Enumerating $A_3(2)$ Blueprints, and an Application

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CONTENTS

We present an enumeration scheme for all blueprints of an $A_2(p)$ building (with p prime). We then provide a computer based proof that the $\widetilde{A}_n(\mathbb{Q}_2)$ buildings do not conform to a blueprint for $n \geq 3$.

J. Tits ([1955b; 1955a; 1956]; see the introduction of [Tits 1974] for a historical discussion) introduced buildings in an attempt to provide a systematic procedure for producing geometries associated to semisimple Lie groups. With this procedure, Tits found geometries associated to the exceptional groups E_6, E_7, and E_8. One of the early hopes for buildings was to provide a way to combinatorially construct the finite groups of Lie type, and in [Ronan and Tits 1987], this was accomplished using blueprints. Ronan and Tits, however, were unable to find blueprints constructing general p-adic affine buildings. At the end of their paper though, they suggested a generalization of a blueprint which might be used.

Abramson [$\geq$ 1998] presented a generalization of a blueprint like that suggested by Ronan and Tits. He used this to construct the p-adic buildings. The question of whether in general p-adic buildings conform to blueprints was left unanswered.

Currently there is only one known case of a p-adic building with a blueprint. Ronan ([1984]; see [Kantor 1986] for a complete discussion) presented four buildings of type $\widetilde{A}_2$ associated to the Frobenius group of order 21. P. Köhler, T. Meixner, and M. Wester proved that one of these corresponded to the classical $\widetilde{A}_2$ building associated to the field $\mathbb{F}_2((t))$ [Köhler et al. 1984] and a second corresponded to the 2-adic building of type $\widetilde{A}_2$ over the

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field $\mathbb{Q}_2$ [Köhler et al. 1985]. These presentations lead naturally to blueprints.

Here we show, using computer evidence, that most $\mathbb{Q}_2$ buildings do not conform to blueprints. To do this we first prove:

Theorem 1. *The only blueprints for the $A_3(2)$ building are relabelings of the natural blueprint.*

From this we obtain:

Theorem 2. *The simply-laced affine buildings of rank at least 4 (with connected diagram) over $\mathbb{Q}_2$ do not conform to blueprints.*

1. DEFINITIONS

A *chamber system* $(\mathcal{C}, \{P_i\}_{i\in I})$ is a set $\mathcal{C}$ of *chambers* and a set $\{P_i\}_{i\in I}$ of partitions of $\mathcal{C}$. We say that two chambers $C \neq D$ of $\mathcal{C}$ are *i-adjacent* if C and D lie in the same part of the partition P_i, and we write $C \underset{i}{\sim} D$. A *gallery* from C to D is a sequence of chambers $C = C_0, C_1, \dots, C_n = D$, such that $C_{j-1} \underset{i_j}{\sim} C_j$ for $j = 1, \dots, n$. The *type* of the gallery $C_0 \underset{i_1}{\sim} C_1 \underset{i_2}{\sim} \cdots \underset{i_n}{\sim} C_n$ is the word $i_1 i_2 \dots i_n$, and its *length* is n. A gallery from C to D is *minimal* if there exists no shorter gallery from C to D.

Let $M = (m_{ij})_{i,j\in I}$ be a symmetric integer matrix with diagonal entries 1 having $m_{ij} \geq 2$ for $i \neq j$. The associated Coxeter diagram is a graph whose vertex set is labeled by the elements of I and whose edge set consists of edges labeled m_{ij} for all $m_{ij} \geq 3$. The diagram is *simply-laced* if $m_{ij} \leq 3$ for all i and j.

Given M as above let W be the Coxeter group

$$W = \langle r_i \mid i \in I,\, r_i^2 = (r_i r_j)^{m_{ij}} = 1 \rangle.$$

We say that W is *spherical* if $|W| < \infty$ and the *rank* of W is $|I|$. For $w \in W$ define the *length*, $l(w)$, to be the least k such that $w = r_{i_1} \dots r_{i_k}$. Given a word $f = i_1 i_2 \dots i_n$ define $r_f = r_{i_1} \dots r_{i_n}$. The word f is *reduced* if $l(r_f) = n$.

