Bounds for the Density of Abundant Integers

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CONTENTS

We say that an integer n is *abundant* if the sum of the divisors of n is at least 2n. It has been known [Wall 1972] that the set of abundant numbers has a natural density A(2) and that $0.244 < A(2) < 0.291$. We give the sharper bounds

$$0.2474 < A(2) < 0.2480.$$

INTRODUCTION

Let x be a positive real number, and n an integer. Let $\sigma(n)$ be the sum of the divisors of n, and set

$$f(n) = \frac{\sigma(n)}{n}, \quad \mathscr{A}(x) = \{n : f(n) \geq x\}. \qquad (0\text{–}1)$$

A number in $\mathscr{A}(x)$ is called x-*abundant*, or simply *abundant* if $x = 2$.

Davenport proved that $\mathscr{A}(x)$ has a natural density $A(x)$, and that $A(x)$ is a continuous function of x; see, for example, [Davenport 1933; Elliott 1979, Chapter 5; Tenenbaum 1995, III.1 and III.2].

Behrend [1933] proved that $0.241 < A(2) < 0.314$, and Wall [1972] improved this to $0.244 < A(2) < 0.291$. We prove here the following:

Theorem 0.1. *The density* $A(2)$ *of the set of abundant numbers satisfies*

$$0.2474 < A(2) < 0.2480.$$

This answers a question asked by Henri Cohen: Is the proportion of abundant numbers more or less than a quarter? The method used is essentially that given by Behrend, the computer allowing us to do more computations. This method in fact gives the density $A(x)$ for every x.

Perhaps it could be worthwile to try an analytic method. Cohen, Deshouillers, Martinet showed in

1058-6458/1998 \$0.50 per page
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[Martinet et al. 1973] that the Mellin transform of $A(x)$ is the function

$$g(s) = \frac{1}{s} \prod_{p \geq 2} \frac{1}{\left(1-\frac{1}{p}\right)^{s-1}} \sum_{l \geq 0} \frac{1}{p^l} \left(1 - \frac{1}{p^{l+1}}\right)^s.$$

Hence, by inversion, we have for every $\sigma > 1$

$$A(x) = \frac{1}{2i\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} x^{-s} g(s)\, ds,$$

but the computation of this integral seems to be difficult; taking $x = 2$ and $\sigma = 2$ we computed the sum between $2 - 10000i$ and $2 + 10000i$, and got the approximate value 0.242. For large values of $\mathrm{Im}(s)$ the computation of $g(s)$ is difficult.

1. EXPRESSING A(x) AS A SUM

We denote by $(p_n)_{n \geq 1}$ the increasing sequence of primes. Let k be a fixed integer. We consider the set

$$\mathscr{A}_k(x) = \{n : f(n) \geq x,\ \gcd(n, p_1 p_2 \ldots p_k) = 1\}. \tag{1–1}$$

This set has a density [Elliott 1979; Tenenbaum 1995], which will be denoted by $A_k(x)$.

Let n be an arbitrary integer. We denote by n_1 the product of the prime factors of n among $\{p_1, p_2, \ldots, p_k\}$ and we write $n = n_1 n_2$. The function f is multiplicative and $f(n) = f(n_1) f(n_2)$ is greater than or equal to x if and only if $f(n_2) \geq x/f(n_1)$. This proves that $\mathscr{A}(x)$ is partitioned as follows:

$$\mathscr{A}(x) = \bigcup_{n_1 = p_1^{\alpha_1} \ldots p_k^{\alpha_k}} n_1 \mathscr{A}_k\left(\frac{x}{f(n_1)}\right).$$

Considering the densities we have:

Proposition 1.1.

$$A(x) = \sum_{n_1 = p_1^{\alpha_1} \ldots p_k^{\alpha_k}} \frac{1}{n_1} A_k\left(\frac{x}{f(n_1)}\right), \tag{1–2}$$

where the sum is taken over all n_1 that are a product of primes belonging to $\{p_1, p_2, \ldots, p_k\}$.

