B in V_g.

Theorem 3.3. *Let B, P, and Q denote the parabolic subgroups of* GL(3) *defined above, and set* $\Gamma = \mathrm{GL}(3, \mathbb{Z})$. *Let $\mathcal{P}$ denote the polynomial ring*

$$\mathbb{F}_p[x, y, z] = \bigoplus_{g\geq 0} V_g.$$

Define $B(s,t) = \prod_{b\in\mathbb{Z}/p}(bs + t)$ *and* $C(r,s,t) = \prod_{a,b\in\mathbb{Z}/p}(ar + bs + t)$. *Then*:

(i) *The elements x, $B(x, y)$ and $B(x, z)$ generate the $(U_Q \cap \Gamma)$-invariants of $\mathcal{P}$.*
(ii) *The elements x, y, and $C(x, y, z)$ generate the $(U_P \cap \Gamma)$-invariants of $\mathcal{P}$.*
(iii) *The elements x^2, $B(x, y)^2$ and $C(x, y, z)^2$ generate the $(B \cap \Gamma)$-invariants of $\mathcal{P}$.*

Moreover, the $L_Q \cap \Gamma$ *action on the invariants in* (i) *can be described by saying that the map that sends* y *to* $B(x,y)$ *and* z *to* $B(x,z)$ *gives an isomorphism of* $L \cap \Gamma$*-modules. The* $L_P \cap \Gamma$ *action on the invariants in* (ii) *is the obvious one.*

Proof. We shall prove statement (i) and leave the others to the reader. First we show that if $k(x,y)$ is a polynomial invariant under $y \mapsto ax+y$ for all $a \in \mathbb{Z}/p$, then k is a polynomial in x and $B(x,y)$.

It suffices to prove this when k is homogeneous, say of degree d. Subtracting off the x^d term if any, we may assume y divides k. Hence $(ax+y)$ divides k for all a. Since the polynomial ring is a unique factorization domain, we conclude that $B(x,y)$ divides k. Now proceed by induction on d, applying the induction hypothesis to $k/B(x,y)$.

Now we make a similar argument for $f(x,y,z)$ invariant under $y \mapsto ax+y$ and $z \mapsto bx+z$ for all $a,b \in \mathbb{Z}/p$. We may assume f is homogeneous of degree d. Write $f(x,y,z) = f_1(x,y) + zf_2(x,y,z)$. Under $y \mapsto ax+y$, any term divisible by z remains so. Hence $f_1(x,y)$ is invariant under $y \mapsto ax+y$ for all a and so f_1 is a polynomial in x and $B(x,y)$. Hence we can subtract it off and assume z divides f. Since f is invariant under $z \mapsto bx+z$ for all b we get that $B(x,z)$ divides f and then we finish by induction as before.

Note that the reason the squares appear in statement iii of Theorem 3.3 is because $B \cap \Gamma$ includes the diagonal matrices with ± 1's down the diagonal. $\square$

Corresponding to our algorithms for computing homology, we will call type B boundary homology "type 1", the part of type P boundary homology generated by x and y alone "type 0", and the rest of type P and all of type Q boundary homology "type 2".

4. SOME REPRESENTATION THEORY

In this section we recall some facts about representations of $\mathrm{GL}(3,\mathbb{F}_p)$ over $\mathbb{F}_p$. Otherwise unreferenced assertions may be found in [Carlisle and Walker 1989] or [Doty and Walker 1992].

The simple $\mathbb{F}_p[\mathrm{GL}(3,\mathbb{F}_p)]$-modules are classified by triples of integers (a,b,c) where

$$0 \le a-b,\, b-c \le p-1 \quad \text{and} \quad 0 \le c < p-1.$$

This triple is the highest weight of the Weyl module whose unique simple submodule is the simple module in question. Twisting (i.e., tensoring) with det adds 1 to a, b, and c.

For example, the module of homogeneous polynomials in 3 variables of degree g (which we have denoted V_g and which Doty and Sullivan call S_g) is irreducible if and only if $g \le p-1$ and the classifying triple is $(g,0,0)$.

Every simple $\mathbb{F}_p[\mathrm{GL}(3,\mathbb{F}_p)]$-module W occurs as a composition factor of V_g for some $g \ge 0$.

From tables in [Doty and Walker 1992] we can compute the multiplicity $m(W,g)$ with which any given W appears as a composition factor in V_g for any given g.

Consider the series $\sum m(W,g)t^g$. It always equals a rational function of the form

$$\frac{t^{a+b+c}\sum d_i(t^{p-1})^i}{(1-t^{p-1})(1-t^{p^2-1})(1-t^{p^3-1})}, \tag{4–1}$$

where W corresponds to the triple (a,b,c) and the coefficients d_i are all 0, 1, or -1. In practice, the vast majority of the nonzero coefficients are $+1$, and the denominator causes a tendency for $m(W,g)$ to exhibit periodic behavior with periods $p-1$, p^2-1 and p^3-1. This periodicity should be compared to the p-adic deformations of the cohomology of congruence subgroups of level p as Hecke modules constructed in [Ash and Stevens 1997]. We have not as yet worked out the details.

Every simple $\mathbb{F}_p[\mathrm{GL}(3,\mathbb{F}_p)]$-module W occurs as a submodule of V_g for some g. This is proved in [Doty and Walker 1996]. In fact, if W is classified by the highest weight (a,b,c), then W is a submodule of V_d where $d = a+bp+cp^2$. Hence, any W has a twist which is a submodule of V_g for some

$$g \le 2p-2+(p-1)p+0 = p^2+p-2.$$

If W is a submodule of V_g, the long exact sequence in homology gives

$$0 = H_4(M,V_g/W) \to H_3(M,W) \to H_3(M,V_g) \to \cdots$$

since M has dimension 3.

As in Lemma 1.1.1 in [Ash and Stevens 1986a], this exact sequence is Hecke equivariant. Taking Γ-invariants gives a Hecke equivariant injective map $H_3^\dagger(M,W) \to H_3^\dagger(M,V_g)$.

We see that to check Conjecture 1.2 for $n = 3$, $* = 3$, and V any admissible module, it suffices to check it for $V = V_g$ with $g \le p^2 + p - 2$. This is because by Lemma 2.1 of [Ash and Stevens 1986a], any package of Hecke eigenvalues appearing in the homology with coefficients in V appears already in the homology with coefficients in some simple submodule W of V. From the exact sequence above, it also appears in the homology with coefficients in V_g as long as W embeds in V_g. Thus it suffices to prove that Conjecture 1.2 is stable under twisting.

To see this, let ρ be a Galois representation attached to a homology class α in $H_3(\Delta, V)$. For any nonnegative integer m let $V(m)$ denote the $M_n(\mathbb{F})$ module $V \otimes \mathbb{F}(m)$, where $\mathbb{F}(m)$ is the one-dimensional $\mathbb{F}$-module on which $M_n(\mathbb{F})$ acts via the m-th power of the determinant. Since the determinants of the elements of Δ are all 1, we can view the same homology class α as being in $H_3(\Delta, V(m))$. However the eigenvalue of $T(l,k)$ on the new class is l^k times the old eigenvalue. It's easy then to see that if ω denotes the cyclotomic character mod p, then $\rho \otimes \omega^m$ is attached to the new class.

However, when $p \ge 5$, we can prove Conjecture 1.2 for $n = 3$ and *any* $*$, and V any admissible module, as long as it holds for $n = 3$, $* = 3$, and $V = V_g$ with $g \le p^2 + p - 2$. This is stated in Theorem 4.2 below. To prove this we need some preliminary remarks.

For the next few paragraphs, we work with general n, so that $\mathbb{H}$ is the Hecke algebra for $\mathrm{GL}(n,\mathbb{Z})$. Recall that an R-valued system of $\mathbb{H}$-eigenvalues is a set function $\sigma : \mathbb{H} \to R$ for some ring R. We say σ occurs in an R-module Y if there exists an eigenvector $y \in Y$ such that $Ty = \sigma(T)y$ for every $T \in \mathbb{H}$. In particular, $y \ne 0$. We also define the dual system of eigenvalues $\sigma^\#$ as follows: If T is the double coset $\Gamma s \Gamma$, let $T^\#$ be the double coset $\Gamma(\det(s)^t s^{-1})\Gamma$. Then $\sigma^\#(T) = \sigma(T^\#)$.

We shall write $\sigma(l,k)$ instead of $\sigma(T(l,k))$. Note that if σ actually occurs in the homology of an admissible module of level p then $l \mapsto \sigma(l,n)$ defines a multiplicative character $\chi : (\mathbb{Z}/p)^\times \to \mathbb{F}^\times$. We shall also denote by χ the corresponding character $\chi : G_\mathbb{Q} \to \mathbb{F}^\times$ via class field theory. That is, $\chi(\mathrm{Frob}_l^{-1}) = \chi(l)$.

Given an $\mathbb{F}$-valued system of $\mathbb{H}$-eigenvalues σ, we say that a representation $\rho : G_\mathbb{Q} \to \mathrm{GL}(n,\mathbb{F})$ is attached to σ if the characteristic polynomials of ρ on Frobenius elements equal the corresponding Hecke polynomials as in Conjecture 1.2, with $\sigma(T(l,k))$ in place of $a(l,k)$.

Given a representation ρ attached to σ as above, we define the representation $\rho^\#$ by setting $\rho^\#(g) = \omega(g)^{(n-1)}\chi(g)\rho({}^t g^{-1})$, where ω is the cyclotomic character mod p and χ depends on σ as explained above.

Lemma 4.1. *If ρ is attached to σ, then $\rho^\#$ is attached to $\sigma^\#$.*

Proof. Note that $T(l,k)^\# = T(l,n)^{k-1}T(l,n-k)$. Therefore, $\sigma(l,k)^\# = \chi(l)^{k-1}\sigma(l,n-k)$. On the other hand, the k-th symmetric polynomial of the eigenvalues of $\rho^\#(\mathrm{Frob}_l^{-1})$ is equal to

$$\left(l^{n-1}\chi(l)\right)^k l^{-n(n-1)/2}\chi(l))^{-1}$$

times the k-th symmetric polynomial of the eigenvalues of $\rho(\mathrm{Frob}_l^{-1})$, since

$$\det \rho(\mathrm{Frob}_l^{-1}) = l^{n(n-1)/2}\sigma(l,n).$$

A simple computation from the definitions now finishes the argument. □

For any $\mathbb{F}_p$-vector space A, A^* denotes the vector space dual. If A is also a right G-module for some group G, we make G act on A^* by $(\varphi g)(a) = \varphi(ag^{-1})$. We also define the right G-module $A^\#$ by keeping the underlying space A the same and letting $g \in G$ act via the outer automorphism $g \mapsto \det(g)^t g^{-1}$. Assume A has a central character χ. Then since $A^* \otimes (\chi \circ \det)$ and $A^{\#}$ have the same Brauer character, they have the same composition factors. In particular, if A is simple, they are isomorphic.

Theorem 4.2. *Suppose $p \geq 5$ and Conjecture 1.2 holds for $H_3^\dagger(\Gamma, V_g)$ for all $g \leq p^2+p-2$. Then Conjecture 1.2 holds for $H_k^\dagger(\Gamma, V)$ for all k and all admissible coefficient modules V.*

Proof. First, by Lemma 2.1 of [Ash and Stevens 1986a] we may assume that V is simple. By [Ash 1992] we know Conjecture 1.2 holds for $k = 0, 1$. It also holds vacuously for $k > 3$ since the homology vanishes in that range. Suppose now we have an eigenclass in $H_2^\dagger(\Gamma, V)$. If it is a boundary class, the conjecture holds by Theorem 3.1. If not, it is quasicuspidal and dual to a class in $H_3^\dagger(\Gamma, V^{\#})$. By Lemma 4.1, if the conjecture holds for this dual class, then it holds for the original class. Thus we only have to worry about the case $k = 3$.

Now if σ is the system of eigenvalues attached to a given eigenclass in $H_3(\Gamma, V)$, we know that σ occurs (up to a twist) in $H_3(\Gamma, V_g)$ for some $g \leq p^2+p-2$. $\square$

In our computations below, we found that for $p \leq 7$ and $g \leq p^2+p-2$, all the homology was boundary homology. Combining this fact with theorems 3.1 and 4.2, we obtain:

Theorem 4.3. (i) *Conjecture 1.2 holds for $H_3^\dagger(\Gamma, V)$ for all $p \leq 7$ and all admissible V.*
(ii) *If $p = 5$ or 7, Conjecture 1.2 holds for $H_k^\dagger(\Gamma, V)$ for all k and all admissible V.*

Remark. For $p = 11$, we do find quasicuspidal homology for V_g with $g \leq p^2+p-2$. However as noted in Section 6 below, our results suggest strongly that each of these quasicuspidal classes does have an attached Galois representation. Thus our evidence strongly supports the assertion that the range of p in (ii) could be extended to include $p = 11$.