Let W be a Coxeter group with index set I. A chamber system $(\mathcal{C}, \{P_i\}_{i\in I})$ is a *building of type W* if there exists a W-distance function

$$\delta : \mathcal{C} \times \mathcal{C} \to W$$

such that for all reduced words f there exists a gallery of type f from C to D if and only if we have $\delta(C, D) = r_f$ [Tits 1981].

Let W be a Coxeter group with index set I, and let S_i be an index set for each $i \in I$. A *labeled word* of type f is a reduced word $f = i_1 i_2 \dots i_t$ together with a sequence $a = (a_1, \dots, a_t)$ where $a_t \in S_{i_t}$ for all t. We write this as

$$f^a = i_1^{a_1} i_2^{a_2} \dots i_t^{a_t}.$$

Define a chamber complex with $\mathcal{C}$ the set of labeled words and i-adjacency given by

$$f^a i^b \underset{i}{\sim} f^a i^c \underset{i}{\sim} f^a$$

where fi is reduced, a is a label for f, and $b, c \in S_i$. Given a word f, define $\mathcal{C}_f = \{f^a \in \mathcal{C}\}$.

Given i and j, let $p(i, j) = \dots ij$ be the word of length m_{ij}. If $I = \{i, j\}$ a *rank 2 equivalence* for $\mathcal{C}$ is a bijection $\varphi : \mathcal{C}_{p(i,j)} \to \mathcal{C}_{p(j,i)}$. Given a rank 2 equivalence φ, define $\mathcal{C}(\varphi)$ to be the chamber complex obtained by identifying $p(i, j)^a$ with $\varphi(p(i, j)^a)$ where for ease of notation, we write $p(i, j)^{\varphi(a)}$ for the latter. More precisely, the chambers of $\mathcal{C}(\varphi)$ are the equivalence classes of chambers of $\mathcal{C}$ where two equivalence classes are i-adjacent if they contain i-adjacent elements. The rank 2 equivalence φ is a *rank 2 blueprint* if $\mathcal{C}(\varphi)$ is a building of type W.

Example A. Consider the case where $m_{12} = 2$ and $\varphi(1^a 2^b) = 2^b 1^a$. Then we have $1^a 2^b \underset{1}{\sim} 2^b 1^c$ for all a, b, c in the appropriate label sets. In this case φ is a rank 2 blueprint.

Example B. Alternatively, if φ is defined by $\varphi(i^a j^b) = j^a i^b$ for all labels a and b, then $1^a 2^b \underset{1}{\sim} 2^b 1^c$ for all c, and φ is not a rank 2 blueprint since the shortest gallery from 1^a to $2^b 1^c$, is of length 3 when $a \neq b$.

Given W, I, S_i, and C as above, a *blueprint* for $\mathcal{C}$ is the collection

$$\mathcal{B} = \{\varphi_{ij} : \mathcal{C}_{p(i,j)} \to \mathcal{C}_{p(j,i)} \mid i \neq j \in I\}$$

satisfying $\varphi_{ij} = \varphi_{ji}^{-1}$ and such that φ_{ij} is a rank 2 blueprint for all 2-element subsets $\{i, j\}$ of I.

Given a blueprint $\mathcal{B}$, define the labeled words $h^a p(j,i)^b g^c$ and $h^a p(i,j)^{\varphi_{ij}(b)} g^c$ to be *elementarily $\mathcal{B}$-equivalent.* Let $\sim$ be the transitive closure of this relation, and define $\mathcal{C}(\mathcal{B}) = \mathcal{C}/\sim$; that is, the chambers of $\mathcal{C}(\mathcal{B})$ are the equivalence classes of chambers of $\mathcal{C}$ with two equivalence classes i-adjacent if they contain elements which are i-adjacent in $\mathcal{C}$. Following the notation of [Ronan and Tits 1987], we define a blueprint to be *realizable* if $\mathcal{C}(\mathcal{B})$ is a building. Conversely, given a building $\Delta = \mathcal{C}(\mathcal{B})$, we say that Δ *conforms to the blueprint* $\mathcal{B}$.

