Constant Mean Curvature Surfaces with Two Ends in Hyperbolic Space

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CONTENTS

We investigate the close relationship between minimal surfaces in Euclidean three-space and surfaces of constant mean curvature 1 in hyperbolic three-space. Just as in the case of minimal surfaces in Euclidean three-space, the only complete connected embedded surfaces of constant mean curvature 1 with two ends in hyperbolic space are well-understood surfaces of revolution: the catenoid cousins.

In contrast to this, we show that, unlike the case of minimal surfaces in Euclidean three-space, there do exist complete connected immersed surfaces of constant mean curvature 1 with two ends in hyperbolic space that are not surfaces of revolution: the genus-one catenoid cousins. These surfaces are of interest because they show that, although minimal surfaces in Euclidean three-space and surfaces of constant mean curvature 1 in hyperbolic three-space are intimately related, there are essential differences between these two sets of surfaces. The proof we give of existence of the genus-one catenoid cousins is a mathematically rigorous verification that the results of a computer experiment are sufficiently accurate to imply existence.

1. INTRODUCTION

The main result presented in this paper is motivated primarily by a result of Schoen [1983], that the only complete connected minimal immersions of finite total curvature in $\mathbb{R}^3$ with two embedded ends are catenoids. In this paper we investigate the closely related case of surfaces of constant mean curvature (CMC) 1 with two ends in hyperbolic space $\mathbb{H}^3$. Other motivations are the results of Kapouleas, Korevaar, Kusner, Meeks, and Solomon. In [Korevaar et al. 1989] it was shown that any complete properly embedded nonminimal CMC surface with two ends in $\mathbb{R}^3$ is a peri-

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odic surface of revolution (Delaunay surface). In [Kapouleas 1990] it was shown that there exist immersed complete nonminimal CMC surfaces with two ends in $\mathbb{R}^3$ with genus $g \geq 2$. And in [Korevaar et al. 1992] it was shown that any complete properly embedded surface of constant mean curvature $c > 1$ with two ends in $\mathbb{H}^3$ is a periodic surface of revolution (hyperbolic Delaunay surface).

Surfaces of constant mean curvature 1 in $\mathbb{H}^3$ are closely related to minimal surfaces in $\mathbb{R}^3$. There is a natural correspondence between them, known as Lawson's correspondence. Let $\mathscr{U}$ be a simply connected region of $\mathbb{C}$. If $x : \mathscr{U} \to \mathbb{R}^3$ is a local representation for a minimal surface in $\mathbb{R}^3$ with first and second fundamental forms I and II, the Gauss and Codazzi equations and the fundamental theorem for surfaces imply that there is a well-defined surface $\tilde{x} : \mathscr{U} \subset \mathbb{C} \to \mathbb{H}^3$ of constant mean curvature 1 with first and second fundamental forms I and $II + I$. In addition, surfaces of constant mean curvature 1 in $\mathbb{H}^3$ have a Weierstrass representation based on a pair of holomorphic functions [Bryant 1987], similar to the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$ (see Section 2).

The following theorem holds for any surface of constant mean curvature, not just when the curvature is 1. We include a proof in Section 3 for the sake of completeness.

Theorem 1.1 [Levitt and Rosenberg 1985]. *Any complete properly embedded surface of constant mean curvature c in $\mathbb{H}^3$ with asymptotic boundary consisting of at most two points is a surface of revolution. In particular, it is homeomorphic to a punctured sphere.*

In the case $c = 1$, this theorem implies that the surface must be a catenoid cousin of genus zero. This was shown in [Umehara and Yamada 1993]; genus-zero catenoid cousins were originally described in [Bryant 1987]. The condition that the surface has asymptotic boundary at most two points implies that $c \geq 1$, as shown by do Carmo, Gomes, and Thorbergsson [do Carmo et al. 1986].

We will show that the condition "embedded" is critical to the above theorem, by giving an immersed counterexample, which we call the genus-one catenoid cousin.