To see this, it is sufficient to prove the following lemma.

Lemma 1.2. *Let p be an integer greater than 1 and $(A_\alpha)_{\alpha \geq 0}$ a sequence of disjoint sets having densities d_α. Set $\mathscr{A} = \bigcup_{\alpha \geq 0} p^\alpha \mathscr{A}_\alpha$. Then $\mathscr{A}$ has a density $d(\mathscr{A})$ and*

$$d(\mathscr{A}) = \sum_{\alpha \geq 0} \frac{1}{p^\alpha} d_\alpha.$$

Proof. Write

$$\mathscr{A} = \left(\bigcup_{0 \leq \alpha \leq r} p^\alpha \mathscr{A}_\alpha\right) \cup \left(\bigcup_{\alpha > r} p^\alpha \mathscr{A}_\alpha\right)$$

The second set in this union is formed of multiples of p^{r+1}. Its upper density is bounded by $1/p^{r+1}$ and

$$\sum_{0 \leq \alpha \leq r} \frac{1}{p^\alpha} d_\alpha \leq \underline{d}(\mathscr{A}) \leq \bar{d}(\mathscr{A}) \leq \sum_{0 \leq \alpha \leq r} \frac{1}{p^\alpha} d_\alpha + \frac{1}{p^{r+1}}$$

where $\underline{d}$ and $\bar{d}$ denote the lower and upper densities. We let $r \to \infty$ and we get the result. □

2. TRIVIAL BOUNDS FOR $A_k(x)$

Proposition 2.1. *For every $k \geq 0$ and every $x > 0$ we have*

$$A_k(x) \leq F_k \tag{2–1}$$

and

$$A_k(x) = F_k \quad \text{if} \quad x \leq 1, \tag{2–2}$$

where $F_k = \prod_{i=1}^k (1 - 1/p_i)$.

Proof. Clear, since $\mathscr{A}_k(x)$ is formed only with integers coprime with $p_1 p_2 \ldots p_k$, and comprises all these integers if $x \leq 1$. □

3. LOWER BOUND FOR A(x)

Let z be a arbitrary positive real parameter. If in (1–2) we just keep the integers $n_1 = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \leq z$, we get a lower bound for $A(x)$. Hence

$$A(x) \geq \sum_{n_1 = p_1^{\alpha_1} \ldots p_k^{\alpha_k}}^{n_1 \leq z} \frac{1}{n_1} A_k\left(\frac{x}{f(n_1)}\right).$$

We still get a lower bound if we just keep those n_1 such that $f(n_1) \geq x$; hence

$$A(x) \geq \sum_{\substack{n_1 = p_1^{\alpha_1} \dots p_k^{\alpha_k} \\ f(n_1) \geq x}}^{n_1 \leq z} \frac{1}{n_1} A_k\left(\frac{x}{f(n_1)}\right).$$

By (2–2), all the $A_k(x/f(n_1))$ are equal to F_k; hence

$$A(x) \geq F_k \sum_{\substack{n_1 = p_1^{\alpha_1} \dots p_k^{\alpha_k} \\ f(n_1) \geq x}}^{n_1 \leq z} \frac{1}{n_1}. \tag{3–1}$$

This lower bound is almost trivial and could have been shown slightly differently. We choose an upper bound z and a set $\{p_1, p_2, \dots, p_k\}$ of small primes. We compute all the integers m less than z, composed of prime factors from $\{p_1, p_2, \dots, p_k\}$, and x-abundant. Every multiple of an abundant number being abundant, all the products of the numbers m thus obtained by some prime factors out of $\{p_1, p_2, \dots, p_k\}$ are still abundant numbers. The lower bound for $A(x)$ is the density of this set, $F_k \sum_m 1/m$.