Associated to any Coxeter group with index set I is a diagram whose vertices correspond to the elements of I and where the vertices i and j are connected by 0, 1, 2, or 3 bonds, as the order of $r_i r_j$ is two, three, four, or six respectively. If the order of $r_i r_j$ is some other integer n, then we represent this on the diagram by joining vertices i and j by a bond with the label n.

The following special case of [Ronan and Tits 1987, Theorem 1] simplifies checking whether a blueprint is realizable.

Theorem 3 [Ronan and Tits 1987]. *A blueprint $\mathcal{B}$ associated to a simply-laced diagram is realizable if and only if its restriction to the following subdiagrams is also realizable*:

1. $A_1 \times A_1 \times A_1$: ○ ○ ○
2. $A_1 \times A_2$: ○ ○—○
3. A_3 : ○—○—○

Moreover in this case there is a unique building Δ conforming to $\mathcal{B}$.

The *natural blueprint* of type $A_2(p)$ (p a prime) is the rank 2 blueprint given by

$$\varphi(1^a 2^b 1^c) = 2^c 1^{ac-b} 2^a$$

where $a, b, c \in \mathbb{F}_p$.

The *natural blueprint* of type $(A_1 \times A_1)(p)$ is given by

$$\varphi(1^a 2^b) = 2^b 1^a,$$

where $a, b \in \mathbb{F}_p$.

2. ENUMERATING BLUEPRINTS

In [Ronan and Tits 1987] natural blueprints are used to construct geometries associated to the Chevalley groups and from this the Chevalley groups themselves. Other blueprints may also give rise to interesting groups.

For example, the Frobenius group of 21 elements gives rise to $A_2(2)$ blueprints (not natural) which can be used to construct a blueprint for the $\widetilde{A}_2(\mathbb{Q}_2)$ building in the following fashion. Define $\gamma : \mathbb{Z}_2^3 \to \mathbb{Z}_2^3$ by

$$\begin{aligned} \gamma(0,0,0) &= (1,0,0), & \gamma(1,0,0) &= (0,1,1),\\ \gamma(0,0,1) &= (0,0,0), & \gamma(1,0,1) &= (1,0,1),\\ \gamma(0,1,0) &= (0,1,0), & \gamma(1,1,0) &= (0,0,1),\\ \gamma(0,1,1) &= (1,1,1), & \gamma(1,1,1) &= (1,1,0). \end{aligned}$$

Now define $\varphi_{1,2}$ by $\varphi_{1,2}(1^a 2^b 1^c) = 1^x 2^y 1^z$ where $\gamma(a,b,c) = (x,y,z)$. Define $\varphi_{2,3}$ and $\varphi_{3,1}$ analogously, and then $\varphi_{2,1}$, $\varphi_{3,2}$, and $\varphi_{1,3}$ are then the respective inverses. By [Tits 1990, Remark 2], this is then a blueprint for the affine building of $\mathrm{SL}_3(\mathbb{Q}_2)$.

In this paper, our goal is to provide a starting point for answering the question of what buildings conform to blueprints. In particular, we will show that no buildings over $\mathbb{Q}_2$ with triangle-free simply-laced diagrams conform to a blueprint.

By Theorem 3, a blueprint is realizable if and only if it is realizable on the appropriate subdiagrams. As a result our strategy is to enumerate all rank 2 blueprints, and then check when three such blueprints combine to form a realizable blueprint.

