The S_5 Extensions of Degree 6 with Minimum Discriminant

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The algebraic number fields of degree 6 having Galois group S_5 and minimum discriminant are determined for signatures $(0,3)$, $(2,2)$ and $(6,0)$. The fields F_0, F_2, F_6 are generated by roots of $f_0(t) = t^6 + 3t^4 + 2t^3 + 6t^2 + 1$, $f_2(t) = t^6 - 2t^4 + 12t^3 - 16t + 8$, and $f_6(t) = t^6 - 18t^4 + 9t^3 + 90t^2 - 70t - 69$ respectively. Each of these fields is unique up to isomorphism. This completes the enumeration of primitive sextic fields with minimum discriminant for all possible combinations of Galois group and signature.

1. INTRODUCTION

The primitive algebraic number fields of a given degree having discriminant within given bounds may be enumerated by the method of [Pohst 1982]. This approach is applied in [Pohst et al. 1982] to determine the sextic fields of minimum discriminant with Galois group S_6 (signatures $(6,0)$, $(4,1)$ and $(0,3)$) and in [Olivier 1990] for Galois groups S_6 (all signatures) and $A_5 \simeq \mathrm{PSL}_2(\mathbb{F}_5)$ and A_6 (signature (2,2) only). As we will show, the method suffices as well to determine the fields of minimum discriminant for the group $S_5 \simeq \mathrm{PGL}_2(\mathbb{F}_5)$, except in the totally real case.

The method is not adequate for investigating primitive totally real sextic fields; too many examples are generated. A refined method was developed to reduce the examples to a manageable number, and the totally real sextic fields of minimum discriminant with Galois groups A_5 [Ford and Pohst 1992] and A_6 [Ford and Pohst 1993] were determined.

The number of examples produced by this improved method is still enormous. In searching for

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fields with alternating Galois groups (A_5 and A_6) it is effective to screen out polynomials with non-square discriminant. For the group S_5 this technique is not available. In its place we use a more costly screening method based on mod-p polynomial factorization, the efficiency of which is critical for the feasibility of the computation.

2. DISTINGUISHING GALOIS GROUP S_5

The group $\mathrm{PGL}_2(\mathbb{F}_5)$ is generated as a permutation group by (1 2 3 4 5) and (1 6)(2 3)(4 5); the cycle types 1·1·1·1·2, 1·1·1·3, 1·2·3 and 2·4 do not occur in $\mathrm{PGL}_2(\mathbb{F}_5)$. So if f is a polynomial with Galois group $\mathrm{PGL}_2(\mathbb{F}_5)$ and p is a prime not dividing the discriminant of f then the degree sequence of the mod p factors of f cannot be among these four types [van der Waerden 1966, Section 8.10].

We generate polynomials of the form

$$f(t) = t^6 + a_1t^5 + a_2t^4 + a_3t^3 + a_4t^2 + a_5t + a_6 \in \mathbb{Z}[t], \tag{2–1}$$

the coefficients being determined in the order a_1, a_2, a_6, a_3, a_4, a_5.

For each triple (a_1, a_2, a_6) and for each p in a suitably chosen set of primes $\{p_1, \dots, p_n\}$ we define a flag I_p and Boolean arrays V_p and W_p.

NAME	SPACE	DEFINITION
I_p	n	Has V_p been initialized (to False)?
$V_p[r_3, r_4, r_5]$	$\sum p^3$	Has $W_p[r_3, r_4, r_5]$ been computed?
$W_p[r_3, r_4, r_5]$	$\sum p^3$	Does $t^6+a_1t^5+a_2t^4+r_3t^3+r_4t^2+r_5t+a_6$ give a cycle type from $\mathrm{PGL}_2(\mathbb{F}_5)$?

The values r_3, r_4, r_5 are the residues of a_3, a_4, a_5 mod p. When p divides the discriminant of $t^6 + a_1t^5+a_2t^4+r_3t^3+r_4t^2+r_5t+a_6$, its mod p factorization gives no information, so we regard $W_p[r_3,r_4,r_5]$ as True.

The polynomial $f(t)$ is excluded if $W_p[r_3, r_4, r_5]$ is False for some p in $\{p_1, \dots, p_n\}$.

When $|a_6|$ is large few triples (a_3, a_4, a_5) are generated. In such cases it is usual that these few polynomials are all excluded using only a few small primes, and it is worthwhile to avoid taking time to initialize V_p for the larger values of p.