Theorem 1.2. *There exists a one-parameter family of complete, properly immersed surfaces of genus 1 and constant mean curvature 1 in $\mathbb{H}^3$ with asymptotic boundary consisting of two points.*

The genus-one catenoid cousin displays a clear difference between surfaces of constant mean curvature 1 in $\mathbb{H}^3$ and minimal surfaces in $\mathbb{R}^3$, since Schoen's result on minimal surfaces in $\mathbb{R}^3$ holds even for immersions.

The genus-one catenoid cousin further shows that the set of surfaces of constant mean curvature 1 in $\mathbb{H}^3$ with embedded ends is in some sense larger than the set of minimal surfaces in $\mathbb{R}^3$ with embedded ends. Loosely speaking, the set of complete minimal surfaces with embedded ends in $\mathbb{R}^3$ can be mapped injectively to a set of (one-parameter families of) corresponding complete surfaces of constant mean curvature 1 with embedded ends in $\mathbb{H}^3$ [Rossman et al. 1997]. The second theorem above shows that we cannot map the set of (one-parameter families of) complete surfaces of constant mean curvature 1 with embedded ends in $\mathbb{H}^3$ injectively to a set of corresponding complete minimal surfaces with embedded ends in $\mathbb{R}^3$, since there does not exist a minimal surface in $\mathbb{R}^3$ corresponding to the genus-one catenoid cousin in $\mathbb{H}^3$.

In Section 4 we give a nonrigorous explanation for why one should expect the genus-one catenoid cousins to exist. The remainder of the paper is then devoted to proving Theorem 1.2 rigorously. The proof has two interesting characteristics:

1. The period problems that must be solved can be reduced to a single period problem, using symmetry properties of the surface, by a fairly direct argument. This kind of dimension reduction of the period problem can usually be done in a geometric and uncomplicated way for minimal surfaces is $\mathbb{R}^3$, but for surfaces of constant mean curvature 1 in $\mathbb{H}^3$

it seems to be inherently more algebraic and less geometrically transparent [Rossman et al. 1997].

2. We then solve the single remaining period numerically, and use a mathematically rigorous analysis of the numerical method to conclude that the numerical results are correct. This kind of "numerical error analysis" has been used before, for example in [Hass et al. 1995] and [Karcher et al. 1988], and it is likely to be used frequently in the future, as it is well suited for solving period problems on surfaces for which no other method of solution can be found. It is easy to imagine how this method could be useful in a very wide variety of situations.

We solve the single period problem by applying the intermediate value theorem. The idea is similar to the way in which the same theorem is used in the conjugate Plateau construction to solve period problems for minimal surfaces in $\mathbb{R}^3$ [Karcher 1989; Berglund and Rossman 1995]. However, the conjugate Plateau construction fails to help us in the study of surfaces of constant mean curvature 1 in $\mathbb{H}^3$; hence we have used numerical analysis instead. (The conjugate Plateau construction is of use in studying *minimal* surfaces in $\mathbb{H}^3$ [Polthier 1991], but it does not appear to be useful for studying surfaces of CMC 1 in $\mathbb{H}^3$.)

The same methods we use here could likely also be applied to produce similar examples with two ends and genus greater than one, without any conceptual additions. However, with genus greater than one, after reducing the period problems to a minimal set, we would still have at least a two-dimensional problem, and thus the computational aspects would become much more involved. As the genus-one example fulfills our goal of finding a counterexample to Schoen's result in the hyperbolic case (and is computationally more easily understandable), we felt it was appropriate to restrict ourselves to genus one.

Although this paper is written from a mathematical viewpoint, the arguments used here became apparent to the authors only by means of a numerical experiment. Hence, from the authors' point of view, experimental results were essential in obtaining the above result.