5. TESTING FOR THE IMAGE OF THE GALOIS REPRESENTATION

For this paper, we have computed many examples of nontrivial quasicuspidal homology classes and the action of some of the Hecke operators on them. By Conjecture 1.2, there should be attached to each Hecke eigenclass a Galois representation ρ. In only a few of the quasicuspidal cases that we computed can we prove the conjecture, as discussed below. However in all cases we can say something about the image of ρ should it exist. This is because the conjecture tells us that our computed Hecke eigenvalues can be used to give us the characteristic polynomials of various elements of the image of ρ, namely the images of Frobenius at l for small l. We then can invoke the classification of subgroups of $\mathrm{GL}(3, \mathbb{Z}/p)$ to make assertions about the possibilities for the image of ρ.

To do this we begin with a paper of H. H. Mitchell [1911]. Although this paper studies the projective special linear group, from his main result we can easily list the maximal proper subgroups up to conjugacy of $J = \mathrm{GL}(3, \mathbb{Z}/p)$ for odd p. To facilitate comparison, for each such subgroup H we indicate the order of

$$\tilde{H} = H/(H \cap Z(J)) \cap \mathrm{PSL}(3, \mathbb{Z}/p)$$

where $Z(J)$ denotes the center of J. We let $\mu = \gcd(3, p-1)$. The possibilities for H are as follows:

1. The stabilizer of a line in $(\mathbb{Z}/p)^3$; then $|\tilde{H}| = (p+1)p^3(p-1)^2/\mu$.
2. The stabilizer of a plane in $(\mathbb{Z}/p)^3$; then $|\tilde{H}| = (p+1)p^3(p-1)^2/\mu$.
3. The stabilizer up to similitudes of a nondegenerate quadratic form in 3 variables with coefficients in $\mathbb{Z}/p$; then $|\tilde{H}| = (p+1)p(p-1)$.
4. The normalizer of the subgroup of diagonal matrices; then $|\tilde{H}| = 6(p-1)^2/\mu$.
5. The normalizer of the units in the cubic extension $\mathbb{F}_{p^3}$ of $\mathbb{Z}/p$ embedded into J by the regular representation (after choosing a basis of $\mathbb{F}_{p^3}$ over $\mathbb{Z}/p$); then $|\tilde{H}| = 3(p^2+p+1)/\mu$.

6. Certain finite groups with $|\tilde{H}| = 36$, 72, 168, 216, or 360. These cases can only occur depending on certain congruence properties of p.

Of the groups of type 6, the group of order 168 only occurs if $p = 7$ or if $p \equiv 1, 2$ or 4 (mod 7). The group of order 216 only occurs if $p \equiv 1 \pmod 9$ and the ones of order 72 and 36 only occur if $p \equiv 1 \pmod 3$. The group of order 360 requires that $p \equiv 1$ or 4 (mod 15).

Now we fix a Hecke eigenclass in $H_3^\dagger(\Gamma, V_g)$ with eigenvalues a_l and b_l for $T(l, 1)$ and $T(l, 2)$, respectively. Since the central character of V_g is raising to the g-th power, we see that the eigenvalue of $T(l, 3)$ is l^g.

As in [Ash and McConnell 1992] we find it more convenient to study a certain twist of ρ, namely $\rho' = \rho \otimes \omega^{-1}$ where ω denotes the cyclotomic character. Then what we know (conjecturally) about ρ' are the characteristic polynomials

$$\begin{aligned} P_l(X) &= \det(X - \rho'(\mathrm{Frob}_l^{-1})) \\ &= X^3 - a_l l^{-1} X^2 + b_l l^{-1} X - l^g. \end{aligned}$$

Note that if $P_l(X)$ has a root of multiplicative order d then d must divide the order of $\rho'(\mathrm{Frob}_l^{-1})$. Similarly, if the companion matrix to $P_l(X)$ in J has order d' for some d' prime to p, then d divides d' and d' divides the order of $\rho'(\mathrm{Frob}_l^{-1})$.

Let us say that ρ or ρ' is "big" if the projection of its image to $\mathrm{PGL}(3, \mathbb{F})$ contains $\mathrm{PSL}(3, \mathbb{F})$. By looking at the roots and companion matrix of $P_l(X)$ as l varies, and using the list above, we can usually determine that ρ must be big if it exists. In those cases where it doesn't seem to be big, we can determine what the image of ρ is likely to be. In the latter cases, if we compute more Hecke data for larger l it would be possible, but not likely, that the putative image might grow, but it could never shrink.

It is hard to predict how many l's are required to be satisfied with the apparent answer. By the effective Tchebotarev theorem, one knows that finitely many l's determine ρ but the bound given by the theorem is not practical. A small number of l's may be misleading. For example, when $g = 44$, $p = 97$, we had initially computed the Hecke polynomials $P_l(X)$ for $l \leq 19$. They were all reducible. This pointed to a possibly reducible ρ. However, examination of the roots of the Hecke polynomials (as discussed further below) showed that in fact ρ had to be big. Yet if the Frobeniuses were behaving randomly, the chances that the first 8 Hecke polynomials should be reducible would be less than 5%. If we were experimental scientists we should have rejected the "bigness" hypothesis. Eventually P_{23} turned out to be irreducible, and so did P_{29}.

(The probability calculation alluded to in the preceding paragraph goes like this: In J we can compute the sizes of the conjugacy classes divided by the order of the group using Jordan canonical form. Since p is large we may estimate the results by keeping only the leading powers of p. We see that the union of the conjugacy classes whose characteristic polynomials factor into three linear terms make up roughly one sixth of the group; those that factor into one linear and one quadratic term make up roughly half the group, and those with irreducible characteristic polynomial contribute approximately one third of the group. Thus the probability that the first eight polynomials should be reducible given that ρ is big and assuming randomness, would be $(2/3)^8 < 0.04$.)

For the rest of this section we will assume that we have a homology eigenclass β such that all the Hecke eigenvalues lie in the prime field, so that $\mathbb{F} = \mathbb{Z}/p$; and a representation ρ associated to β as in Conjecture 1.2.

We will prove a theorem in the case when we have an irreducible characteristic polynomial, and we will make some remarks about cases in which all the computed characteristic polynomials are reducible.

Let $J = \mathrm{GL}(3, \mathbb{Z}/p)$ as above, and set

$$\begin{aligned} \bar{J} &= \mathrm{PGL}(3, \mathbb{Z}/p), \\ S\bar{J} &= \mathrm{PSL}(3, \mathbb{Z}/p). \end{aligned}$$

Let G denote the image of ρ' in J, $\bar{G}$ its projection to $\bar{J}$ and $S\bar{G} = \bar{G} \cap S\bar{J}$. Since $[\bar{J} : S\bar{J}] = \mu = \gcd(p-1, 3)$, we have that $[\bar{G} : S\bar{G}] = \mu$ or 1.

Let L be a finite set of rational primes and let there be given the characteristic polynomial

$$P_l(X) = \det(X - \rho'(\mathrm{Frob}_l^{-1}))$$

for each $l \in L$.

Theorem 5.1. *Assume $p \geq 11$. Suppose there exist $l, m \in L$ satisfying these conditions*:

1. *$P_l(X)$ is irreducible.*
2. *$\rho'(\mathrm{Frob}_l^{-1})^3$ is not a scalar matrix.*
3. *One of the following holds*:
   a. *$P_m(X) = L(X)Q(X)$, where L is linear and Q is an irreducible quadratic and the companion matrix of Q raised to the $2^e 3^f$-th power is not scalar, where 2^e and 3^f are the highest powers respectively of 2 and 3 dividing $p+1$; or*
   b. *$P_m(X) = L(X)Q(X)$, where L is linear and Q is an irreducible quadratic and the coefficient of X^2 is nonzero and $\rho'(\mathrm{Frob}_l^{-1})^{168\mu}$ is not a scalar matrix.*

Then the image of ρ' is big.

Remark. $P_l(x)$ has three distinct roots, so that its companion matrix is conjugate to $\rho'(\mathrm{Frob}_l^{-1})$. Thus one can check condition 2 using the companion matrix of $P_l(X)$. A similar remark applies to $P_m(x)$ and the second part of condition 3b.

Using this theorem one checks easily that ρ' is big where indicated in Table 11. We used Maple for computing the orders of roots of companion matrices.

Proof. Referring to the list of possible maximal subgroups, we see that condition 1 rules out types 1, 2 and 3 immediately, since the characteristic polynomials of elements of these subgroups are always reducible.

Let a be a root of $P_l(X)$. Write the multiplicative order d of a as $d = d_1 d_2$ where $d_2 = \gcd(p-1, d)$. Then d_1 divides $(p^3-1)/(p-1) = p^2+p+1$. Note that d must be the order of $\rho'(\mathrm{Frob}_l^{-1})$ since the latter is semisimple with eigenvalues equal to three conjugate elements of $\mathbb{F}_{p^3}/\mathbb{F}_p$.

Now $(a^{d_1})^{p-1} = 1$ so that $(a^{d_1}) \in (\mathbb{Z}/p)^\times$. Therefore the eigenvalues of $\rho'(\mathrm{Frob}_l^{-1})^{d_1}$ lie in the prime field. Since $\rho'(\mathrm{Frob}_l^{-1})$ is semisimple with eigenvalues conjugate over the prime field, it follows that $\rho'(\mathrm{Frob}_l^{-1})^{d_1}$ is a scalar matrix, i.e., lies in the center of J.

Suppose $d_1 = 2^x 3^y 5^z$. But p^2+p+1 is not divisible by 2, 9 or 5. Hence $d_1 = 1$ or 3. It follows from condition 2 and the preceding paragraph that this alternative is impossible, so d_1 has a prime divisor $q \geq 7$. Since $\gcd(p-1, p^2+p+1) = 1$ or 3, q must be prime to $p-1$.

Let g denote the image of $\rho'(\mathrm{Frob}_l^{-1})^{d/q}$ in $\bar{J}$. If $g = 1$, $\rho'(\mathrm{Frob}_l^{-1})^{d/q}$ would be a scalar matrix, and hence $\rho'(\mathrm{Frob}_l^{-1})^{d(p-1)/q} = 1$. But

$$\gcd(d(p-1)/q, d) = d/q,$$

so that the order of $\rho'(\mathrm{Frob}_l^{-1})$ would divide q/d, a contradiction.

Thus $g \neq 1$ and g has order q. Hence g^3 is an element of order q in $S\bar{G}$. This rules out type 4 and all the groups of type 6 except the ones with $|\tilde{H}| = 168$.

Assume now condition 3a. Let h be the companion matrix to $P_m(X)$, so that h is conjugate to $\rho'(\mathrm{Frob}_m^{-1})$ (which has three distinct eigenvalues and hence is semi-simple). Suppose the rational eigenvalue of h is b and the conjugate irrational eigenvalues are α and $\bar{\alpha} \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$.

Let $\bar{h}$ be the image of h in $\bar{J}$, and let t be the order of $\bar{h}$. So h^t is scalar and t divides p^2-1. Write $t = t_1 t_2$ where $t_2 = \gcd(p-1, t)$. Then t_1 divides $p+1$. One of the eigenvalues of $h^{t_1(p-1)}$ is $(b^{(p-1)})^{t_1} = 1$. But it is scalar, so it is the identity matrix. It follows that h^{t_1} has all rational eigenvalues.

Suppose now that t_1 is divisible by no prime greater than 3. Then t_1 is a multiple of $2^e 3^f$. Thus $h^{2^e 3^f}$ has rational eigenvalues. In particular, $\alpha^{2^e 3^f}$ and $\bar{\alpha}^{2^e 3^f}$ are rational and hence equal. But by hypothesis, the companion matrix to Q, which is semisimple with eigenvalues α and $\bar{\alpha}$ when raised to the $2^e 3^f$ power would not be scalar, and this is a contradiction.

Hence t_1 is divisible by some prime r greater than 3. Thus the image of ρ' in $S\bar{J}$ contains an element of order r. Since r divides $p+1$ it must be prime to $\mu(p^2+p+1)$ and this rules out type 5. If $r \neq 7$ we also can rule out the remaining group of type 6 (with 168 elements). But if $r = 7$, then $p \equiv -1 \pmod 7$ and $S\bar{J}$ doesn't have a maximal subgroup of order 168 anyway.