4. UPPER BOUNDS FOR $A_k(x)$

As in the previous section, we introduce a real positive parameter z and write

$$A(x) = \sum_{n_1 = p_1^{\alpha_1} \dots p_k^{\alpha_k}}^{n_1 \leq z} \frac{1}{n_1} A_k\left(\frac{x}{f(n_1)}\right) + \sum_{n_1 = p_1^{\alpha_1} \dots p_k^{\alpha_k}}^{z < n_1} \frac{1}{n_1} A_k\left(\frac{x}{f(n_1)}\right).$$

In the second sum, each value of A_k is bounded from above by F_k; thus the second sum is bounded from above by

$$F_k \sum_{n_1 = p_1^{\alpha_1} \dots p_k^{\alpha_k}}^{z < n_1} \frac{1}{n_1} = F_k \sum_{n_1 = p_1^{\alpha_1} \dots p_k^{\alpha_k}}^{1 \leq n_1 \leq \infty} \frac{1}{n_1} - F_k \sum_{n_1 = p_1^{\alpha_1} \dots p_k^{\alpha_k}}^{n_1 \leq z} \frac{1}{n_1}$$
$$= 1 - F_k \sum_{n_1 = p_1^{\alpha_1} \dots p_k^{\alpha_k}}^{n_1 \leq z} \frac{1}{n_1},$$

so

$$A(x) \leq \sum_{n_1 = p_1^{\alpha_1} \dots p_k^{\alpha_k}}^{n_1 \leq z} \frac{1}{n_1} A_k\left(\frac{x}{f(n_1)}\right) + 1 - F_k \sum_{n_1 = p_1^{\alpha_1} \dots p_k^{\alpha_k}}^{n_1 \leq z} \frac{1}{n_1}. \tag{4–1}$$

It remains to bound the values of A_k that appear in the sum (4–1). If we just use the trivial upper bound $A_k \leq F_k$ we will get $A(x) \leq 1$, so we need a nontrivial upper bound for $A_k(x)$. This is the subject of the next section.

5. MEAN VALUES OF $f(n)^r$ AND UPPER BOUNDS FOR $A_k(x)$

Let f_k be the multiplicative function that takes the value 1 for p^α with $p \leq p_k$ and the value $f(p^\alpha)$ for $p > p_k$. We fix an integer r and we consider $g = f_k^r$ and the mean value of g computed on the first n integers:

$$M_n = \frac{1}{n} \sum_{m=1}^{n} g(m).$$

Let ρ be the convolution product of g and the Möbius μ function:

$$\rho(m) = \sum_{d \mid m} \mu\left(\frac{m}{d}\right) g(d). \tag{5–1}$$

The Möbius inversion formula gives

$$\begin{aligned} M_n &= \frac{1}{n} \sum_{m=1}^{n} g(n) = \frac{1}{n} \sum_{m=1}^{n} \sum_{d \mid m} \rho(d) \\ &= \frac{1}{n} \sum_{d=1}^{n} \rho(d) \left[\frac{n}{d}\right] \leq \frac{1}{n} \sum_{d=1}^{n} \rho(d) \frac{n}{d} \\ &\leq \sum_{d=1}^{\infty} \frac{\rho(d)}{d} = \Lambda_k(r). \end{aligned}$$

The function $\rho(d)/d$ is multiplicative, so $\Lambda_k(r)$ is also equal to the value of the Euler product

$$\Lambda_k(r) = \prod_p \left(1 + \frac{\rho(p)}{p} + \frac{\rho(p^2)}{p^2} + \cdots + \right). \tag{5–2}$$

Using the definition equation (5–1) of ρ, we have

$$\rho(p^\alpha) = g(p^\alpha) - g(p^{\alpha-1}) = \left(1+\frac{1}{p}+\cdots+\frac{1}{p^\alpha}\right)^r - \left(1+\frac{1}{p}+\cdots+\frac{1}{p^{\alpha-1}}\right)^r$$

when $p > p_k$ and $\alpha > 0$, otherwise $\rho(p^\alpha) = 0$.