Let $I = \{1, 2\}$, and let $S_1 = S_2 = \{0, \ldots, n\}$. Given a non-empty word $f = \ldots i$, let Σ_f denote the symmetric group on S_j where $j \neq i$. Define X to be the *set*

$$X = \mathrm{Sym}(S_1) \times \mathrm{Sym}(S_2) \times \prod_{\substack{f^a \in \mathcal{C} \\ l(f) < m_{1,2}}} \Sigma_f.$$

If $m_{1,2} = 3$, an arbitrary element of X has the form

$$\begin{aligned}\sigma = (&\sigma_1, \sigma_2, \sigma_{1^0}, \dots, \sigma_{1^n}, \sigma_{2^1}, \dots, \sigma_{2^n}, \\ &\sigma_{1^0 2^0}, \dots, \sigma_{1^0 2^n}, \sigma_{1^1 2^0}, \dots, \sigma_{1^n 2^n}, \\ &\sigma_{2^0 1^0}, \sigma_{2^0 1^1}, \dots, \sigma_{2^n 1^n}).\end{aligned}$$

Let W be a spherical Coxeter group of rank 2, and suppose $\{S_1, S_2\}$ is a labeling. Then for $\sigma \in X$ as above, define a bijection ψ_σ on $\mathcal{C}$ by

$$\psi_\sigma(i^a j^b \dots) = i^{\sigma_i(a)} j^{\sigma_{i^a}(b)} \dots.$$

The set $\Psi = \{\psi_\sigma \mid \sigma \in X\}$ is then a group under composition. Given a rank 2 blueprint φ, define $\psi_\sigma(\varphi) = \varphi^\sigma$ by

$$\varphi^\sigma(f^a) = \psi_\sigma(\varphi(\psi_\sigma^{-1}(f^a)))$$

where f is any reduced word of length $m_{i,j}$. This defines an action of G on the set $\{\psi_\sigma(\varphi) \mid \psi_\sigma \in G\}$.

Lemma 4. *With W, S_1, S_2, $\mathcal{C}$, σ, ψ_σ, and φ as above, the map φ^σ is a rank 2 blueprint for $\mathcal{C}$. Moreover, ψ_σ induces an isomorphism from $\mathcal{C}(\varphi)$ to $\mathcal{C}(\varphi^\sigma)$.*

Proof. We first show that ψ_σ maps equivalent elements of $\mathcal{C}(\varphi)$ to equivalent elements of $\mathcal{C}(\varphi^\sigma)$. If the equivalence class has only one element, there is nothing to check. Else, suppose $f^a \sim h^b$ in $\mathcal{C}(\varphi)$. Then $l(f) = m_{1,2}$ and $\varphi(f^a) = h^b$. By definition,

$$\varphi^\sigma(\psi_\sigma(f^a)) = \psi_\sigma(\varphi(f^a)) = \psi_\sigma(h^b).$$

Since ψ_σ maps i-adjacent elements of $\mathcal{C}(\varphi)$ to i-adjacent elements of $\mathcal{C}(\varphi^\sigma)$, it follows that ψ_σ induces an isomorphism of these two chamber systems. If φ is a rank 2 blueprint then $\mathcal{C}(\varphi)$ is a building and hence so is $\mathcal{C}(\varphi^\sigma)$. □

Lemma 5. *Given W, S_1, S_2, $\mathcal{C}$, Ψ, as above, if φ and γ are two rank 2 blueprints and $\alpha : \mathcal{C}(\varphi) \to \mathcal{C}(\gamma)$ is a type preserving isomorphism of chamber complexes fixing the base chamber, (the trivial labeled word $\varnothing$), then $\gamma = \varphi^\sigma$ for some $\psi_\sigma \in \Psi$ completely determined by α. In particular, if $\alpha : \mathcal{C}(\varphi) \to \mathcal{C}(\varphi)$, then $\alpha = \psi_\sigma$ for some $\sigma \in X$ with $\varphi^\sigma = \varphi$.*