Now assume instead condition 3b. Since

$$\rho'(\mathrm{Frob}_l^{-1})^{168\mu}$$

is not a scalar matrix, the remaining groups of type 6 are eliminated. The only possibility left is type 5.

Every element in a group of type 5 is conjugate over $\mathbb{F}_{p^3}$ to a matrix of one of the following forms:

$$\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & \beta_3 \\ \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \\ \beta_1 & 0 & 0 \end{pmatrix},$$

where β_1, β_2 and β_3 are the three Galois images of an element (not necessarily irrational) of $\mathbb{F}_{p^3}$. The first type of matrix is either scalar or has irreducible characteristic polynomial, so $\rho'(\mathrm{Frob}_m^{-1})$ can't be conjugate to it. But our hypothesis implies that the trace of $\rho'(\mathrm{Frob}_m^{-1})$ isn't zero, so that rules out the other two types of matrices. □

Symmetric Squares and Other Classes with Small Image

When there are no irreducible Hecke polynomials $P_l(x)$ for a relatively long list of l's, it is likely that the image of ρ is not big. Reference to Table 11 bears this out.

When $(g,p) = (42,29)$, $(54,37)$ or $(60,41)$, our Hecke data is compatible with the equality $\rho' = \mathrm{ad}^0(\tau)\otimes\chi$. Here χ is the nontrivial chracter of $G_{\mathbb{Q}}$ of order 2 unramified outside p, and τ is the Galois representation attached to a cusp form of weight 2, level p and nebentype χ, and $\mathrm{ad}^0 : \mathrm{GL}(2) \to \mathrm{GL}(3)$ is the homomorphism given by conjugation on matrices of trace 0. In this case ρ and ρ' would have their images contained in the orthogonal group $GO(3, \mathbb{F}_p)$.

In fact, using congruences mod p for classical modular forms, the symmetric squares lift from the theory of automorphic representations, the theory of modular representations of $\mathrm{GL}(3, \mathbb{F}_p)$ and some auxilliary computations, it can be proved for the values of (g,p) listed above that there does indeed exist a ρ of this form attached to β as in Conjecture 1.2. This will be the subject of a paper by the second author with P. H. Tiep, under preparation. We would like to thank Richard Taylor and Luiz Figueiredo for helping us to figure out what was going on in these cases.

When $(g,p) = (112,229)$ or $(126,257)$, our Hecke data is compatible with the equality

$$\rho' = (\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}\psi)\otimes\omega^{-1}\oplus 1,$$

where 1 is the trivial character, ω the cyclotomic character mod p, $K = \mathbb{Q}(\sqrt{p})$ (which has class number 3) and ψ a nontrivial cubic character of G_K unramified everywhere.

When $(g,p) = (90,277)$, our Hecke data is compatible with the equality $\rho' = \sigma\omega^{-1}\oplus 1$, where ω is the cyclotomic character mod p and σ has image isomorphic to $\hat{A}_4$. More specifically, let K be the unique cyclic cubic extension of $\mathbb{Q}$ ramified only at p, and let Q be the quartic extension of $\mathbb{Q}$ generated by a root of the irreducible polynomial $x^4+x^3-16x^2-3x+1$. Let L be the compositum KQ. The class group of K is $\mathbb{Z}/2\times\mathbb{Z}/2$ and L is the Hilbert class field of K. Then $\mathrm{Gal}(L/\mathbb{Q}) \simeq A_4$ and the splitting of primes in L agrees with our Hecke data under the assumption that L can be embedded into an $\hat{A}_4$ extension M of $\mathbb{Q}$ and $\sigma : G_{\mathbb{Q}} \twoheadrightarrow \hat{A}_4 \subset \mathrm{GL}(2,\mathbb{F}_p)$ has fixed field M, which is unramified outside p. An abstract argument with Brauer groups shows that σ exists. Thanks to help from Jordi Quer and Warren Sinnot, we constructed σ and checked the splitting of primes from L to M against our Hecke data.

We also computed a number of Hecke polynomials for eigenclasses with $p \leq 11$. They were not included in Table 11 because they are better discussed in the context of the following section.

6. CASES WITH $p \leq 11$

In this section we discuss computations we made of Hecke eigenvalues when p was odd and less than 12. We did not list them in Table 11 because they are more easily described in terms of GL(1) and GL(2) phenomena.

First consider $p = 5$ or 7. By Theorem 4.3 above and its proof, we know that every Hecke eigenclass in $H_3^{\dagger}(\Gamma, V_g(\mathbb{F}_p))$ has an attached Galois representation ρ always of boundary type. Thus ρ is reducible and can be written as

$$\rho \approx \omega^a \oplus \omega^b \sigma,$$

where ω is the cyclotomic character mod p and σ is a 2-dimensional mod p Galois representation attached to a cohomology eigenclass α for $\mathrm{GL}(2,\mathbb{Z})$ with coefficients in U_h = homogeneous polynomials in two variables over $\mathbb{F}_p$, for some h. Equivalently, by a theorem of Eichler and Shimura, σ is attached to a classical modular form of weight $h + 2$ and level 1. A direct applicaton of Corollary 3.6 of [Ash and Stevens 1986b] shows that after possible twisting of σ, we may take $h < p$. In this range, all such σ are themselves reducible.

So we have

$$\rho \approx \omega^a \oplus \omega^b \oplus \omega^c$$

for some a, b, c. We checked this numerically for the quasicuspidal classes with $(g, p) = (52, 5)$, $(54, 5)$, $(58, 7)$, $(64, 7)$, $(70, 7)$ and $(106, 7)$. That is, for each class we found (a, b, c) such that

$$P_l(x) = (x - l^a)(x - l^b)(x - l^c)$$

for all small values of l that we tested.

In the case $p = 3$, the results of [Ash and Stevens 1986b] and Theorem 4.2 above are no longer applicable. They are probably still true, but since 3 divides the torsion of Γ, the methods we used are not sufficient to establish their truth. We still expect that any eigenclass in $H_3^{\dagger}(\Gamma, V_g(\mathbb{F}_3))$ will have attached a ρ of the form $\omega^a \oplus \omega^b \oplus \omega^c$. We checked this for $g = 36$, $p = 3$ and $l \leq 19$. In this case, there is no quasicuspidal homology, but there are five linearly independent boundary classes, 4 in F^0 and one of type 2. We upper triangularized this 5-dimensional space and verified that each Hecke eigenclass has attached to it the Galois representation

$$\rho' \approx 1 \oplus \omega \oplus \omega.$$

Finally, suppose $p = 11$. By Theorem 4.2, if Conjecture 1.2 holds for any eigenclass in $H_3^{\mathrm{qc}}(\Gamma, V_g)$ for $g \leq 130$, then it holds for any eigenclass in $H_*(\Gamma, W)$ for any $*$ and any admissible W.

We computed T_l for $l = 2$, 3, 5, 7 on $H_3^{\mathrm{qc}}(\Gamma, V_g)$ for $g \leq 130$. There was a strong "$p-1$ regularity", that is, every package of Hecke eigenvalues for V_g also occured for V_{g+p-1}. (When $p = 7$, we also verified such a $p-1$ regularity for $g = 52$, 58, 64, and 70.) See Section 4 for a heruistic explanation of this pattern.

In all, there are 9 distinct packages of Hecke eigenvalues in $\bigoplus_{g \leq 130} H_3^{\mathrm{qc}}(\Gamma, V_g)$. Of these, 3 are consistent with a Galois representation

$$\rho \approx \omega^a \oplus \omega^b \oplus \omega^c$$

for some a, b, c and 5 are consistent with

$$\rho \approx \omega^t \oplus \omega^s \tau$$

for some s, t where τ = the 2-dimensional mod 11 representation associated to Δ, the classical holomorphic cusp form of weight 12 on $\mathrm{SL}(2,\mathbb{Z})$. The remaining package is consistent with

$$\rho \approx \mathrm{ad}^0(\tau).$$

These results strongly support Conjecture 1.2 for $p = 11$.

7. COMPUTATION OF HOMOLOGY

Theory

Let $\Gamma = \mathrm{GL}(3, \mathbb{Z})$ and $\Gamma_2 = \mathrm{GL}(2, \mathbb{Z})$. Let $V = V_g(\mathbb{F}_p)$ denote the set of homogeneous polynomials in three variables x, y, and z of total degree g with coefficients in $\mathbb{F}_p$. Given a matrix $m = (m_{ij})$ in Γ, we let it act on $f \in V$ as follows:

$$f(\boldsymbol{x}) \cdot m = f(m\boldsymbol{x}), \quad \text{where } \boldsymbol{x} = {}^t(x, y, z).$$

Suppose $p > 2$. Let Z be the subspace of solutions in V of equations (i)–(iii) in Theorem 2.1. From equation (i), any solution must consist of terms of even degree in each variable. From equation (ii), a solution must be antisymmetric in the three variables.

A nice basis for the polynomials in V satisfying these two equations are the antisymmetric polynomials of the form

$$A(\boldsymbol{x}^\lambda) := \sum_{\sigma \in S_3} (\operatorname{sgn} \sigma)(\sigma x)^a (\sigma y)^b (\sigma z)^c,$$

where $\lambda = (a, b, c)$ ranges over unordered partitions of g into exactly three distinct nonnegative even parts. Any two unordered partitions are the same so long as they contain the same elements, each with the same multiplicities. We will follow the convention that unordered partitions are always written in descending sequence. The symmetric group S_3 acts on the set of variables $\{x, y, z\}$ by permuting the set.

On the other hand, suppose $p = 2$. The only diagonal matrix is 1 so we no longer get a restriction on the degree of the variables in each term. Since

$$\operatorname{sgn} \sigma \equiv 1 \pmod 2,$$

the antisymmetries $A(\boldsymbol{x}^\lambda)$ which are the solutions to equation (ii) in Theorem 2.1 reduce to symmetric polynomials. In the $p = 2$ case, a nice basis for the solutions to the first two equations are the symmetric polynomials of the form

$$S(\boldsymbol{x}^\lambda) := \begin{cases} \displaystyle\sum_{\sigma \in S_3} (\sigma x)^a (\sigma y)^b (\sigma z)^c & \text{if } a > b > c, \\ \displaystyle\sum_{\sigma \in A_3} (\sigma x)^a (\sigma y)^b (\sigma z)^c & \text{if } a = b > c \\ & \text{or } a > b = c, \\ x^a y^a z^a & \text{if } a = b = c, \end{cases}$$

where $\lambda = (a, b, c)$ ranges over all unordered partitions of g into three nonnegative parts.

Boundary homology. The homology group $H_3^\dagger(\Gamma, V)$ consists of boundary homology and quasicuspidal homology. We divided the boundary homology into four overlapping subspaces designated as types 0, 1, 2a, and 2b, for which we give explicit generator polynomials below. We let T^0, T^1, T^{2a} and T^{2b} denote these four subspaces and define a filtration as follows:

$$\begin{aligned} F^0 &= T^0, \\ F^1 &= F^0 \vee T^1, \\ F^{2a} &= F^1 \vee T^{2a}, \\ F^{2b} &= F^{2a} \vee T^{2b}, \end{aligned}$$

where $\vee$ denotes the linear span. Since F^{2b} is precisely the boundary homology, we obtain the quasicuspidal homology $H_3^{\mathrm{qc}}(\Gamma, V)$ as the quotient $H_3(\Gamma, V)/F^{2b}$.

Type 0 homology. Let $U = U_g(\mathbb{F}_p)$ denote the set of homogeneous polynomials of degree g in two variables x and y with coefficients in $\mathbb{F}_p$. The antisymmetrization map A above is a linear map from U into V. Further, if $f(x, y)$ is a solution in U to the system of equations in Theorem 2.3, then $A(f(x, y))$ is a solution in V to the system in Theorem 2.1. The type 0 solutions are all such antisymmetrizations:

$$F^0 = T^0 = \{A(f(x, y, z)) : f(x, y) \in H_1^\dagger(\Gamma_2, U)\}.$$

One particular solution in U to equations (i), (ii), (iii) of Theorem 2.3 is $x^g - y^g$. This solution vanishes in F^0 under the antisymmetry map A. Moreover let $f(x, y)$ be a GL_2 solution and let

$F(x,y,z) = A(f(x,y))$. There exists $a \in \mathbb{F}_p$ such that

$$f(x,y) - F(x,y,0) = a(x^g - y^g).$$

We can accordingly view F^0 as a quotient:

$$F^0 \simeq H_1^\dagger(\Gamma_2, U)/(x^g - y^g).$$

(In characteristic 0, F^0 would correspond to classical cusp forms of weight $g+2$.)