When $|a_6|$ is small, many triples (a_3, a_4, a_5) are generated and all the primes are used. For these cases it is advantageous to have the number of primes as large as possible.

3. SIGNATURE (6, 0)

We are to generate at least one defining polynomial $f(t)$ of the form (2.1) for each primitive sextic algebraic number field F with signature (6,0) and discriminant $d_F \le B = 767431973$. Taking $a_1 \in \{0, 1, 2, 3\}$ and ρ a root of f we have

$$\mathrm{Tr}(\rho^2) \le \tfrac{1}{6}a_1^2 + \left(\tfrac{4}{3}B\right)^{1/5}$$

by [Cohen 1993, Theorem 6.4.2], which for successive values of a_1 gives bounds of 63.386, 63.553, 64.053 and 64.886 for $\mathrm{Tr}(\rho^2)$. Because F is totally real we have

$$\mathrm{Tr}(\rho^2) = a_1^2 - 2a_2 \equiv a_1 \mod 2,$$

so that $\mathrm{Tr}(\rho^2)$ is bounded by 62, 63, 64, 63 for $a_1 =$ 0, 1, 2, 3 respectively. Bounds on the coefficients a_2, a_6, a_3, a_4, a_5 are determined as in [Ford and Pohst 1992].

The polynomials are screened for cycle-type compatibility with $\mathrm{PGL}_2(\mathbb{F}_5)$ using the technique of section 2 with the twenty-five primes in the range $2 \le p \le 97$.

The computation required about 13720 CPU-hours on a Digital VAXstation 4000-90 in the Computer Science Department at Concordia University (the same system used in [Ford and Pohst 1993]). The cases with $\mathrm{Tr}(\rho^2) \ge 55$ were independently confirmed on a network of thirty UNIX workstations at the Technische Universität Berlin.

Theorem 3.1. *The minimum possible discriminant for a totally real S_5 extension of degree* 6 *is* $d_6 = 767431973 = 7^3 11^3 41^2$. *There is, up to isomorphy,* *exactly one field F_6 of that discriminant with Galois group S_5. It is generated by a root ρ_6 of the polynomial*

$$f_6(t) = t^6 - 18t^4 + 9t^3 + 90t^2 - 70t - 69.$$

The class number of F_6 is 1. An integral basis for F_6 is given by

$$1, \quad \rho_6, \quad \rho_6^2, \quad \rho_6^3, \quad \rho_6^4, \quad \rho_6^5.$$

A system of fundamental units for F_6 is

$$\begin{aligned}
\varepsilon_{61} &= 2 - \rho_6,\\
\varepsilon_{62} &= 8 + 8\rho_6 - 10\rho_6^2 + \rho_6^4,\\
\varepsilon_{63} &= 71 + 102\rho_6 - 47\rho_6^2 - 27\rho_6^3 + 5\rho_6^4 + 2\rho_6^5,\\
\varepsilon_{64} &= 104 + 90\rho_6 - 50\rho_6^2 - 26\rho_6^3 + 5\rho_6^4 + 2\rho_6^5,\\
\varepsilon_{65} &= 101 + 129\rho_6 - 63\rho_6^2 - 38\rho_6^3 + 7\rho_6^4 + 3\rho_6^5.
\end{aligned}$$

4. SIGNATURES (2, 2) AND (0, 3)

We generate at least one defining polynomial $f(t)$ for each primitive sextic algebraic number field F with $|d_F| \le B = 2299968$ as in section 3 of [Pohst 1982], with slight variations. Taking $\rho^{(1)}, \dots, \rho^{(6)}$ to be the algebraic conjugates of a root ρ of $f(t)$ and $m > 0$, we define

$$S_m(\rho) = \sum_{j=1}^{6} \bigl(\rho^{(j)}\bigr)^m \quad \text{and} \quad T_m(\rho) = \sum_{j=1}^{6} \bigl|\rho^{(j)}\bigr|^m.$$

For $a_1 = 0$, 1, 2, 3 the respective bounds on $T_2(\rho)$ given by [Cohen 1993, Theorem 6.4.2] are 19.830, 19.997, 20.497 and 21.330. Bounds on $\lfloor T_3(\rho)\rfloor$, $\lfloor T_4(\rho)\rfloor$, $\lfloor T_5(\rho)\rfloor$ and $\lfloor T_6(\rho)\rfloor$ follow according to [Pohst 1982, Theorem 4], and bounds for a_2, a_6, a_3, a_4, a_5 are then determined in the usual way, using the facts that $S_m(\rho) \in \mathbb{Z}$ and $|S_m(\rho)| \le \lfloor T_m(\rho)\rfloor$.