Proof. Let α be as above. Then α lifts to an isomorphism $\hat\alpha$ on $\mathcal{C}$ defined by $\hat\alpha(f^a) = f^b$ where $\alpha([f^a]_\varphi) = [f^b]_\gamma$, where $[f^a]_\varphi$ denotes the equivalence class of f^a in $\mathcal{C}(\varphi)$; hence given $a_j \in S_{i_j}$, there exist $b_j \in S_{i_j}$ such that

$$\hat\alpha(i_1^{a_1} i_2^{a_2} \dots) = i_1^{b_1} i_2^{b_2} \dots,$$

and we have $\gamma \circ \hat\alpha = \hat\alpha \circ \varphi$. Let σ_1 be the permutation of S_1 such that $1^{\sigma_1(a)} = \hat\alpha(1^a)$, and let σ_2 be the permutation of S_2 such that $2^{\sigma_2(a)} = \hat\alpha(2^a)$. Given $\sigma_1(a) = b$, define σ_{1^a} by $1^b 2^{\sigma_{1^a}(c)} = \hat\alpha(1^a 2^c)$. Continue inductively so that if $\hat\alpha(f^a) = f^b$ then σ_{f^a} is defined by $\hat\alpha(f^a i^c) = f^b i^{\sigma_{f^a}(c)}$, and let $\sigma \in X$ be the element with first coordinate σ_1, second coordinate σ_2 and f^ath coordinate σ_{f^a}. By the definition of $\hat\alpha$, we have $\hat\alpha(\varphi(1^a 2^b 1^c)) = \gamma(\hat\alpha(1^a 2^b 1^c))$. By the definition of σ, we have $\alpha = \psi_\sigma$. Hence $\gamma = \varphi^\sigma$. □

What Lemmas 4 and 5 imply is that given a rank 2 building Δ and one blueprint, φ, realizing it, every other blueprint realizing Δ is of the form φ^σ for some $\sigma \in X$. Moreover, if there is a non-trivial automorphism of the rank 2 building fixing the base chamber then this automorphism corresponds to an element $\sigma \in X$ such that $\varphi^\sigma = \varphi$. Similarly, given an element $\sigma \in X$ such that $\varphi^\sigma = \varphi$, the map ψ_σ must be an automorphism of the building fixing $\varnothing$. Let H be the stabilizer in Ψ of the blueprint φ under the action $\psi_\sigma(\varphi) = \varphi^\sigma$. Then an efficient enumeration scheme for blueprints realizing $\mathcal{C}(\varphi)$ would pick out exactly one element of each coset of H.

Theorem 6. *Let Δ be a rank 2 building with $m_{1,2} = 3$ and parameter p a prime. Suppose $(\{S_1, S_2\}, \varphi)$ is a rank 2 blueprint realizing Δ. Then every blueprint realizing Δ is given by φ^σ, where*

$$\sigma = (\sigma_1, \dots) \in X$$

can be chosen uniquely so that

$$\begin{aligned}\sigma_1(0) = \sigma_{1^0}(0) = \sigma_{1^0 2^0}(0) = 0,\\ \sigma_1(1) = \sigma_{1^0}(1) = 1.\end{aligned} \tag{2–1}$$

Proof. By Lemma 5 every blueprint realizing Δ is of the form φ^σ for some σ. It remains to show that σ can be uniquely chosen satisfying equations (2–1). Suppose σ satisfies these equations and $\varphi^\sigma = \varphi$. Then the map ψ_σ fixes the apartment of $\mathcal{C}(\varphi)$ containing the chambers $[\varnothing]$, $[1^0]$, $[1^0 2^0]$, $[1^0 2^0 1^0]$, $[1^1]$, and $[1^0 2^1]$. Moreover, since this is the $A_2(p)$ building and ψ_σ fixes $[1^1]$ and $[1^0 2^1]$, then ψ_σ fixes every chamber of form $[1^a]$ and every chamber of form $[1^0 2^a]$. By [Tits 1974, Theorem 4.1.1], this implies that ψ_σ is the identity on Δ. In fact, the requirement that ψ_σ fixes the given apartment and the base chamber implies ψ_σ is a diagonal matrix under the appropriate basis for $\mathbb{F}_p^3$. The requirements that $[1^1]$ and $[1^0 2^1]$ are fixed imply that ψ_σ corresponds to the matrix kI for some $k \in \mathbb{F}_p$. But then ψ_g acts as the identity on Δ. As a result, if γ is any blueprint for Δ, then there exists at most one element $\sigma \in X$ satisfying equations (2–1) with $\gamma = \varphi^\sigma$.