For some g, $H_3^\dagger(\Gamma, V)/F^0$ contains nonzero solutions that can be written explicitly:

Example 7.1. If $p > 2$, then $A(x^{p^3-1}y^{p^2-1}z^{p-1})$ is a solution to the equations in Theorem 2.1.

Proof. It is easy to see that the first two equations are satisfied. To verify the third, let

$$f(x,y,z) = A(x^{p^3-1}y^{p^2-1}z^{p-1})$$

and use the fact that $(\alpha+\beta)^p = \alpha^p + \beta^p$ to show that

$$(x-y)\big(f(x,y,z) - f(x-y,y,z) - f(x,x-y,z)\big) = 0.$$

Since $x - y \neq 0$, the third equation is satisfied. $\square$

Type 1 solutions. The solutions of type 1 are constructed as follows. First let p be odd. Set

$$\begin{aligned} b(x,y) &= \left(y^p - x^{p-1}y\right)^2, \\ c(x,y,z) &= \left(b(x,z)^{p/2} - b(x,y)^{(p-1)/2} b(x,z)\right)^2, \\ f_{klm}(x,y,z) &= x^{2k} b(x,y)^l c(x,y,z)^m, \end{aligned}$$

so that $c(x,y,z)$ is a fundamental invariant (modulo $p > 2$) for $\mathrm{GL}(3,\mathbb{Z})$; see [Dixon 1911].

For $p = 2$, this scheme is modified as follows:

$$\begin{aligned} b(x,y) &= y^p - x^{p-1}y, \\ c(x,y,z) &= b(x,z)^2 - b(x,y)b(x,z), \\ f_{klm}(x,y,z) &= x^k b(x,y)^l c(x,y,z)^m. \end{aligned}$$

Here $c(x,y,z)$ is the corresponding fundamental invariant for $p = 2$.

Example 7.2. $F_{klm}(x,y,z) := A(f_{klm}(x,y,z))$ is a solution.

If $m = 0$, then such a solution will be type 0. Linear combinations of these solutions ($m > 0$) give the type 1 solutions of weight g:

$$T^1 = \begin{cases} \bigvee\{F_{klm} : g = 4m + 2l + k,\ m > 0\} & \text{if } p = 2; \\ \bigvee\{F_{klm} : g = 2mp^2 + 2lp + 2k,\ m > 0\} & \text{if } p > 2. \end{cases}$$

From the inequality $2mp^2 \le g$, it follows for the range of our calculations ($g \le 200$) that type 1 solutions can only exist for $p \le 7$.

Since all homology for $p = 2$ and $g \le 100$ turned out to be type 1, we will assume $p \ge 3$ for the rest of this section.

Type 2a solutions. Let $\varphi(x,y)$ be a Γ_2 solution of total degree d, i.e., $\varphi \in H_1^\dagger(\Gamma_2, U_d(\mathbb{F}_p))$. Let $c(x,y,z)$ be the fundamental invariant as above.

Example 7.3. If $g = d + 2mp^2$, a boundary solution is given by

$$G_{1m\varphi}(x,y,z) := \sum_{\sigma \in A_3} \sigma\big(\varphi(x,y)c(x,y,z)^m\big).$$

If $m = 0$, then the solution is a type 0 solution. The type 2a solutions, T^{2a}, are the linear span of solutions $G_{1m\varphi}$ of weight g:

$$T^{2a} = \bigvee\{G_{1m\varphi} : \varphi \in H_1^\dagger(\Gamma_2, U_d(\mathbb{F}_p)),\\ g = d + 2mp^2,\ m > 0\}.$$

For $p > 2$, the smallest Γ_2 solution is of weight $d = 10$. This gives rise to the inequality $g \ge 10 + 2p^2$. For $g \le 200$, we have $p \le 7$ for all type 2a solutions.

Type 2b solutions. Set

$$B(x,y) := y^p - x^{p-1}y = \prod_{\alpha \in \mathbb{Z}/p} (\alpha x + y).$$

As before, let $\varphi(x,y)$ be a Γ_2 solution of total degree d.

Example 7.4. If $g = pd + 2m$, a boundary solution is given by

$$G_{2m\varphi}(x,y,z) := \sum_{\sigma \in A_3} \sigma\left(x^{2m}\varphi\left(B(x,y), B(x,z)\right)\right).$$

The type 2b solutions, T^{2b}, are the linear span of solutions $G_{2m\varphi}$ of weight g:

$$T^{2b} = \bigvee\{G_{2m\varphi} : \varphi \in H_1^\dagger(\Gamma_2, U_d(\mathbb{F}_p)),\ g = pd + 2m\}$$

For type 2b solutions, since $d \geq 10$, we have the inequality $g \geq 10p$. Hence we have $p \leq 19$ for any type 2b solution with $g \leq 200$.

Algorithms

Fix a prime p and a weight g and let $V = V_g(\mathbb{F}_p)$ of degree g with coefficients in $\mathbb{F}_p$. Let $D(V)$ be the span of the set of monomials $x^a y^b z^c$ of degree g with a, b and c all even. Define $A(V) := \bigcup_{v \in V} A(v)$. For $p > 2$, $A(V)$ is the set of solutions to equation (ii) of Theorem 2.1.

Let $\Lambda = \{\lambda = (a, b, c) : a > b > c \geq 0,\ a+b+c = g,\ a \equiv b \equiv c \equiv 0 \pmod 2\}$. An indexed basis for $A(V) \cap D(V)$ is given by $\mathcal{B} = \{A(\boldsymbol{x}^\lambda) : \lambda \in \Lambda\}$. For $p \geq 3$, we may take $A(V) \cap D(V)$ as a superspace of $H_3^\dagger(\Gamma, V)$. Recall that for $p > 3$, $H_3^\dagger(\Gamma, V) \simeq H_3(\Gamma, V)$.

Assume now that $p > 2$. To compute $H_3^\dagger(\Gamma, V)$, we construct a matrix m with rows indexed by monomials in V and columns indexed by Λ. Given a monomial $v \in V$ and a partition $\lambda \in \Lambda$, the entry $m_{v,\lambda}$ in row v column λ is the coefficient of v in $A(\boldsymbol{x}^\lambda)(1 + h + h^2)$. Setting $f(x, -y, z) = A(\boldsymbol{x}^\lambda)$, we have

$$\begin{aligned}
&A(\boldsymbol{x}^\lambda)(1+h+h^2)\\
&\quad = f(x, -y, z)(1+h+h^2)\\
&\quad = f(x, -y, z) + f(y, -x+y, z) + f(-x+y, x, z)\\
&\qquad\qquad \text{(by the action of } h \text{ on } f)\\
&\quad = f(x, y, z) + f(y, x-y, z) + f(x-y, x, z)\\
&\qquad\qquad \text{(since } f \text{ is in } D(V))\\
&\quad = f(x, y, z) - f(x-y, y, z) - f(x, x-y, z)\\
&\qquad\qquad \text{(since } f \text{ is in } A(V)).
\end{aligned}$$

We used this last form for computational purposes.

The number of rows in the matrix is larger than the number of columns so there are necessarily dependencies. One simple application of symmetry reduced the number of rows by more than half. Consider the monomials $v = x^a y^b z^c$ and $v' = x^b y^a z^c$ obtained by permuting x and y in v. Given any partition $\lambda \in \Lambda$, the associated matrix entries satisfy $m_{v',\lambda} = -m_{v,\lambda}$. The rows associated with v and v' are thus dependent. In particular, if $v = v'$ (that is, if $a = b$), the row is zero.

We then used row reduction and back-substitution to obtain a basis for $H_3^\dagger(\Gamma, V)$ consisting of linear combinations of the basis elements for $A(V) \cap D(V)$. To obtain F^0, we restricted Λ to partitions into two parts:

$$\Lambda_2(g) = \{\lambda = (a, b, 0) : a > b > 0,\ a + b = g,\ a \equiv b \equiv 0 \pmod 2\}.$$

To obtain $H_3^\dagger(\Gamma, V)/F^0$, what we call *excess homology*, we restricted Λ to partitions into three nonzero parts:

$$\Lambda_3(g) = \{\lambda = (a, b, c) : a > b > c > 0,\ a + b + c = g,\ a \equiv b \equiv c \equiv 0 \pmod 2\}.$$

We illustrate the calculation of type 1 homology with the case $g = 36$, $p = 3$. The 19 partitions in $\Lambda_3(36)$ act as labels for a basis of $H_3^\dagger(\Gamma, V_{36}(\mathbb{F}_3))/F_0$. For example, the partition $(30, 4, 2)$ corresponds to the basis polynomial $A(x^{30}y^4z^2)$. We construct a matrix with columns indexed by these nineteen partitions as follows. The rows are indexed by the $\binom{g+1}{2} = 666$ *ordered* partitions $\mu = (a, b, c)$ of 36. The entry in row μ column λ is the coefficient of $x^a y^b z^c$ in the computational form of

$$u_\lambda = A(\boldsymbol{x}^\lambda)(1 + h + h^2).$$

For example, since the coefficient of $x^6 y^{16} z^{14}$ in $u_{(14,12,10)}$ is 714 or $0 \pmod 3$, the entry in row $(6, 16, 14)$ and column $(14, 12, 10)$ is 0. We find a basis for the nullspace of $\{u_\lambda : \lambda \in \Lambda_3(36)\}$ by performing Gaussian elimination on our matrix. For $g = 36$, $p = 3$, we obtain the basis $\{A(x^{26}y^8z^2)\}$ for $H_3^\dagger(\Gamma, V_{36}(\mathbb{F}_3))/F_0$.

As mentioned, we were able to exploit symmetry to reduce the number of rows in our matrix.

To calculate a basis for type 1 boundary homology, we used the same set of partitions $\Lambda_3(g)$ to index the columns of the matrix. The rows were indexed by the generators F_{klm} of the type one solutions. Given a partition $\lambda = (a, b, c) \in \Lambda_3(g)$ and a generator $F = F_{klm}$, the entry in row F column λ is given by the coefficient of $x^a y^b z^c$ in F. Row reducing this matrix and deleting rows of all zeros gives a basis for the type 1 solutions.

Bases for type 2a and type 2b homology were calculated in a similar fashion.

For $p = 2$, we made appropriate modifications to the procedures above.

Implementation

We used Mathematica to do the necessary polynomial manipulation to obtain the matrix m for fixed g, without reducing entries modulo p. A C language implementation of Gaussian elimination read the entries, reducing them modulo p, row reduced the matrix and used back substitution to get a basis for $H_3^\dagger(\Gamma, V)$ which could be checked against the original equations. For larger g, back substitution and verification were omitted to reduce program running time. In this case, we used the corank of the row reduced matrix to obtain the dimension of $H_3^\dagger(\Gamma, V)$.

All our arithmetic was exact arithmetic. Our primes p all satisfied $p^2 < 2^{31} - 1$, this number being our largest machine integer. Thus integer overflow was not a significant concern in writing our C language code for row reduction and back substitution.

Consistency Checks

To check against programming errors, the program verified that solutions obtained by the matrix calculations satisfied the equations. Of course, this only verifies our results to be lower bounds of the true dimensions of the solution space. As this checking is time consuming for large g, we omitted this check for $g \geq 100$. For small g, we used hand calculations as an exact check. The success of the Hecke operator calculations (see Section 9) is a very strong indication of correctness of the homology programs.

Examples

Table 1 shows some specific examples of the homology computations. The program output has been converted to tabular form for compactness.

$g = 36$, $p = 3$						$g = 42$, $p = 29$ (exc.)	
Λ	nullspace basis					Λ	n.b.
(30, 4, 2)						(36, 4, 2)	
(28, 6, 2)						(34, 6, 2)	
(26, 8, 2)	1					(32, 8, 2)	1
(26, 6, 4)						(32, 6, 4)	26
(24, 10, 2)						(30, 10, 2)	
(24, 8, 4)						(30, 8, 4)	
(22, 12, 2)						(28, 12, 2)	22
(22, 10, 4)						(28, 10, 4)	10
(22, 8, 6)						(28, 8, 6)	5
(20, 14, 2)						(26, 14, 2)	15
(20, 12, 4)						(26, 12, 4)	22
(20, 10, 6)						(26, 10, 6)	27
(18, 16, 2)						(24, 16, 2)	4
(18, 14, 4)						(24, 14, 4)	6
(18, 12, 6)						(24, 12, 6)	9
(18, 10, 8)						(24, 10, 8)	2
(16, 14, 6)						(22, 18, 2)	27
(16, 12, 8)						(22, 16, 4)	12
(14, 12, 10)						(22, 14, 6)	6
(34, 2, 0)		1				(22, 12, 8)	18
(32, 4, 0)			1			(20, 18, 4)	19
(30, 6, 0)				1		(20, 16, 6)	10
(28, 8, 0)					1	(20, 14, 8)	3
(26, 10, 0)		1	2		1	(20, 12, 10)	11
(24, 12, 0)			2	1		(18, 16, 8)	15
(22, 14, 0)						(18, 14, 10)	7
(20, 16, 0)						(16, 14, 12)	7

TABLE 1. Output of sample homology computations. Λ is the index set, consisting (in the case $g = 36$) of the 27 partitions of 36 into distinct even parts. "Nullspace basis" gives a basis for our realization of the homology; each column represents a vector in the basis (vector entries not shown are zero). See next page for details.