The polynomials are screened for cycle-type compatibility with $\mathrm{PGL}_2(\mathbb{F}_5)$ and tested for irreducibility. Due to system restrictions the screening technique of section 2 was applied only for the twenty primes in the range $2 \le p \le 71$.

The computation required about 441 CPU-hours on a Digital AlphaServer 2100 4/200 in the Department of Computing Services at Concordia University (for polynomial generation and screening), plus a small amount of time on other systems (for computing signatures, field discriminants, Galois groups, class groups and fundamental units).

Theorem 4.1. *The minimum discriminant for an S_5 extension of degree 6 and signature $(2,2)$ is $d_2 = 2299968 = 2^6 3^3 11^3$. There is, up to isomorphy, exactly one field F_2 of that discriminant with Galois group S_5. It is generated by a root ρ_2 of the polynomial*

$$f_2(t) = t^6 - 2t^4 + 12t^3 - 16t + 8.$$

The class number of F_2 is 1. An integral basis for F_2 is given by

$$1, \quad \rho_2, \quad \omega_2 = \tfrac{1}{2}\rho_2^2, \quad \rho_2\omega_2, \quad \omega_2^2, \quad \rho_2\omega_2^2.$$

A system of fundamental units for F_2 is

$$\begin{aligned}
\varepsilon_{21} &= \omega_2,\\
\varepsilon_{22} &= 1 - 3\rho_2 + 6\omega_2 - \omega_2^2 + \rho_2\omega_2,\\
\varepsilon_{23} &= -3 + 3\rho_2 + 5\omega_2 - \rho_2\omega_2 + \omega_2^2 + \rho_2\omega_2^2.
\end{aligned}$$

Theorem 4.2. *The discriminant of minimum absolute value for a totally complex S_5 extension of degree 6 is $d_0 = -1778112 = -2^6 3^4 7^3$. There is, up to isomorphy, exactly one field F_0 of that discriminant with Galois group S_5. It is generated by a root ρ_0 of the polynomial*

$$f_0(t) = t^6 + 3t^4 + 2t^3 + 6t^2 + 1.$$

The class number of F_0 is 1. An integral basis for F_0 is given by

$$1, \quad \rho_0, \quad \rho_0^2, \quad \rho_0^3, \quad \rho_0^4, \quad \omega_0 = \tfrac{1}{3}(1 - \rho_0 + \rho_0^2 + \rho_0^3 - \rho_0^4 + \rho_0^5).$$

A system of fundamental units for F_0 is

$$\varepsilon_{01} = \rho_0, \qquad \varepsilon_{02} = -1 + \rho_0 + \rho_0^2 + \rho_0^3.$$

This result is reported in [Haddad 1996].

GROUP	SIGNATURE	DISCRIM.	GENERATING POLYNOMIAL	REFERENCE
A_5	$(6,0)$	30991489	$t^6-10t^4+7t^3+15t^2-14t+3$	[Ford and Pohst 1992]
	$(2,2)$	287296	$t^6+2t^5+t^4+4t^3+2t^2-4t+1$	[Olivier 1990]
A_6	$(6,0)$	170067681	$t^6-24t^4+21t^2+9t+1$	[Ford and Pohst 1993]
	$(2,2)$	287296	$t^6+2t^5-t^4+2t^2-1$	[Olivier 1990]
S_5	$(6,0)$	767431973	$t^6-18t^4+9t^3+90t^2-70t-69$	—
	$(2,2)$	2299968	$t^6-2t^4+12t^3-16t+8$	—
	$(0,3)$	-1778112	$t^6+3t^4+2t^3+6t^2+1$	—
S_6	$(6,0)$	592661	$t^6-5t^5+2t^4+18t^3-11t^2-19t+1$	[Pohst et al. 1982]
	$(4,1)$	-92779	$t^6+t^5-2t^4-3t^3-t^2+2t+1$	[Pohst et al. 1982]
	$(2,2)$	29077	$t^6+2t^5-t^4-t^2-t+1$	[Olivier 1990]
	$(0,3)$	-14731	$t^6+t^5-t^3-t^2+1$	[Pohst et al. 1982]

Primitive sextic fields of minimal discriminant.

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