# Embedding the $3x+1$ Conjecture in a $3x+d$ Context

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**CONTENTS**



Recall the well-known 3x + 1 conjecture: if T(n) = (3n + 1)/2 for n odd and T(n) = n/2 for n even, repeated application of T to any positive integer eventually leads to the cycle

$$\{1 \to 2 \to 1\}.$$

We study a natural generalization of the function T, where instead of 3n + 1 one takes 3n + d, for d equal to $-1$ or to an odd positive integer not divisible by 3. With this generalization new cyclic phenomena appear, side by side with the general convergent dynamics typical of the 3x + 1 case. Nonetheless, experiments suggest the following conjecture: For any odd d $\geq -1$ not divisible by 3 there exists a finite set of positive integers such that iteration of the 3x + d function eventually lands in this set.

Along with a new boundedness result, we present here an improved formalism, more clear-cut and better suited for future experimental research.

## 1. INTRODUCTION

The well-known $3x+1$ problem deals with the iterative behavior of the function $T : \mathbb{N}^* \to \mathbb{N}^*$ (where $\mathbb{N}^*$ is the set of positive integers) defined as follows:

$$T(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

All known numerical checks, as well as a few interesting heuristic arguments [Lagarias 1985], indicate that a typical trajectory (sequence of iterates) of $T$ degenerates into repetitions of the finite cycle $\{1 \to 2 \to 1\}$. The $3x+1$ conjecture asserts that this is true for any positive integer $n$.

Since the problem became known about sixty years ago, many interesting and deep facts concerning the iteration of $T$ have been discovered; most are reported in [Lagarias 1985], where one can find 70 relevant references. See also [Lagarias 1990;

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Lagarias and Weiss 1992; Applegate and Lagarias 1995]. Still, the $3x+1$ conjecture remains open. One can only marvel at how such a straightforward and primitive in extremis rule can produce such an immensely rich and balanced dynamical pattern!

**Remark 1.1.** Generally speaking, a trajectory of a map $Z : \mathbb{N}^* \to \mathbb{N}^*$ can be either divergent (that is, $\limsup Z^k(n) = \infty$) or ultimately $t$-periodic (after a finite number of initial iterations, the transformation enters into a cycle of length $t$). In the 1-periodic case we say the trajectory terminates at a fixed point. The $3x+1$ conjecture is equivalent to the conjunction of the two following conjectures:

**(CD)** *$T$ has no divergent trajectories.*
**(CC)** *The only cycle of $T$ is $\{1 \to 2 \to 1\}$.*

**Remark 1.2.** Statements (CD) and (CC), simple and natural as they are, might well turn out to be *algorithmically undecidable*, as is their rather straightforward arithmetical generalization due to John H. Conway [1972]; hence the problem:

**(PAD)** *Is the $3x+1$ conjecture algorithmically decidable?*

Past and present research on the $3x+1$ problem has centered around the three themes (CD), (CC), and (PAD), with their quite different and almost unrelated methods and techniques. This paper attempts to contribute to our understanding of all the aspects of the $3x+1$ dynamics by extending it to a more general $3x+d$ case. This extension was originally studied in [Lagarias 1990], in a somewhat different context, for $d \geq 1$; the case $d=-1$ was briefly mentioned in [Böhm and Sontacchi 1978].

For reasons described in the next section, it is more convenient to express the generalization as a function involving odd numbers only. For $n \in \mathbb{N}$, let $\mathrm{odd}(n)$ be the number obtaining by factoring out the highest possible power of 2; thus $\mathrm{odd}(n)$ is odd and $n = 2^k \,\mathrm{odd}(n)$ for some $k$.

Now let $d \geq -1$ be an odd integer not divisible by 3, and define the $3x+d$ function $S_d$ as follows:

$$S_d(n) = \mathrm{odd}(3n+d).$$

Notice that $S_d(d) = d$, and thus the fact that $d$ is a fixed point of $S_d$ is the $3x+d$ analogue of the fixed point 1 of the $3x+1$ transformation. But $\{d\}$ is, generally speaking, not the only cycle, and even not the only fixed point, of the mapping $S_d$. For example, 5, 13 and 65 are the fixed points of $S_{65}$ (see Proposition 3.1 and Example 3.3). Here are examples of cycles of length 2, 3, and 7:

$$\begin{aligned} d=-1:&\ \{5\to 7\to 5\}\\ &\ \{17\to 25\to 37\to 55\to 41\to 61\to 91\to 17\}\\ d=5:&\ \{19\to 31\to 49\to 19\}\\ &\ \{23\to 37\to 29\to 23\}\end{aligned}$$