The left half of the table shows the case $g = 36$, $p = 3$. Our program outputs the index set Λ, consisting of the 27 partitions of 36 into distinct even parts; this is shown in the first column. It then outputs five 27-tuples representing a basis for our realization of $H_3^\dagger(\Gamma, V_{36}(\mathbb{F}_3))$. These vectors are shown here as columns in the next section of the table. For instance, the first basis vector has a single nonzero entry at the position $(26, 8, 2)$, and so corresponds to the polynomial

$$A(x^{26}y^8z^2) = x^{26}y^8z^2 + y^{26}z^8x^2 + z^{26}x^8y^2 - y^{26}x^8z^2 - z^{26}y^8x^2 - x^{26}z^8y^2,$$

while the second vector corresponds to $A(x^{34}y^2) + A(x^{26}y^{10})$ and so on.

The last four of these five vectors are Γ_2 solutions. Restricting Λ to partitions of $g = 36$ into distinct even positive parts gives the excess homology. The program can be queried about the excess homology separately; it responds essentially with the information shown above the dashed line in the table — that is, a 19-element index set and one 19-tuple, which corresponds to a basis of the nullspace. This solution, $A(x^{26}y^8z^2)$, turns out to be type 1 boundary homology.

To isolate the nontrivial Γ_2 solutions, needed for calculation of type 2a and 2b solutions, our program has a Γ_2 homology mode. Choosing this mode and running the program on the same example yields the information shown below the dashed line in the table: an eight-element index set and four solutions, each corresponding to a solution found in the earlier realization of $H_3^\dagger(\Gamma, V_{36}(\mathbb{F}_3))$ (which is why we need not display the output separately). This simple correspondence of solutions always occurs as a result of our ordering of the basis vectors. The trivial solution $x^{36} - y^{36}$ needs to be added to this list to obtain a basis for the realization of $H_1^\dagger(\Gamma_2, U_{36}(\mathbb{F}_3))$.

The right half of the table shows the excess homology in the case $g = 42$, $p = 29$. The interpretation of the columns is similar; the nullspace turns out to be one-dimensional.

8. HOMOLOGY TABLES

The next several tables list the dimension of solution spaces by prime p and by weight g. See Definition 1.1 and Section 7 for an explanation of the notations $H^\dagger$, F^*, T^*, H^{qc}.

Homology for $p = 2$

We computed excess homology for $p = 2$ and $g \leq 100$. All this homology turned out to be type 1 boundary homology. We list the dimension of type 1 boundary homology for each $g \leq 100$.

g	dim	g	dim	g	dim	g	dim	g	dim
11	1	29	13	47	41	65	84	83	143
12	1	30	14	48	43	66	87	84	147
13	1	31	15	49	45	67	90	85	150
14	2	32	17	50	47	68	93	86	154
15	2	33	18	51	49	69	96	87	158
16	2	34	19	52	51	70	99	88	162
17	3	35	21	53	54	71	102	89	166
18	4	36	22	54	56	72	105	90	170
19	4	37	23	55	58	73	108	91	174
20	5	38	25	56	61	74	112	92	178
21	6	39	27	57	63	75	115	93	182
22	6	40	28	58	65	76	118	94	186
23	7	41	30	59	68	77	122	95	191
24	8	42	32	60	71	78	125	96	195
25	9	43	33	61	73	79	128	97	199
26	10	44	35	62	76	80	132	98	204
27	11	45	37	63	79	81	136	99	208
28	12	46	39	64	81	82	139	100	212

TABLE 2. $p = 2$: Excess homology (and also type 1 boundary homology) for $H_3^\dagger(\mathrm{GL}(3,\mathbb{Z}), V_g(\mathbb{F}_2))/F^0$ by weight g, for $g \leq 100$.

Homology for $p > 2$

Tables 3 and 4 list the dimension of the solution spaces for all pairs (g, p) with $3 \leq p \leq 541$, $g \leq 200$ and such that this dimension is nonzero. (For primes $p > 3$, the solution spaces are isomorphic to the homology groups without the $\dagger$.) The tables are organized by p and g, respectively. The dimension appears as a superscript when greater than 1 and is implicit when equal to 1.

p	g: a superscript indicates the dimension of the solution space when greater than 1
3	$36, 38, 40, 42, 44^2, 46^2, 48^2, 50^2, 52^3, 54^3, 56^3, 58^3, 60^4, 62^5, 64^5, 66^5, 68^6, 70^7, 72^7, 74^7, 76^8, 78^9, 80^9, 82^9, 84^{10}, 86^{11}, 88^{12}, 90^{12}, 92^{13}, 94^{14}, 96^{15}, 98^{15}, 100^{16}, 102^{17}, 104^{18}, 106^{18}, 108^{19}, 110^{20}, 112^{21}, 114^{22}, 116^{23}, 118^{24}, 120^{25}, 122^{26}, 124^{27}, 126^{28}, 128^{29}, 130^{30}, 132^{31}, 134^{32}, 136^{33}, 138^{34}, 140^{36}, 142^{37}, 144^{38}, 146^{39}, 148^{41}, 150^{42}, 152^{43}, 154^{44}, 156^{46}, 158^{47}, 160^{48}, 162^{49}, 164^{51}, 166^{53}, 168^{54}, 170^{55}, 172^{57}, 174^{59}, 176^{60}, 178^{61}, 180^{63}, 182^{65}, 184^{66}, 186^{67}, 188^{69}, 190^{71}, 192^{73}, 194^{74}, 196^{76}, 198^{78}, 200^{80}$
5	$52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72^2, 74, 76^2, 78^2, 80^3, 82^2, 84^3, 86^2, 88^3, 90^3, 92^4, 94^3, 96^4, 98^3, 100^5, 102^4, 104^5, 106^4, 108^5, 110^5, 112^6, 114^6, 116^7, 118^6, 120^8, 122^6, 124^8, 126^7, 128^8, 130^8, 132^9, 134^8, 136^9, 138^9, 140^{11}, 142^{10}, 144^{11}, 146^{10}, 148^{11}, 150^{12}, 152^{14}, 154^{13}, 156^{14}, 158^{13}, 160^{15}, 162^{15}, 164^{16}, 166^{15}, 168^{16}, 170^{16}, 172^{18}, 174^{17}, 176^{19}, 178^{18}, 180^{20}, 182^{19}, 184^{20}, 186^{20}, 188^{21}, 190^{21}, 192^{23}, 194^{21}, 196^{23}, 198^{22}, 200^{26}$
7	$58, 64, 70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100^2, 102^2, 104^2, 106^3, 108^3, 110^2, 112^3, 114^4, 116^3, 118^3, 120^4, 122^4, 124^4, 126^4, 128^4, 130^4, 132^5, 134^4, 136^5, 138^5, 140^5, 142^5, 144^6, 146^5, 148^6, 150^6, 152^6, 154^8, 156^8, 158^7, 160^8, 162^8, 164^9, 166^8, 168^9, 170^{10}, 172^{10}, 174^9, 176^9, 178^{10}, 180^{10}, 182^{10}, 184^{11}, 186^{12}, 188^{11}, 190^{11}, 192^{12}, 194^{11}, 196^{13}, 198^{13}, 200^{13}$
11	$68, 78, 88, 90, 92, 98, 100, 102, 108, 110^2, 112^2, 114, 116, 118^2, 120^2, 122^2, 124, 126, 128^2, 130^3, 132^2, 134^2, 136^2, 138^3, 140^3, 142^3, 144^2, 146^2, 148^3, 150^3, 152^3, 154^2, 156^3, 158^3, 160^3, 162^4, 164^3, 166^3, 168^4, 170^4, 172^4, 174^3, 176^3, 178^5, 180^5, 182^4, 184^4, 186^4, 188^6, 190^5, 192^5, 194^4, 196^4, 198^6, 200^6$
13	$80, 92, 104, 106, 108, 116, 118, 120, 126, 128, 130^2, 132^3, 134^2, 136^2, 138^2, 140^2, 142^2, 144^3, 146^2, 148^2, 150^2, 152^2, 154^2, 156^4, 158^2, 160^2, 162^2, 164^2, 166^2, 168^4, 170^2, 172^2, 174^2, 176^2, 178^2, 180^4, 182^3, 184^4, 186^4, 188^4, 190^4, 192^6, 194^3, 196^5, 198^4, 200^4$
17	$94, 102, 104^2, 110, 118, 120^2, 126, 134^2, 136^2, 138, 140, 142, 150^2, 152^2, 154, 156, 158, 166^2, 168^2, 170^2, 172^3, 174^3, 176^2, 178, 180, 182^3, 184^3, 186^2, 188^3, 190^3, 192^2, 194, 196, 198^3, 200^3$
19	$66, 76, 84, 94, 102, 112, 116, 120, 130, 134, 138^2, 148^2, 150, 152^2, 154^2, 156^4, 166^2, 168, 170^2, 172^2, 174^4, 178, 184^2, 186, 188^3, 190^4, 192^6, 194^2, 196^3, 198, 200$
23	$92, 114, 136, 138, 140^2, 158, 160, 162^2, 174, 176, 180, 182, 184^3, 186^2, 188^2, 196, 198$
29	$42, 70, 98, 116, 126, 144, 154, 160, 168, 172, 174^2, 176^3, 182, 188, 196, 200$
31	$108, 124^2, 138, 154^2, 168, 184^2, 186, 188^2, 198$
37	$54, 88, 90, 124^2, 126, 134, 140, 148^2, 160^2, 162, 170, 176, 184^2, 196^2, 198$
41	$50, 60, 70, 90, 96, 100, 110, 130, 136, 140, 150, 164^2, 168, 170, 176, 178, 180, 190$
43	$66, 96, 108, 116, 138, 150^3, 158, 172^3, 180, 184, 192^3, 200$
47	$138, 164, 184, 188^3$
53	$52, 78^2, 104^2, 114, 130^2, 156^2, 166, 178, 182^2, 192, 198$
61	$60, 80, 90^2, 100, 120^3, 140, 150^3, 160, 174, 180^3, 200$
67	$102, 160, 168, 196$
71	$128, 176, 198$
73	$108^2, 124, 180^2, 196$
79	$78, 104, 130, 156^2, 182$
83	136
89	$88, 132^3, 148, 176^2$
97	$44, 140, 144^3$
101	150^4
103	$84, 186$
107	$88, 146, 194$

p	g^{dim}	p	g^{dim}
109	$126, 162^4, 198$	229	112
113	$78, 140, 168^4, 190, 196$	239	60
127	146	257	126
137	186	277	$90, 182$
139	188	307	124
181	$54, 84$	397	188
191	48	401	$100, 198^2$
199	150	463	88
211	52	523	106

TABLE 3. Excess homology $H_3^{\dagger}(\mathrm{GL}(3,\mathbb{Z}), V_g(\mathbb{F}_p))/F^0$ by prime p, for $3 \leq p \leq 541$, $g \leq 200$. If a combination (g, p) in this range is not shown, the space has dimension 0. In particular, this happens for all $g \leq 200$ in the case of the primes $p =$ 59, 131, 149, 151, 157, 163, 167, 173, 179, 193, 197, 223, 227, 233, 241, 251, 263, 269, 271, 281, 283, 293, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 467, 479, 487, 491, 499, 503, 509, 521, 541.