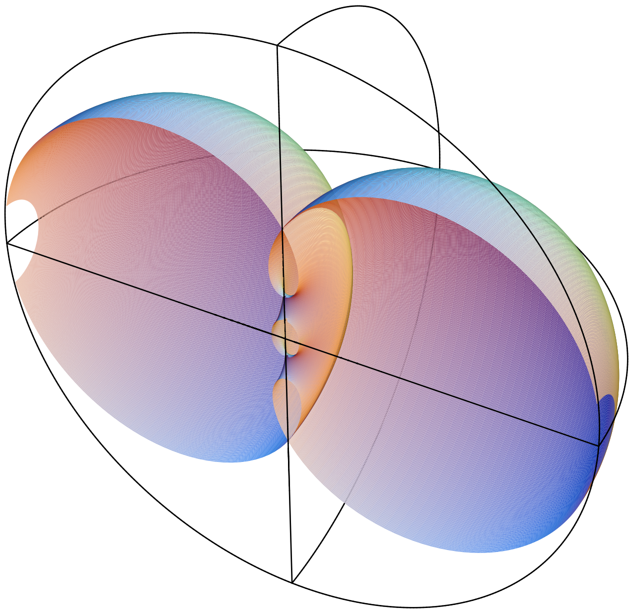

FIGURE 1. A genus-one catenoid cousin in the Poincaré model for $\mathbb{H}^3$. (Half of the surface has been cut away.)

2. THE WEIERSTRASS REPRESENTATION

Both minimal surfaces in $\mathbb{R}^3$ and surfaces of mean curvature 1 in $\mathbb{H}^3$ can be described parametrically by a pair of meromorphic functions on a Riemann surface, via a Weierstrass representation. First we describe the well-known Weierstrass representation for minimal surfaces in $\mathbb{R}^3$. We will incorporate into this representation the fact that any complete minimal surface of finite total curvature is conformally equivalent to a Riemann surface Σ with a finite number of points $\{p_j\}_{j=1}^k \subset \Sigma$ removed [Osserman 1969]:

Lemma 2.1. *Let Σ be a Riemann surface. Let $\{p_j\}_{j=1}^k$ be a finite set of points of Σ, which will represent the ends of the minimal surface defined in this lemma. Let z_0 be a fixed point in $\Sigma \setminus \{p_j\}$. Let g be a meromorphic function from $\Sigma \setminus \{p_j\}$ to the complex plane $\mathbb{C}$. Let f be a holomorphic function* *from* $\Sigma \setminus \{p_j\}$ *to* $\mathbb{C}$. *Assume that, for any point in* $\Sigma \setminus \{p_j\}$, f *has a zero of order* $2k$ *at some point if and only if* g *has a pole of order* k *at that point, and assume that* f *has no other zeroes on* $\Sigma \setminus \{p_j\}$. *Then*

$$\Phi(z) = \operatorname{Re} \int_{z_0}^{z} \begin{pmatrix} (1-g^2)\, f\, d\zeta \\ i\,(1+g^2)\, f\, d\zeta \\ 2gf\, d\zeta \end{pmatrix}$$

is a conformal minimal immersion of the universal cover $\Sigma \setminus \{p_j\}$ *into* $\mathbb{R}^3$. *Furthermore, any complete minimal surface in* $\mathbb{R}^3$ *can be represented in this way.*

The map g can be geometrically interpreted as the stereographic projection of the Gauss map. The first and second fundamental forms and the intrinsic Gaussian curvature for the surface $\Phi(z)$ are

$$\begin{aligned} ds^2 &= (1 + g\bar{g})^2 f\bar{f}\, dz\, \overline{dz}, \\ I\!I &= -2 \operatorname{Re} Q, \\ K &= -4 \left(\frac{|g'|}{|f|(1+|g|^2)^2} \right)^2, \end{aligned}$$

where $Q = fg'\,dz^2$ is the Hopf differential.

To get a surface whose total curvature $\int_\Sigma -K\, dA$ is finite, we must choose f and g so that Φ is well defined on $\Sigma \setminus \{p_j\}$ itself. Usually this involves adjusting some real parameters in the descriptions of f and g and $\Sigma \setminus \{p_j\}$ so that the real part of the above integral about any nontrivial loop in $\Sigma \setminus \{p_j\}$ is zero.

We now describe a Weierstrass-type representation for surfaces of constant mean curvature c in $\mathbb{H}^3(-c^2)$. (This notation stands for a simply connected complete three-dimensional space with constant sectional curvature $-c^2$; thus $\mathbb{H}^3 := \mathbb{H}^3(-1)$.) This lemma is a composition of results found in [Bryant 1987; Umehara and Yamada 1996; 1997].