We return to the sum

$$nM_n = \sum_{m=1}^{n} g(m).$$

Let B_n be the number of integers m between 1 and n such that $f_k(m) \geq x$, or equivalently $g(m) \geq x^r$. We collect the terms of this sum in two classes, first those terms for which $g(m) \geq x^r$, that are bounded from below by x^r, and the other terms, that are bounded from below by 1. We get

$$x^r B_n + n - B_n \leq nM_n \leq n\Lambda_k(r);$$

dividing by n and letting $n \to \infty$ we get

$$B_k(x) \leq \frac{\Lambda_k(r) - 1}{x^r - 1},$$

where $B_k(x)$ is the density of the set of all m such that $f_k(m) \geq x$. This set is the disjoint union of the $p_1^{\alpha_1} \ldots p_k^{\alpha_k} \mathscr{A}_k(x)$, and we deduce the following upper bound, proved by Behrend [1933].

Proposition 5.1. *For every integer $r \geq 1$ and every k,*

$$A_k(x) \leq F_k \frac{\Lambda_k(r) - 1}{x^r - 1}. \tag{5–3}$$

Table 1 gives the upper bounds for $\Lambda_{95}(r) - 1$ for $r = 1, 2, 4, 8, 16, \ldots, 4096$.

r	$\Lambda_{95}(r)-1 \leq$	r	$\Lambda_{95}(r)-1 \leq$
1	0.000284	64	0.0189
2	0.000568	128	0.0395
4	0.00114	256	0.0866
8	0.00228	512	0.213
16	0.00458	1024	0.726
32	0.00925	2048	12.3
		4096	1.37×10^{17}

TABLE 1. Upper bounds for $\Lambda_{95}(r) - 1$.

When x is very close to 1, almost every integer is x-abundant and the trivial upper bound (2–1) is better than the upper bound (5–3). Table 2 shows this phenomenon. It gives for some values of x the best upper bound for $A_k(x)$ obtained by formula (5–3) choosing the right value for r. The value $r = 0$ on the first line means that, for this $x = 1.0001$, the trivial upper bound (2–1) is the better one.

x	r	$A_{95}(x) \leq$	x	r	$A_{95}(x) \leq$
1.0001	0	0.0897	1.005	2048	4.35×10^{-5}
1.001	1	0.0254	1.01	2048	1.68×10^{-9}
1.002	1024	0.0096	1.02	4096	9.21×10^{-20}

TABLE 2. Some upper bounds for $A_{95}(r)$ obtained using Table 1.

6. UPPER BOUNDS FOR THE EULER PRODUCTS $\Lambda_k(r)$

In this section we give some effective upper bounds used to get upper bounds for the Euler products $\Lambda_k(r)$. In all this section we write

$$\rho(p^\alpha) = \left(1+\frac{1}{p}+\cdots+\frac{1}{p^\alpha}\right)^r - \left(1+\frac{1}{p}+\cdots+\frac{1}{p^{\alpha-1}}\right)^r = \sum_{d \mid p^\alpha} \mu(d) \left(f\left(\frac{p^\alpha}{d}\right)\right)^r.$$

This is the ρ function defined by (5–1) for $k = 0$.

We gave in [Deléglise and Nicolas 1994] a method to quickly compute a good approximate value of an Euler product $\prod_p g(1/p)$, when g is a holomorphic function around 0 whose first Taylor series coefficients are not too large. This method could have been used to get some very accurate values for the first $\Lambda_k(r)$. For very large values of r the accuracy would not be so good. Since we just need an upper bound for each $\Lambda_k(r)$, we will just use the trivial method: find upper bounds for the partial products, and for the tails of the products.