Alternatively, if γ is a blueprint, Lemma 5 implies that $\gamma = \psi_\sigma(\varphi)$ for some $\sigma \in X$. Since $\mathrm{PSL}_3(p)$ is transitive on apartments of $\mathcal{C}(\varphi)$ (which follows from [Ronan 1989, Theorem 5.3]), there exists an automorphism α such that $\alpha(\varnothing) = \varnothing$ and $\alpha(\psi_\sigma(1^0 2^0 1^0)) = 1^0 2^0 1^0$. Since the stabilizer of an apartment is transitive on ordered pairs of chambers of the form $([1^a], [1^0 2^a])$, with $a, b \neq 0$, we can choose α such that $\alpha(\varnothing) = \varnothing$, $\alpha(\psi_\sigma(1^0 2^0 1^0)) = 1^0 2^0 1^0$, $\alpha(\psi_\sigma(1^1)) = 1^1$, and $\alpha(\psi_\sigma(1^0 2^1)) = 1^0 2^1$. By Lemma 5, $\alpha = \psi_\tau$ for some $\tau \in X$ and $\psi_\tau(\gamma) = \gamma$. Since Ψ is a group, $\psi_\tau \circ \psi_\sigma = \psi_\eta$ for some $\eta \in X$. Hence $\gamma = \varphi^\eta$ where η satisfies (2–1). □

Since there exist efficient algorithms for producing permutations (in [Rosen 1991, p. 284], for example), this theorem together with such an algorithm provides a way of enumerating all the blueprints of $A_2(p)$, where p is a prime. The following corollary provides a count for the number of $A_2(p)$ blueprints:

Corollary 7. *The number of blueprints for the $A_2(p)$ building is*

$$\frac{(p!)^{(2p^2+2p+2)}}{p^3(p-1)^2}.$$

Proof. The number of blueprints is clearly $|\Psi|/|H|$, where H is the stabilizer of the natural blueprint. Above, we proved that the stabilizer of H has order $p^3(p-1)^2$. □

For $p = 2$ this number is 2048, for $p = 3$ the number is approximately 1.57×10^{18}. As a result, pure enumeration schemes are only reasonable for the $p = 2$ case.

One can generalize Theorem 6 to other buildings, but the results are not as easily stated. For this paper, we also need the $(A_1 \times A_1)(p)$ case. In this case the building corresponds to the complete bipartite graph $K_{p+1,p+1}$, and the stabilizer of a chamber in the automorphism group is isomorphic to $S_p \times S_p$. The number of blueprints is $(p!)^{2p}$ giving 16 blueprints for $p = 2$.

We now turn to the question of blueprints for higher-rank buildings. If we are given a blueprint $\mathcal{B}$ for the building Δ, then corresponding to any element of $\coprod_i \mathrm{Sym}(S_i)$ we get a blueprint simply by permuting the elements of S_i, that is by mapping $i_1^{a_1} \dots i_k^{a_k}$ to the element $i_1^{\tau_{i_1}(a_1)} \dots i_k^{\tau_{i_k}(a_k)}$. In general, this blueprint will not be equal to the original blueprint, although in the natural $A_1 \times A_1$ case it is. Consider the case of the natural blueprint φ for $A_2(p)$, and let $\sigma \in \mathrm{Sym}(S_1)$ and $\tau \in \mathrm{Sym}(S_2)$. Since

$$\varphi(1^a 2^b 1^c) = 2^c 1^{ac-b} 2^a,$$

for (σ, τ) to fix φ would require $\sigma(a) = \tau(a)$ for all $a \in S_i$, and hence that $\sigma = \tau$. Moreover, we would have $\sigma(ac - b) = \sigma(a)\sigma(c) - \sigma(b)$ for all $a, b, c \in S_1 = S_2$, implying σ is a field automorphism of $\mathbb{F}_p$. Hence σ is trivial. We have shown:

Theorem 8. *There are $(p!)^3$ distinct relabelings of the natural $A_3(p)$ blueprint.*

3. APPLICATIONS

In this section we summarize results from computer analysis on $A_3(2)$-blueprints. All programs were run using MATLAB on a Silicon Graphics Indy computer. To find all $A_3(2)$ blueprints, we enumerated the rank 2 blueprints of type $A_2(2)$ and $A_1(2) \times A_1(2)$. We then checked all triples $(\varphi_{1,2}, \varphi_{2,3}, \varphi_{1,3})$, where $\varphi_{1,2}$ and $\varphi_{2,3}$ are $A_2(2)$ blueprints and $\varphi_{1,3}$ is an $(A_1 \times A_1)(2)$ blueprint. These maps then produce a blueprint $\mathcal{B}$. To see if this blueprint realizes the $A_3(2)$ building, by [Ronan 1989, Theorem 7.2] it suffices to check that if f^a is equivalent to f^b in $\mathcal{C}(\mathcal{B})$ then $a = b$. In turn to check this it suffices as in [Ronan 1989, p. 93] to see that the following *self-homotopy* induces a self-equivalence:

$$\begin{aligned}
1^a2^b1^c3^d2^e1^f &\sim 2^{a_1}1^{b_1}2^{c_1}3^d2^e1^f \\
&\qquad \text{(where } \varphi_{12}(1^a2^b1^c) = 2^{a_1}1^{b_1}2^{c_1}) \\
&\sim 2^{a_1}1^{b_1}3^{c_2}2^{d_1}3^{e_1}1^f \\
&\qquad \text{(where } \varphi_{23}(2^{c_1}3^d2^e) = 3^{c_2}2^{d_1}3^{e_1}) \\
&\sim 2^{a_1}3^{b_2}1^{c_3}2^{d_1}1^{e_2}3^{f_1} \\
&\qquad \text{(where } \varphi_{13}(1^{b_1}3^{c_2}) = 3^{b_2}1^{c_3}, \text{ etc.)} \\
&\sim 2^{a_1}3^{b_2}2^{c_4}1^{d_2}2^{e_3}3^{f_1} \\
&\sim 3^{a_2}2^{b_3}3^{c_5}1^{d_2}2^{e_3}3^{f_1} \\
&\sim 3^{a_2}2^{b_3}1^{c_6}3^{d_3}2^{e_3}3^{f_1} \\
&\sim 3^{a_2}2^{b_3}1^{c_6}2^{d_4}3^{e_4}2^{f_2} \\
&\sim 3^{a_2}1^{b_4}2^{c_7}1^{d_5}3^{e_4}2^{f_2} \\
&\sim 1^{a_3}3^{b_5}2^{c_7}3^{d_6}1^{e_5}2^{f_2} \\
&\sim 1^{a_3}2^{b_6}3^{c_8}2^{d_7}1^{e_5}2^{f_2} \\
&\sim 1^{a_3}2^{b_6}3^{c_8}1^{d_8}2^{e_6}1^{f_3} \\
&\sim 1^{a_3}2^{b_6}1^{c_9}3^{d_9}2^{e_6}1^{f_3}.
\end{aligned}$$

Since this word has length 6, 2^6 checks must be performed to prove a particular blueprint realizes the $A_3(2)$ building. From our results we obtained:

Theorem 1. *The only blueprints realizing the $A_3(2)$ building are relabelings of the natural blueprint for $A_3(2)$.*

This theorem is proved by exhaustive computer search. Given our basic algorithm this case required roughly 120 hours of computer time on a Silicon Graphics Indy computer, and the computer produced exactly the 8 blueprints that are relabelings of the natural blueprint.