Lemma 2.2. *Let* Σ, $\Sigma \setminus \{p_j\}$, z_0, f, *and* g *be the same as in the previous lemma. Choose a null holomorphic immersion* F *from the universal cover* X *of* $\Sigma \setminus \{p_j\}$ *to* $\mathrm{SL}(2, \mathbb{C})$ *so that* $F(z_0)$ *is the identity matrix and so that* F *satisfies*

$$F^{-1} dF = c \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} f\, dz. \qquad (2\text{–}1)$$

Then the map $\Phi : X \to H^3(-c^2)$ *defined by*

$$\Phi = \frac{1}{c} F^{-1}\, \overline{F^{-1}}^{\,t} \qquad (2\text{–}2)$$

is a conformal immersion of constant mean curvature c *into* $\mathbb{H}^3(-c^2)$ *with the Hermitian model. Further, any surface of constant mean curvature* c *in* $\mathbb{H}^3(-c^2)$ *can be represented in this way.*

A description of the Hermitian model can be found in any of [Bryant 1987; Umehara and Yamada 1992; 1993], but we also briefly describe it here. If $\mathscr{L}^4$ denotes the standard Lorentzian 4-space of signature $-{+}{+}{+}$, the Minkowski model for $\mathbb{H}^3(-c^2)$ is

$$\mathbb{H}^3(-c^2) = \left\{ (t, x_1, x_2, x_3) \in \mathscr{L}^4 : \sum_{j=1}^{3} t^2 - x_j^2 = \frac{1}{c^2} \right\}.$$

We can identify each point (t, x_1, x_2, x_3) in the Minkowski model with a point

$$\begin{pmatrix} t + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & t - x_3 \end{pmatrix}$$

in the space of 2×2 Hermitian matrices. Thus the Hermitian model for $\mathbb{H}^3(-c^2)$ is

$$\mathbb{H}^3(-c^2) = \left\{ \pm \frac{1}{c} a \cdot \bar{a}^t : a \in \mathrm{SL}(2, \mathbb{C}) \right\}.$$

We call g the *hyperbolic Gauss map* of Φ. As its name suggests, the map $g(z)$ has a geometric interpretation for this case as well. It is the image of the composition of two maps. The first map is from each point z on the surface to the point at the sphere at infinity in the Poincaré model that is at the opposite end of the oriented perpendicular geodesic ray starting at z on the surface. The second map is stereographic projection of the sphere at infinity to the complex plane $\mathbb{C}$ [Bryant 1987].

The first fundamental form and the intrinsic Gaussian curvature of the surface are

$$ds^2 = (1+G\bar{G})^2 \frac{fg'}{G'}\overline{\left(\frac{fg'}{G'}\right)}\, dz\, \overline{dz},$$
$$K = -4\left(\frac{|G'|^2}{|g'|\,|f|(1+|G|^2)^2}\right)^2,$$

where G is defined as the multi-valued meromorphic function $dF_{11}/dF_{21} = dF_{12}/dF_{22}$ on $\Sigma \setminus \{p_j\}$, with $F = (F_{ij})_{i,j=1,2}$. The reason G is multi-valued is that F itself can be multi-valued on $\Sigma\setminus\{p_j\}$ (even if Φ is well defined on $\Sigma\setminus\{p_j\}$ itself). The function G is called the *secondary Gauss map* of Φ [Bryant 1987]. The second fundamental form is given by

$$I\!I = -2\,\mathrm{Re}\, Q + c\, ds^2,$$

where in this case the Hopf differential is $Q = -fg'dz^2$. (The sign change in Q is due to the fact that we are considering the "dual" surface; see [Umehara and Yamada 1997] for an explanation of this.)