Lemma 6.1. *Let r be an integer ≥ 1 and $p \geq 2r$. Then*

$$\left(1+\frac{1}{p}\right)^r - 1 < 1.3\,\frac{r}{p}.$$

Proof. We have

$$\frac{(1+1/p)^r-1}{r/p}=\frac{\exp\bigl(r\ln(1+1/p)\bigr)-1}{r/p}$$
$$<\frac{\exp(r/p)-1}{r/p}$$
$$\le\frac{e^{1/2}-1}{1/2}<1.3. \qquad\square$$

Lemma 6.2. *Let r an integer ≥ 2 and $p\ge 2r$. Then*

$$\left(\frac{1}{1-1/p}\right)^{r-1}<\frac{16}{9}<1.78.$$

Proof. Let $u=1/p$. Then

$$y=\left(\frac{1}{1-1/p}\right)^{r-1}<\left(\frac{1}{1-1/p}\right)^{r}\le\left(\frac{1}{1-u}\right)^{1/2u};$$

hence

$$\ln(y)=\frac{1}{2u}\ln\left(\frac{1}{1-u}\right)\le 2\ln\left(\frac{1}{1-\frac14}\right)=\ln\frac{16}{9},$$

since the function $(1/u)\ln\bigl(1/(1-u)\bigr)$ is increasing for $0<u\le\frac{1}{2r}\le\frac14$. $\square$

Lemma 6.3. *For every integer r and every prime p,*

$$\sum_{\alpha\ge0}\frac{\rho(p^\alpha)}{p^\alpha}$$
$$\le 1+\frac{(1+1/p)^r-1}{p}+r\left(\frac{1}{1-1/p}\right)^{r-1}\frac{1}{p^4-p^2}.$$

Proof. Set

$$Y=1+\frac1p+\cdots+\frac{1}{p^{\alpha-1}},\qquad X=Y+\frac{1}{p^\alpha}.$$

We get, for $\alpha\ge1$,

$$\frac{\rho(p^\alpha)}{p^\alpha}=\frac{1}{p^\alpha}(X^r-Y^r)$$
$$=\frac{1}{p^{2\alpha}}(X^{r-1}+X^{r-2}Y+\cdots+Y^{r-1})$$
$$\le\frac{r}{p^{2\alpha}}X^{r-1}\le\frac{r}{p^{2\alpha}}\left(\frac{1}{1-1/p}\right)^{r-1}.$$

Using this upper bound for $\alpha\ge2$ in the sum

$$\sum_{\alpha\ge0}\frac{\rho(p^\alpha)}{p^\alpha}$$

we get the conclusion. $\square$

Lemma 6.4. *For every integer $r\ge1$ and every $p\ge\max(2r,15)$ we have*

$$\sum_{\alpha\ge0}\frac{\rho(p^\alpha)}{p^\alpha}<1+1.31\frac{r}{p^2}.$$

Proof. The preceding three lemmas give, for every $r\ge2$,

$$\sum_{\alpha\ge0}\frac{\rho(p^\alpha)}{p^\alpha}\le 1+1.3\,\frac{r}{p^2}+1.78\,\frac{r}{p^4-p^2}$$
$$=1+\frac{r}{p^2}\Bigl(1.3+1.78\,\frac{1}{p^2-1}\Bigr)$$
$$\le 1+1.31\,\frac{r}{p^2}\quad\text{ if } p\ge15.$$

For $r=1$ this upper bound is still true, because

$$\sum_{\alpha\ge0}\frac{\rho(p^\alpha)}{p^\alpha}=\sum_{\alpha\ge0}\frac{1}{p^{2\alpha}}=1+\frac{1}{p^2-p^4}. \qquad\square$$

Lemma 6.5. *For every integer r with $1\le r\le 10000$ we have*

$$\prod_{p>10^6}\left(\sum_{\alpha\ge0}\frac{\rho(p^\alpha)}{p^\alpha}\right)\le 1+\frac{r}{10^7}.$$

Proof. Set

$$u=\prod_{p>10^6}\left(\sum_{\alpha\ge0}\frac{\rho(p^\alpha)}{p^\alpha}\right).$$