From this theorem we obtain:

Corollary 9. *Let Δ be a simply-laced building. Suppose the Coxeter diagram of Δ is connected, contains at least 4 vertices, and contains no triangles. If every panel of Δ lies on exactly 3 chambers (the $p = 2$ case), and Δ conforms to a blueprint $\mathcal{B} = \{\varphi_{i,j}\}$, then all $\varphi_{i,j}$ are natural up to possible relabelings of the S_i's.*

Proof. If the vertices i and j in the Coxeter diagram are connected, then there is a vertex k such that i, j, and k form an A_3 diagram. Hence by Theorem 1, $\varphi_{i,j}$ is natural up to a relabeling. Moreover, if k and k' are both adjacent to i, then the relabeling of i must be the same since i, k, and k' form an A_3-subdiagram. Next, suppose i and j are not connected. The vertices i and j are connected by a path of length m. Inducting on m; if $m = 2$, then i and j are part of an A_3 subdiagram and Theorem 1 yields the result. Else, consider the vertex k adjacent to i at distance $m-1$ from j. Then i, j, and k form an $A_1 \times A_2$ subdiagram. Moreover, we know that $\varphi_{i,k}$ and $\varphi_{j,k}$ are natural by the induction hypothesis. It remains to show that $\varphi_{i,j}$ is natural. Without loss of generality, assume $i = 1$, $k = 2$, and $j = 3$ and consider the labeled word $1^a2^b1^c3^d$. Then, $2^a3^d \simeq 3^d2^a$ since the $\varphi_{2,3}$ blueprint is natural. Similarly, $1^a2^b1^c \simeq 2^a1^{ac-b}2^a$ since the $\varphi_{1,2}$ blueprint is natural. Hence we obtain the following chain of elementary equivalences:

$$\begin{aligned}
1^a2^b1^c3^d &\simeq 2^c1^{ac-b}2^a3^d \\
&\simeq 2^c1^{b'}3^d2^a \qquad (b' = ac-b) \\
&\simeq 2^c3^{d'}1^{b''}2^a && \text{(3–1)} \\
&\simeq 3^{d'}2^a1^{b''}2^c \\
&\simeq 3^{d'}1^a2^{b'''}1^c \qquad (b''' = ac-b'') \\
&\simeq 1^{a'}3^{d''}2^{b'''}1^c && \text{(3–2)} \\
&\simeq 1^{a'}2^{b'''}3^{d''}1^c \\
&\simeq 1^{a'}2^{b'''}1^{c'}3^{d'''}. && \text{(3–3)}
\end{aligned}$$

Since we have a blueprint, by [Ronan 1989, Theorem 7.2] $a = a'$, $b = b'''$, $c = c'$, and $d = d'''$. Since $c' = c$ and $d''' = d$, equation (3–3) implies for all a that $1^a 3^x \simeq 3^y 1^a$. In turn we obtain $b' = b''$ from equation (3–1). Let $b = ac - a$. Since $b' = a$, by equation (3–1) $1^a 3^d \simeq 3^{d'} 1^a$. Now in equation (3–2) we have $3^{d'} 1^a \simeq 1^a 3^{d''}$. This implies $d = d''$. But then the last line implies $1^c 3^d \simeq 3^d 1^c$. Since c and d are free, we obtain that $\varphi_{1,3}$ is natural. This completes the proof that $\mathcal{B}$ must be a relabeling of a natural blueprint. □

As a Corollary to this we obtain Theorem 2.

Proof of Theorem 2. By the above, if there exists a simply-laced affine building over $\mathbb{Q}_2$ of rank at least 4 (with connected diagram) conforming to a blueprint then this buildings would conform to the natural blueprint. From [Ronan and Tits 1987, Example 6.1], we know that these blueprints realize the affine buildings over $\mathbb{F}_2((t))$. Hence they do not realize the $\mathbb{Q}_2$ buildings. □

One result of this last corollary is that to use the idea of blueprints to construct $\mathbb{Q}_2$ buildings requires some generalization of the idea as was done in [Abramson $\geq$ 1998]. An interesting question raised by this is whether the lack of such blueprints is due to the smallness of 2 or the restrictions placed on a blueprint by the building.

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