In Lemma 2.2, we have changed the notation slightly from the notation used in [Bryant 1987], because we wish to use the same symbol g both for the map g for minimal surfaces in $\mathbb{R}^3$ and for the hyperbolic Gauss map for surfaces of constant mean curvature c in $\mathbb{H}^3(-c^2)$. We use a separate notation G for the secondary Gauss map used in the hyperbolic case. We do this to emphasize that the g in the Euclidean case is more closely related to the hyperbolic Gauss map g in the $\mathbb{H}^3$ case than to the geometric Gauss map G.

We now describe some simple examples:

- The horosphere is a surface of constant mean curvature 1 in $\mathbb{H}^3$. It has these Weierstrass data: $\Sigma \setminus \{p_j\} = \mathbb{C}$, $g = 1$, $f = 1$.
- The Enneper cousins of [Rossman et al. 1997] (see Figure 2, right) have the Weierstrass data $\Sigma \setminus \{p_j\} = \mathbb{C}$, $g = z$, $f = \lambda \in \mathbb{R}$.
- The catenoid cousins [Bryant 1987; Umehara and Yamada 1993] (see Figure 3) have Weierstrass data $\Sigma \setminus \{p_j\} = \mathbb{C} \setminus \{0\}$, $g = z$, $f =$

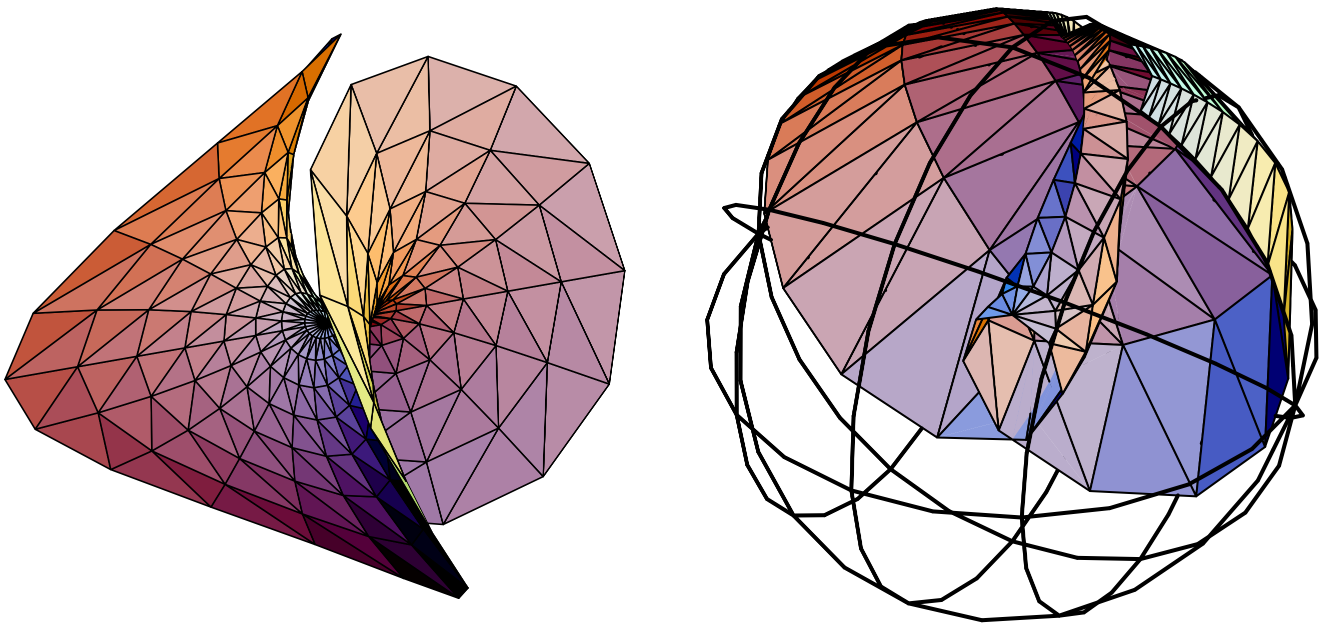

FIGURE 2. A minimal Enneper surface in $\mathbb{R}^3$, and half of an Enneper cousin in the Poincaré model for $\mathbb{H}^3$. The entire Enneper cousin consists of the piece on the right and its reflection across the plane containing the planar geodesic boundary.