Using Lemma 6.4 we get

$$\ln(u)\le 1.31r\sum_{p>10^6}\frac{1}{p^2}.$$

The sum of $1/p^2$ can be computed as explained in [Deléglise and Nicolas 1994, pp. 331–332], or it can be found in [Glaisher 1891]:

$$\sum_p\frac{1}{p^2}=0.452247420041\ldots.$$

Interval		Interval		Interval		Interval	
$[1, 10^1]$	1	$[1, 10^6]$	24799	$[10^9, 10^9 + 10^7)$	2476049	$[10^{14}, 10^{14} + 10^7)$	2476150
$[1, 10^2]$	24	$[1, 10^7]$	2476741	$[10^{10}, 10^{10} + 10^7)$	2476372	$[10^{15}, 10^{15} + 10^7)$	2476212
$[1, 10^3]$	249	$[1, 10^8]$	24760673	$[10^{11}, 10^{11} + 10^7)$	2476154	$[10^{16}, 10^{16} + 10^7)$	2476247
$[1, 10^4]$	2492	$[1, 10^9]$	247610965	$[10^{12}, 10^{12} + 10^7)$	2476199	$[10^{17}, 10^{17} + 10^7)$	2476098
				$[10^{13}, 10^{13} + 10^7)$	2476213	$[10^{18}, 10^{18} + 10^7)$	2476304

TABLE 3. Frequency of abundant numbers in different intervals.

Hence, subtracting $\sum_{p\le 10^6} 1/p^2$, we have

$$\sum_{p>10^6} \frac{1}{p^2} = 0.0000000677\ldots$$

and

$$\ln(u) < 0.9\,\frac{r}{10^7} < 10^{-3},$$

and finally

$$u = e^{\ln u} < 1 + \frac{r}{10^7},$$

using the estimate $e^t < 1 + \frac{10}{9}t$ for $t < 0.001$. $\square$

We get an upper bound for the Euler product (5–2), writing

$$\prod_{p>p_k}\Bigl(\sum_{\alpha\ge 0}\frac{\rho(p^\alpha)}{p^\alpha}\Bigr) = \prod_{p_k<p\le 10^6}\Bigl(\sum_{\alpha\ge 0}\frac{\rho(p^\alpha)}{p^\alpha}\Bigr)\prod_{p>10^6}\Bigl(\sum_{\alpha\ge 0}\frac{\rho(p^\alpha)}{p^\alpha}\Bigr).$$

The first product is bounded by Lemma 6.3 and the second by Lemma 6.5.

Table 1 gives the upper bounds for $\Lambda_{95}(r) - 1$ for $r = 1, 2, 4, 8, 16, \ldots, 4096$. These are the values used for bounding the values A_k that appear in formula (4–1).

7. NUMERICAL RESULTS

We have bounded $A(2)$ using (3–1) and (4–1) with $x = 2$, $k = 95$ (which is the number of primes less than 500), and $z = 10^{14}$. For the upper estimate each term

$$A_k\Bigl(\frac{x}{f(p_1^{\alpha_1}\ldots p_k^{\alpha_k})}\Bigr)$$

in (4–1) is bounded using formula (5–3) with $r = 1, 2, 4, 8, \ldots, 4096$ and the trivial bound (2–1); we keep the best result obtained. This requires the enumeration of all the $p_1^{\alpha_1}\ldots p_k^{\alpha_k}$ not greater than z, which is done by a backtracking procedure. The total number of these n less than 10^{14} whose prime factors are less than 500 is 23581230171.

The computation was performed on an HP900-730 workstation, using about 100 hours of CPU time. It yields

$$0.2474 < A(2) < 0.2480; \tag{7–1}$$

in particular $A(2) = 0.247\ldots$.

8. OTHER EXPERIMENTAL RESULTS

We computed the number of abundant numbers less than N for $N = 1, 10, 10^2, \ldots, 10^9$, and also the number of abundant numbers in the intervals $[N, N{+}10^7)$ for $N = 10^9, 10^{10}, \ldots, 10^{18}$. The results are given in Table 3 and seem to show that the next digit of $A(2)$ is a 6.

We thank the referee for remarking to us that the number of abundant numbers given above in the intervals of size 10^7 are compatible with a binomial law with parameters $m_p = 2476200$ and $s = 1365$.

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Received January 21, 1997; accepted August 11, 1997