g	$p^{\dim}$	g	$p^{\dim}$
36	3	120	$3^{25}, 5^8, 7^4, 11^2, 13, 17^2, 19, 61^3$
38	3	122	$3^{26}, 5^6, 7^4, 11^2$
40	3	124	$3^{27}, 5^8, 7^4, 11, 31^2, 37^2, 73, 307$
42	$3, 29$	126	$3^{28}, 5^7, 7^4, 11, 13, 17, 29, 37, 109, 257$
44	$3^2, 97$	128	$3^{29}, 5^8, 7^4, 11^2, 13, 71$
46	3^2	130	$3^{30}, 5^8, 7^4, 11^3, 13^2, 19, 41, 53^2, 79$
48	$3^2, 191$	132	$3^{31}, 5^9, 7^5, 11^2, 13^3, 89^3$
50	$3^2, 41$	134	$3^{32}, 5^8, 7^4, 11^2, 13^2, 17^2, 19, 37$
52	$3^3, 5, 53, 211$	136	$3^{33}, 5^9, 7^5, 11^2, 13^2, 17^2, 23, 41, 83$
54	$3^3, 5, 37, 181$	138	$3^{34}, 5^9, 7^5, 11^3, 13^2, 17, 19^2, 23, 31, 43, 47$
56	$3^3, 5$	140	$3^{36}, 5^{11}, 7^5, 11^3, 13^2, 17, 23^2, 37, 41, 61, 97, 113$
58	$3^3, 5, 7$	142	$3^{37}, 5^{10}, 7^5, 11^3, 13^2, 17$
60	$3^4, 5, 41, 61, 239$	144	$3^{38}, 5^{11}, 7^6, 11^2, 13^3, 29, 97^3$
62	$3^5, 5$	146	$3^{39}, 5^{10}, 7^5, 11^2, 13^2, 107, 127$
64	$3^5, 5, 7$	148	$3^{41}, 5^{11}, 7^6, 11^3, 13^2, 19^2, 37^2, 89$
66	$3^5, 5, 19, 43$	150	$3^{42}, 5^{12}, 7^6, 11^3, 13^2, 17^2, 19, 41, 43^3, 61^3, 101^4, 199$
68	$3^6, 5, 11$	152	$3^{43}, 5^{14}, 7^6, 11^3, 13^2, 17^2, 19^2$
70	$3^7, 5, 7, 29, 41$	154	$3^{44}, 5^{13}, 7^8, 11^2, 13^2, 17, 19^2, 29, 31^2$
72	$3^7, 5^2, 7$	156	$3^{46}, 5^{14}, 7^8, 11^3, 13^4, 17, 19^4, 53^2, 79^2$
74	$3^7, 5, 7$	158	$3^{47}, 5^{13}, 7^7, 11^3, 13^2, 17, 23, 43$
76	$3^8, 5^2, 7, 19$	160	$3^{48}, 5^{15}, 7^8, 11^3, 13^2, 23, 29, 37^2, 61, 67$
78	$3^9, 5^2, 7, 11, 53^2, 79, 113$	162	$3^{49}, 5^{15}, 7^8, 11^4, 13^2, 23^2, 37, 109^4$
80	$3^9, 5^3, 7, 13, 61$	164	$3^{51}, 5^{16}, 7^9, 11^3, 13^2, 41^2, 47$
82	$3^9, 5^2, 7$	166	$3^{53}, 5^{15}, 7^8, 11^3, 13^2, 17^2, 19^2, 53$
84	$3^{10}, 5^3, 7, 19, 103, 181$	168	$3^{54}, 5^{16}, 7^9, 11^4, 13^4, 17^2, 19, 29, 31, 41, 67, 113^4$
86	$3^{11}, 5^2, 7$	170	$3^{55}, 5^{16}, 7^{10}, 11^4, 13^2, 17^2, 19^2, 37, 41$
88	$3^{12}, 5^3, 7, 11, 37, 89, 107, 463$	172	$3^{57}, 5^{18}, 7^{10}, 11^4, 13^2, 17^3, 19^2, 29, 43^3$
90	$3^{12}, 5^3, 7, 11, 37, 41, 61^2, 277$	174	$3^{59}, 5^{17}, 7^9, 11^3, 13^2, 17^3, 19^4, 23, 29^2, 61$
92	$3^{13}, 5^4, 7, 11, 13, 23$	176	$3^{60}, 5^{19}, 7^9, 11^3, 13^2, 17^2, 23, 29^3, 37, 41, 71, 89^2$
94	$3^{14}, 5^3, 7, 17, 19$	178	$3^{61}, 5^{18}, 7^{10}, 11^5, 13^2, 17, 19, 41, 53$
96	$3^{15}, 5^4, 7, 41, 43$	180	$3^{63}, 5^{20}, 7^{10}, 11^5, 13^4, 17, 23, 41, 43, 61^3, 73^2$
98	$3^{15}, 5^3, 7, 11, 29$	182	$3^{65}, 5^{19}, 7^{10}, 11^4, 13^3, 17^3, 23, 29, 53^2, 79, 277$
100	$3^{16}, 5^5, 7^2, 11, 41, 61, 401$	184	$3^{66}, 5^{20}, 7^{11}, 11^4, 13^4, 17^3, 19^2, 23^3, 31^2, 37^2, 43, 47$
102	$3^{17}, 5^4, 7^2, 11, 17, 19, 67$	186	$3^{67}, 5^{20}, 7^{12}, 11^4, 13^4, 17^2, 19, 23^2, 31, 103, 137$
104	$3^{18}, 5^5, 7^2, 13, 17^2, 53^2, 79$	188	$3^{69}, 5^{21}, 7^{11}, 11^6, 13^4, 17^3, 19^3, 23^2, 29, 31^2, 47^3, 139, 397$
106	$3^{18}, 5^4, 7^3, 13, 523$	190	$3^{71}, 5^{21}, 7^{11}, 11^5, 13^4, 17^3, 19^4, 41, 113$
108	$3^{19}, 5^5, 7^3, 11, 13, 31, 43, 73^2$	192	$3^{73}, 5^{23}, 7^{12}, 11^5, 13^6, 17^2, 19^6, 43^3, 53$
110	$3^{20}, 5^5, 7^2, 11^2, 17, 41$	194	$3^{74}, 5^{21}, 7^{11}, 11^4, 13^3, 17, 19^2, 107$
112	$3^{21}, 5^6, 7^3, 11^2, 19, 229$	196	$3^{76}, 5^{23}, 7^{13}, 11^4, 13^5, 17, 19^3, 23, 29, 37^2, 67, 73, 113$
114	$3^{22}, 5^6, 7^4, 11, 23, 53$	198	$3^{78}, 5^{22}, 7^{13}, 11^6, 13^4, 17^3, 19, 23, 31, 37, 53, 71, 109, 401^2$
116	$3^{23}, 5^7, 7^3, 11, 13, 19, 29, 43$	200	$3^{80}, 5^{26}, 7^{13}, 11^6, 13^4, 17^3, 19, 29, 43, 61$
118	$3^{24}, 5^6, 7^3, 11^2, 13, 17$		

TABLE 4. Excess homology $H_3^{\dagger}(\mathrm{GL}(3,\mathbb{Z}), V_g(\mathbb{F}_p))/F^0$ by weight g, for $3 \le p \le 541$, $g \le 200$

Next we break down the dimensions given in Tables 3 and 4 into type 1 homology F^1/F^0 on the one hand and type 2 plus quasicuspidal homology $H_3^\dagger(\mathrm{GL}(3,\mathbb{Z}), V_g(\mathbb{F}_p))/F^1$ on the other. The results for F^1/F^0 are shown in Table 5.

g	$p=3$	$p=5$	g	$p=3$	$p=5$	$p=7$	g	$p=3$	$p=5$	$p=7$	g	$p=3$	$p=5$	$p=7$
36	1		78	9	2		120	25	8	1	162	49	15	3
38	1		80	9	3		122	26	6	1	164	51	16	4
40	1		82	9	2		124	27	8	1	166	53	15	4
42	1		84	10	3		126	28	7	2	168	54	16	5
44	2		86	11	2		128	29	8	2	170	55	16	5
46	2		88	12	3		130	30	8	1	172	57	18	4
48	2		90	12	3		132	31	9	2	174	59	17	5
50	2		92	13	4		134	32	8	2	176	60	19	5
52	3	1	94	14	3		136	33	9	2	178	61	18	4
54	3	1	96	15	4		138	34	9	2	180	63	20	5
56	3	1	98	15	3		140	36	11	2	182	65	19	5
58	3	1	100	16	5	1	142	37	10	3	184	66	20	6
60	4	1	102	17	4	1	144	38	11	3	186	67	20	6
62	5	1	104	18	5	1	146	39	10	2	188	69	21	5
64	5	1	106	18	4	1	148	41	11	3	190	71	21	6
66	5	1	108	19	5	1	150	42	12	3	192	73	23	6
68	6	1	110	20	5	1	152	43	14	3	194	74	21	5
70	7	1	112	21	6	1	154	44	13	4	196	76	23	7
72	7	2	114	22	6	1	156	46	14	3	198	78	22	7
74	7	1	116	23	7	1	158	47	13	4	200	80	26	8

TABLE 5. Dimension of type 1 homology F^1/F^0, for $g \leq 200$ and $3 \leq p \leq 541$. Combinations (g, p) not shown have dimension zero.

g	dim	g	dim	g	dim	g	dim	g	dim	g	dim	g	dim
58	1	86	1	106	2	126	2	146	3	166	4	186	6
64	1	88	1	108	2	128	2	148	3	168	4	188	6
70	1	90	1	110	1	130	3	150	3	170	5	190	5
72	1	92	1	112	2	132	3	152	3	172	6	192	6
74	1	94	1	114	3	134	2	154	4	174	4	194	6
76	1	96	1	116	2	136	3	156	5	176	4	196	6
78	1	98	1	118	2	138	3	158	3	178	6	198	6
80	1	100	1	120	3	140	3	160	4	180	5	200	5
82	1	102	1	122	3	142	2	162	5	182	5		
84	1	104	1	124	3	144	3	164	5	184	5		

TABLE 6. Dimension of type 2 plus quasicuspidal homology $H_3^\dagger(\mathrm{GL}(3,\mathbb{Z}), V_g(\mathbb{F}_p))/F^1$, for $p = 7$ and $g \leq 200$.

g	$p=3$	$p=5$	$p=7$	$p=11$	g	$p=3$	$p=5$	$p=7$	$p=11$	$p=13$	$p=17$	$p=19$
36	1				120	25	8	4	1			
38	1				122	26	6	3	1			
40	1				124	27	8	4	1			
42	1				126	28	7	4	1			
44	2				128	29	8	4	1			
46	2				130	30	8	4	1	1		
48	2				132	31	9	5	1	1		
50	2				134	32	8	4	1	1		
52	3	1			136	33	9	5	1	1		
54	3	1			138	34	9	5	1	1		
56	3	1			140	36	11	5	1	1		
58	3	1			142	37	10	5	1	1		
60	4	1			144	38	11	6	1	1		
62	5	1			146	39	10	5	1	1		
64	5	1			148	41	11	6	1	1		
66	5	1			150	42	12	6	1	1		
68	6	1			152	43	14	6	1	1		
70	7	1	1		154	44	13	8	2	1		
72	7	2	1		156	46	14	8	2	1		
74	7	1	1		158	47	13	7	2	1		
76	8	2	1		160	48	15	8	2	1		
78	9	2	1		162	49	15	8	2	1		
80	9	3	1		164	51	16	8	2	1		
82	9	2	1		166	53	15	8	2	1		
84	10	3	1		168	54	16	9	2	1		
86	11	2	1		170	55	16	9	2	1	1	
88	12	3	1		172	57	18	9	2	1	1	
90	12	3	1		174	59	17	9	2	1	1	
92	13	4	1		176	60	19	9	3	1	1	
94	14	3	1		178	61	18	9	3	1	1	
96	15	4	1		180	63	20	10	3	1	1	
98	15	3	1		182	65	19	10	3	2	1	
100	16	5	2		184	66	20	11	3	2	1	
102	17	4	2		186	67	20	12	3	2	1	
104	18	5	2		188	69	21	11	3	2	1	
106	18	4	2		190	71	21	11	3	2	1	1
108	19	5	3		192	73	23	12	3	2	1	1
110	20	5	2	1	194	74	21	11	3	2	1	1
112	21	6	3	1	196	76	23	13	3	2	1	1
114	22	6	4	1	198	78	22	13	4	2	1	1
116	23	7	3	1	200	80	26	13	4	2	1	1
118	24	6	3	1								

TABLE 7. Dimension of type 1 and 2 boundary homology $(T^1 \vee T^{2a} \vee T^{2b})/F^0$, for $g \leq 200$ and $3 \leq p \leq 541$. Combinations (g, p) not shown have dimension zero.

It turns out that the dimensions for $p = 3$ and $p = 5$ are the same as in Table 3, so there is no type 2 plus quasicuspidal homology for these values of p (always for $g \leq 200$). Further, for $p \geq 11$, there is no type 1 homology F^1/F^0, so the dimension of the type 2 plus quasicuspidal homology can be read off from Tables 3 and 4. There remains the case $p = 7$, where both types of homology are present: this is shown in Table 6. In the range of these tables, T^1 is always contained in the span of T^{2a} and T^{2b}.

Table 7 shows the type 1 and 2 boundary homology, and Table 8 the excess boundary homology. Tables 9 and Tables 10 show the quasicuspidal homology, organized by p and g, respectively.

g	1	2a	2b	2	∂	g	1	2a	2b	2	∂	g	1	2a	2b	2	∂
	$p = 7$						$p = 7$						$p = 7$				
70–98	0	0	1	1	1	136	2	2	4	5	5	168	5	5	8	9	9
100–106	1	0	2	2	2	138	2	2	4	5	5	170	5	5	8	9	9
108	1	1	2	3	3	140	2	2	4	5	5	172	4	5	7	9	9
110	1	0	2	2	2	142	3	2	5	5	5	174	5	5	8	9	9
112	1	1	3	3	3	144	3	3	5	6	6	176	5	5	8	9	9
114	1	1	3	4	4	146	2	2	4	5	5	178	4	5	7	9	9
116	1	0	3	3	3	148	3	3	5	6	6	180	5	6	8	10	10
118	1	0	3	3	3	150	3	3	5	6	6	182	5	5	9	10	10
120	1	1	3	4	4	152	3	3	5	6	6	184	6	6	10	11	11
122	1	0	3	3	3	154	4	4	7	8	8	186	6	7	10	12	12
124	1	1	3	4	4	156	3	4	6	8	8	188	5	6	9	11	11
126	2	1	4	4	4	158	4	3	7	7	7	190	6	6	10	11	11
128	2	1	4	4	4	160	4	4	7	8	8	192	6	7	10	12	12
130	1	1	3	4	4	162	3	4	6	8	8	194	5	6	9	11	11
132	2	2	4	5	5	164	4	4	7	8	8	196	7	8	11	13	13
134	2	1	4	4	4	166	4	4	7	8	8	198	7	7	11	13	13

g	1	2a	2b	2	∂
	$p = 7$				
200	8	7	12	13	13
	$p = 11$				
110–152	0	0	1	1	1
154–174	0	0	2	2	2
176–196	0	0	3	3	3
198–200	0	0	4	4	4
	$p = 13$				
130–180	0	0	1	1	1
182–200	0	0	2	2	2
	$p = 17$				
170–200	0	0	1	1	1
	$p = 19$				
190–200	0	0	1	1	1

TABLE 8. The columns labeled ∂ show the dimension of the boundary homology modulo F^0, for $g \leq 200$ and $7 \leq p \leq 541$. Note that it is all type 2. The remaining columns show the composition of the excess boundary homology. Combinations (g, p) not shown have dimension zero.

p	g: a superscript indicates the dimension of the solution space when greater than 1
7	$58, 64, 106, 122, 164, 170, 172, 178$
11	$68, 78, 88, 90, 92, 98, 100, 102, 108, 110, 112, 118, 120, 122, 128, 130^2, 132, 134, 136, 138^2, 140^2, 142^2, 144, 146, 148^2, 150^2, 152^2, 156, 158, 160, 162^2, 164, 166, 168^2, 170^2, 172^2, 174, 178^2, 180^2, 182, 184, 186, 188^3, 190^2, 192^2, 194, 196, 198^2, 200^2$
13	$80, 92, 104, 106, 108, 116, 118, 120, 126, 128, 130, 132^2, 134, 136, 138, 140, 142, 144^2, 146, 148, 150, 152, 154, 156^3, 158, 160, 162, 164, 166, 168^3, 170, 172, 174, 176, 178, 180^3, 182, 184^2, 186^2, 188^2, 190^2, 192^4, 194, 196^3, 198^2, 200^2$
17	$94, 102, 104^2, 110, 118, 120^2, 126, 134^2, 136^2, 138, 140, 142, 150^2, 152^2, 154, 156, 158, 166^2, 168^2, 170, 172^2, 174^2, 176, 182^2, 184^2, 186, 188^2, 190^2, 192, 198^2, 200^2$
19	$66, 76, 84, 94, 102, 112, 116, 120, 130, 134, 138^2, 148^2, 150, 152^2, 154^2, 156^4, 166^2, 168, 170^2, 172^2, 174^4, 178, 184^2, 186, 188^3, 190^3, 192^5, 194, 196^2$

TABLE 9. Quasicuspidal homology $H_3^{\mathrm{qc}}(\mathrm{GL}(3,\mathbb{Z}), V_g(\mathbb{F}_p))$ by prime p, for $p \leq 19$, $g \leq 200$. For the range $23 \leq p \leq 541$, $g \leq 200$, the results are identical with those shown in Table 3.

g	$p^{\dim}$	g	$p^{\dim}$	g	$p^{\dim}$
42	29	106	$7, 13, 523$	156	$11, 13^3, 17, 19^4, 53^2, 79^2$
44	97	108	$11, 13, 31, 43, 73^2$	158	$11, 13, 17, 23, 43$
48	191	110	$11, 17, 41$	160	$11, 13, 23, 29, 37^2, 61, 67$
50	41	112	$11, 19, 229$	162	$11^2, 13, 23^2, 37, 109^4$
52	$53, 211$	114	$23, 53$	164	$7, 11, 13, 41^2, 47$
54	$37, 181$	116	$13, 19, 29, 43$	166	$11, 13, 17^2, 19^2, 53$
58	7	118	$11, 13, 17$	168	$11^2, 13^3, 17^2, 19, 29, 31, 41, 67, 113^4$
60	$41, 61, 239$	120	$11, 13, 17^2, 19, 61^3$	170	$7, 11^2, 13, 17, 19^2, 37, 41$
64	7	122	$7, 11$	172	$7, 11^2, 13, 17^2, 19^2, 29, 43^3$
66	$19, 43$	124	$31^2, 37^2, 73, 307$	174	$11, 13, 17^2, 19^4, 23, 29^2, 61$
68	11	126	$13, 17, 29, 37, 109, 257$	176	$13, 17, 23, 29^3, 37, 41, 71, 89^2$
70	$29, 41$	128	$11, 13, 71$	178	$7, 11^2, 13, 19, 41, 53$
76	19	130	$11^2, 13, 19, 41, 53^2, 79$	180	$11^2, 13^3, 23, 41, 43, 61^3, 73^2$
78	$11, 53^2, 79, 113$	132	$11, 13^2, 89^3$	182	$11, 13, 17^2, 23, 29, 53^2, 79, 277$
80	$13, 61$	134	$11, 13, 17^2, 19, 37$	184	$11, 13^2, 17^2, 19^2, 23^3, 31^2, 37^2, 43, 47$
84	$19, 103, 181$	136	$11, 13, 17^2, 23, 41, 83$	186	$11, 13^2, 17, 19, 23^2, 31, 103, 137$
88	$11, 37, 89, 107, 463$	138	$11^2, 13, 17, 19^2, 23, 31, 43, 47$	188	$11^3, 13^2, 17^2, 19^3, 23^2, 29, 31^2, 47^3, 139, 397$
90	$11, 37, 41, 61^2, 277$	140	$11^2, 13, 17, 23^2, 37, 41, 61, 97, 113$	190	$11^2, 13^2, 17^2, 19^3, 41, 113$
92	$11, 13, 23$	142	$11^2, 13, 17$	192	$11^2, 13^4, 17, 19^5, 43^3, 53$
94	$17, 19$	144	$11, 13^2, 29, 97^3$	194	$11, 13, 19, 107$
96	$41, 43$	146	$11, 13, 107, 127$	196	$11, 13^3, 19^2, 23, 29, 37^2, 67, 73, 113$
98	$11, 29$	148	$11^2, 13, 19^2, 37^2, 89$	198	$11^2, 13^2, 17^2, 23, 31, 37, 53, 71, 109, 401^2$
100	$11, 41, 61, 401$	150	$11^2, 13, 17^2, 19, 41, 43^3, 61^3, 101^4, 199$	200	$11^2, 13^2, 17^2, 29, 43, 61$
102	$11, 17, 19, 67$	152	$11^2, 13, 17^2, 19^2$		
104	$13, 17^2, 53^2, 79$	154	$13, 17, 19^2, 29, 31^2$		

TABLE 10. Quasicuspidal homology $H_3^{\mathrm{qc}}(\mathrm{GL}(3,\mathbb{Z}), V_g(\mathbb{F}_p))$ by weight g, for $p \leq 541,\ g \leq 200$.

9. THE HECKE ACTION ON $H_3^\dagger(\Gamma, V)$

Define V^* as the contragredient of V. So $V^* = \mathrm{Hom}(V, \mathbb{F})$, and if $s \in S = \mathrm{GL}_3(\mathbb{Z}_p)$ and $v^* \in V^*$ we have $(v^* s)(v) = v^*(vs^{-1})$. Then the pairing $\langle\,\cdot\,,\cdot\,\rangle : V \times V^* \to \mathbb{F}$ defined by $\langle v, v^*\rangle = v^*(v)$ is S-equivariant, hence also $\mathrm{GL}_3(\mathbb{F})$-equivariant. Let $\alpha \in H_3^\dagger(\Gamma, V)$. As in Theorem 2.1, denote by $v = v(x, y, z) \in V$ the vector corresponding to α.

Let $B_i(k) := B_i$ be such that

$$\coprod_i \Gamma B_i = \begin{cases} \Gamma \begin{pmatrix} 1 & & \\ & 1 & \\ & & l \end{pmatrix} \Gamma & \text{if } k = 1, \\[2ex] \Gamma \begin{pmatrix} 1 & & \\ & l & \\ & & l \end{pmatrix} \Gamma & \text{if } k = 2. \end{cases}$$

See Section 3 for notations concerning modular symbols and the definition of the pairing between homology and modular symbols. As in [Ash et al. 1984] we define the action of a Hecke operator on homology to be the contragredient of its action on the modular symbols. Then:

Lemma 9.1. *For $v \in V$ and $v^* \in V^*$ we have*

$$\langle T(l,k)\alpha, [I, v^*]\rangle = \sum_{i,j} \langle v M_{ij} B_i, v^*\rangle,$$

where the M_{ij} are unimodular matrices such that $\sum_j [M_{ij}]$ is homologous to $[B_i]$.

Proof. For $k = 1, 2$,

$$
\begin{aligned}
\langle T(l,k)\alpha,\,[I,v^*]\rangle &= \sum_i \langle \alpha,\,[I,v^*]B_i^{-1}\rangle \\
&= \sum_i \langle \alpha,\,[B_i^{-1},v^*B_i^{-1}]\rangle \\
&= \sum_{i,j} \langle \alpha,\,[M_{ij},v^*B_i^{-1}]\rangle \\
&= \sum_{i,j} \langle \alpha,\,[I,v^*B_i^{-1}M_{ij}^{-1}]M_{ij}\rangle \\
&= \sum_{i,j} \langle \alpha,\,[I,v^*B_i^{-1}M_{ij}^{-1}]\rangle \\
&= \sum_{i,j} \langle v,\,v^*B_i^{-1}M_{ij}^{-1}\rangle \quad \text{(by (3–1))} \\
&= \sum_{i,j} \langle vM_{ij}B_i,v^*\rangle. \qquad \square
\end{aligned}
$$

Computation and Results

The unimodular matrices M_{ij} were first computed using the algorithm presented in [Ash and Rudolph 1979]. Later this was changed to the algorithm of [van Geemen et al. 1997] because it produced a shorter chain of unimodular symbols for each Hecke operator. This was critical for larger primes l as it greatly reduced the number of arithmetic computations being done. Both algorithms were implemented in C.

The Hecke action was then computed using the formula stated in Lemma 9.1. The time consuming step was finding vs in terms of the standard basis of monomials, given $v \in V_g$ and $s \in M_3(\mathbb{Z})$. The naive approach of just expanding the polynomials worked in a reasonable amount of time only for small degree g and for small primes l.

Numerical evaluation was then tried where $p > g$. This required choosing multiple random points in $\mathbb{F}_p^3$, evaluating both $vM_{ij}B_i$ and each basis element v_i of the homology space at these points. Row reduction over $\mathbb{F}_p$ was then used to determine the coefficients of each basis element in v and hence the Hecke action. This worked significantly better, but still ran into time constraints.

The last method involved taking multiple partial derivatives. Due to the row reduction used to find a basis for the homology space, each basis polynomial had a unique "leading term" with coefficient one. After applying the linear transformations $M_{ij}B_i$ to the variables x, y, z in v, one variable was set to 1 (there was no loss of information because the polynomials are homogeneous and this reduced the number of derivatives that were necessary). Derivatives were taken to reduce the specified term to a constant and then the other two variables were set to zero to eliminate higher order terms. Taking this constant and dividing by two factorials to counter the effects of the derivatives isolated the coefficient of the leading term of the basis polynomial in v. This was then the correct coefficient for the entire basis polynomial. Here the polynomial was treated as if over $\mathbb{Q}$ for the purpose of the derivatives and dividing by factorials, but was then reduced modulo p. This worked relatively quickly and was the preferred method because it worked for both small and large p.

All the polynomial manipulation (numerical evaluation and the derivatives) was done using Mathematica, but the row reduction was done in C.

The programs to compute the unimodular symbols were checked by hand for numerous small examples to verify accuracy. The results were also checked in many cases where the degree g was small by verifying that the transformations $T(l,1)$ and $T(l,2)$ commuted. This check was not employed when the derivative algorithm was used, but the derivative algorithm was also run on cases of lesser degree to duplicate previous results and thus verify accuracy.

Other strong evidence of the accuracy of the program is that candidates for Galois representations can often be found. Also the eigenvalues for $p = 7, 11$ repeated as the weight g increased by steps of $p-1$ as was suggested by equation (4–1). For the symmetric squares cases the results obtained from the program matched those predicted by the theory.

The results of our Hecke computations are shown in Table 11. The complexity of the computations increased with g and with l; Therefore we were limited to computing a representative sample of what

g	p	l	a_l	b_l	$P_l(x)$	G
42	29	2	26	3	$(1+x)(1+15x+x^2)$	ss
		3	27	2	$(1+x)(8+x)(11+x)$	
		5	4	4	$(28+x)(1+6x+x^2)$	
		7	26	26	$(23+x)(24+x)(28+x)$	
		11	6	23	$(1+x)(1+9x+x^2)$	
		13	17	17	$(9+x)(13+x)(28+x)$	
		17	26	3	$(1+x)(1+6x+x^2)$	
		19	19	10	$(1+x)(28+x)^2$	
44	97	2	58	86	$(19+x)(70+x)(76+x)$	b
		3	66	86	$(57+x)(5+18x+x^2)$	
		5	49	22	$(8+x)(9+x)(12+x)$	
		7	59	93	$(39+x)(48+x)(57+x)$	
		11	46	32	$(51+x)(13+33x+x^2)$	
		13	17	36	$(87+x)(30+46x+x^2)$	
		17	73	55	$(90+x)(45+94x+x^2)$	
		19	9	80	$(33+x)(1+38x+x^2)$	
		23	76	0	$24+22x^2+x^3$	
		29	62	31	$9+78x+38x^2+x^3$	
48	191	2	164	145	$57+168x+109x^2+x^3$	b
		3	76	151	$(101+x)(13+x+x^2)$	
		5	164	169	$(188+x)(59+123x+x^2)$	
		7	131	55	$(70+x)(24+75x+x^2)$	
		11	90	23	$84+141x+96x^2+x^3$	
		13	33	96	$(122+x)(68+184x+x^2)$	
50	41	2	7	11	$(8+x)(36+9x+x^2)$	b
		3	40	32	$(36+x)(10+19x+x^2)$	
		5	19	17	$(17+x)(29+12x+x^2)$	
		7	25	26	$(26+x)(17+29x+x^2)$	
		11	39	36	$(21+x)(23+9x+x^2)$	
		13	25	24	$(2+x)(16+15x+x^2)$	
		17	3	28	$9+33x+36x^2+x^3$	
		19	16	24	$(7+x)(13+31x+x^2)$	
52	53	2	23	26	$(1+x)(52+14x+x^2)$	b
		3	27	24	$(32+x)(48+12x+x^2)$	
		5	1	1	$(52+x)(1+22x+x^2)$	
		7	50	50	$(13+x)(49+x)(52+x)$	
		11	1	1	$(33+x)(45+x)(52+x)$	
		13	36	1	$(26+x)(2+12x+x^2)$	
		17	22	22	$(52+x)(1+34x+x^2)$	
		19	42	33	$(31+x)(41+17x+x^2)$	
		23	17	14	$52+49x+20x^2+x^3$	
52	211	2	119	25	$(138+x)(154+119x+x^2)$	b
		3	45	142	$(95+x)(108+x)(204+x)$	
		5	84	155	$198+31x+152x^2+x^3$	
		7	112	72	$(43+x)(166+152x+x^2)$	
		11	57	105	$27+163x+14x^2+x^3$	
		13	49	101	$(140+x)(58+51x+x^2)$	
54	37	2	35	2	$(1+x)(6+x)(31+x)$	ss
		3	35	35	$(21+x)(30+x)(36+x)$	
		5	1	36	$(1+x)(2+x)(19+x)$	
		7	2	2	$(36+x)(1+6x+x^2)$	
		11	35	35	$(36+x)(1+18x+x^2)$	
		13	14	23	$(1+x)(23+x)(29+x)$	
54	181	2	68	0	$(22+x)(146+125x+x^2)$	b
		3	146	7	$46+123x+72x^2+x^3$	
		5	29	24	$(15+x)(21+x)(103+x)$	
		7	55	106	$1+41x+18x^2+x^3$	
		11	164	127	$(30+x)(26+169x+x^2)$	
		13	127	145	$(12+x)(161+20x+x^2)$	
60	41	2	40	40	$(40+x)(1+22x+x^2)$	ss
		3	36	5	$(1+x)(34+x)(35+x)$	
		5	40	40	$(16+x)(18+x)(40+x)$	
		7	40	1	$(1+x)(17+x)(29+x)$	
		11	3	38	$(1+x)(12+x)(24+x)$	
		13	22	19	$(1+x)(13+x)(19+x)$	
60	61	2	27	34	$(26+x)(7+52x+x^2)$	b
		3	4	51	$(1+x)(60+18x+x^2)$	
		5	7	54	$(17+x)(27+x)(40+x)$	
		7	17	38	$(38+x)(48+x)(51+x)$	
		11	33	26	$(14+x)(13+44x+x^2)$	
		13	50	3	$(40+x)(32+50x+x^2)$	
		17	60	32	$(11+x)(21+x)(47+x)$	
		19	31	3	$(16+x)(23+x)(30+x)$	
		23	4	6	$(42+x)(45+48x+x^2)$	
		29	21	54	$(8+x)(38+6x+x^2)$	
		31	3	31	$60+x+55x^2+x^3$	
60	239	2	101	54	$(237+x)(169+71x+x^2)$	b
		3	76	10	$(217+x)(169+156x+x^2)$	
		5	19	39	$(226+x)(223+57x+x^2)$	
		7	50	92	$94+218x+27x^2+x^3$	
		11	78	88	$(38+x)(166+x)(180+x)$	
		13	63	116	$86+156x+179x^2+x^3$	

TABLE 11. Hecke eigenvalues on $H_3^\dagger(\Gamma, V)$ by weight g. (Continued on next page.)

g	p	l	a_l	b_l	$P_l(x)$	G
78	79	2	78	78	$(7+x)(34+x)(78+x)$	b
		3	70	7	$(4+x)^2(74+x)$	
		5	12	59	$78+75x+45x^2+x^3$	
		7	5	68	$(16+x)(74+51x+x^2)$	
		11	64	17	$(8+x)(69+58x+x^2)$	
		13	10	57	$(63+x)(5+76x+x^2)$	
78	113	2	51	27	$30+70x+31x^2+x^3$	b
		3	23	25	$13+46x+30x^2+x^3$	
		5	74	12	$(26+x)(1+27x+x^2)$	
		7	91	73	$(4+x)(98+x)(111+x)$	
		11	106	43	$(58+x)(54+107x+x^2)$	
84	103	2	64	63	$(31+x)(79+40x+x^2)$	b
		3	97	19	$69+75x+2x^2+x^3$	
		5	47	32	$(8+x)(18+x)(47+x)$	
		7	90	82	$(93+x)(42+56x+x^2)$	
84	181	2	158	1	$(28+x)(10+74x+x^2)$	b
		3	97	68	$36+83x+28x^2+x^3$	
		5	170	25	$46+5x+147x^2+x^3$	
		7	78	8	$180+27x+144x^2+x^3$	
		11	79	173	$154+98x+108x^2+x^3$	
		13	175	179	$64+153x+84x^2+x^3$	
88	89	2	20	67	$88+78x+79x^2+x^3$	b
		3	33	54	$(2+x)(44+76x+x^2)$	
		5	23	70	$88+14x+31x^2+x^3$	
		7	24	51	$(1+x)(55+x)^2$	
		11	13	70	$88+63x+15x^2+x^3$	
		13	41	46	$(29+x)(57+x)(82+x)$	
88	107	2	33	1	$(1+x)(47+x)(96+x)$	b
		3	10	7	$77+38x+68x^2+x^3$	
		5	71	11	$(1+x)(103+49x+x^2)$	
		7	32	49	$(73+x)^2(94+x)$	
		11	95	52	$45+102x+40x^2+x^3$	
		13	95	70	$60+63x+75x^2+x^3$	
88	463	2	36	44	$(146+x)(353+299x+x^2)$	b
		3	251	351	$(274+x)(116+414x+x^2)$	
		5	216	313	$(72+x)(185+x)(348+x)$	
		7	135	354	$(235+x)(213+341x+x^2)$	
		11	369	34	$(322+x)(254+360x+x^2)$	
		13	311	189	$145+157x+261x^2+x^3$	

g	p	l	a_l	b_l	$P_l(x)$	G
90	277	2	2	139	$(30+x)(247+x)(276+x)$	$\hat{A}_4$
		3	119	77	$(185+x)(238+x)(276+x)$	
		5	122	251	$(143+x)(166+x)(276+x)$	
		7	167	137	$(198+x)(254+x)(276+x)$	
		11	171	95	$(126+x)(212+x)(276+x)$	
		13	13	64	$(38+x)(239+x)(276+x)$	
100	401	2	365	347	$(24+x)(117+395x+x^2)$	b
		3	154	322	$381+241x+216x^2+x^3$	
		5	62	364	$(83+x)(130+x)(336+x)$	
		7	240	400	$(33+x)(158+391x+x^2)$	
		11	80	399	$1+255x+175x^2+x^3$	
		13	65	93	$(249+x)(327+147x+x^2)$	
106	523	2	254	264	$399+132x+396x^2+x^3$	b
		3	343	164	$(398+x)(342+185x+x^2)$	
		5	3	104	$(241+x)(440+x)(469+x)$	
		7	497	266	$98+38x+452x^2+x^3$	
		11	178	250	$(97+x)(161+77x+x^2)$	
		13	329	278	$(57+x)(368+320x+x^2)$	
112	229	2	2	114	$(114+x)(115+x)(228+x)$	S_3
		3	2	152	$(45+x)(108+x)(228+x)$	
		5	4	45	$(19+x)(27+x)(228+x)$	
		7	7	98	$(98+x)(131+x)(228+x)$	
		11	10	124	$(158+x)(196+x)(228+x)$	
		13	13	88	$(88+x)(141+x)(228+x)$	
124	307	2	267	198	$(81+x)(128+246x+x^2)$	b
		3	238	171	$(302+x)(197+28x+x^2)$	
		5	136	49	$74+194x+157x^2+x^3$	
		7	254	4	$(286+x)(162+204x+x^2)$	
		11	165	46	$(103+x)(220+x)(276+x)$	
		13	266	168	$(189+x)(69+192x+x^2)$	
126	257	2	1	128	$(256+x)(193+129x+x^2)$	S_3
		3	3	171	$(86+x)(171+x)(256+x)$	
		5	5	154	$(103+x)(154+x)(256+x)$	
		7	7	110	$(110+x)(147+x)(256+x)$	
		11	10	186	$(256+x)(17+187x+x^2)$	
		13	12	177	$(256+x)(73+178x+x^2)$	
		17	16	120	$(256+x)(249+121x+x^2)$	
		19	19	27	$(27+x)(230+x)(256+x)$	

TABLE 11 (continued). Hecke eigenvalues on $H_3^{\dagger}(\Gamma, V)$ by weight g.

we considered to be interesting cases with small g and over a small range of l's. The size of p was no obstacle, but when $p < g$ the numerical evaluation method of computing vs had to be avoided.

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