These facts illustrate how subtle, unique and, apparently, extremely difficult is the $3x+1$ periodicity conjecture (CC). In this light conjecture (CC) may seem too optimistic; a weaker version, called the *finite cycles conjecture* in [Lagarias 1985], may turn out to be the right one:

**(FCC)** *$T$ has only a finite number of cycles.*

In contrast, the plausibility of the $3x+1$ divergence conjecture (CD) is not weakened by the $3x+d$ dynamics. These facts suggest the following $3x+d$ generalization of the $3x+1$ conjecture (compare [Lagarias 1990]):

**The $3x+d$ Conjecture.** *For any odd $d \geq -1$ not divisible by 3, there exists a finite set $\mathbb{T}_d \in \mathbb{N}$ such that, for any odd positive integer $n$ not divisible by 3, the iterates $S_d^k(n)$ lie in $\mathbb{T}_d$, for all high enough $k$ (depending on $n$). The set $\mathbb{T}_d$ is called the termination set.*

Similarly to the $3x+1$ case, the $3x+d$ conjecture is the conjunction of two weaker statements:

**($CD_d$)** *$S_d$ has no divergent trajectories.*
**($FCC_d$)** *$S_d$ has only a finite number of cycles.*

## 2. REDUCTION FROM T TO S

The function $T$ defined above acts surjectively on the set $\mathbb{N}^*$ of positive integers, but the action is neither "regular" nor "simple". Any positive integer

$m$ is the image of $2m$ under $T$, and if $m = 3a + 2$ for integer $a$ then $m$ is also the image of $2a + 1$. Thus $T^{-1}(m)$ has one element if $m \not\equiv 2 \bmod 3$, but two elements otherwise.

The set of numbers not divisible by 3, denoted (somewhat abusively) $3\mathbb{N}^* \pm 1$, is stable under $T$:

$$T(3\mathbb{N}^* \pm 1) = 3\mathbb{N}^* \pm 1.$$

Moreover, $T$ sends odd numbers divisible by 3 into numbers not divisible by 3. This implies that $T$ sends the subset $6\mathbb{N} + 3$ into its complement "forever":

$$T^k(6\mathbb{N} + 3) \cap (6\mathbb{N} + 3) = \varnothing \quad \text{for any } k \geq 1.$$

In particular, no $3x + 1$ cycle starts at $6\mathbb{N} + 3$.

Such peculiarities obscure the iterative behavior of $T$ and motivate our search for normalized or irreducible versions of $T$. To simplify the notation, we put

$$\mathbb{D} = (6\mathbb{N} + 1) \cup (6\mathbb{N} + 5).$$

With this notation, an irreducible version of $T$ is given by the transformation $S : \mathbb{D} \to \mathbb{D}$ defined by

$$S(n) = \operatorname{odd}(3n + 1) = \operatorname{odd}\big(T(n)\big).$$

Thus $S$ is the trace of $T^k$ on $\mathbb{D}$. A normalized version of $T$, the *periodically linear* transformation $W$, will be defined in Section 4.

The function $S$ has the advantage of an immediate and natural generalization to the $3x+d$ context. We define $S_d : \mathbb{D} \to \mathbb{D}$, for all $d \in \mathbb{D} \cup \{-1\}$, by setting

$$S_d(n) = \operatorname{odd}(3n + d).$$

Thus $S_1 = S$.

## 3. FIXED POINTS, LOOPS AND CYCLES OF 3x + d MAPPINGS

Clearly, $S_1$ has only one fixed point, namely $n = 1$. This is no longer true in the general case; however, the number of fixed points of $S_d$ is always finite:

**Proposition 3.1.** *For any $d \in 6\mathbb{N} \pm 1$, the number of fixed points of $S_d$ is finite. More precisely, $n$ is a fixed point of $S_d$ if and only if $n = d/(2^k - 3)$, for some integer $k > 1$. In particular, $n = d = d/(2^2 - 3)$ is a fixed point, and there are no others if $d$ has no divisors of the form $2^k - 3$ (other than 1).*

*Proof.* Immediate. $\square$

We say that $d$ is the *trivial* fixed point of $S_d$.

**Example 3.2.** The smallest composite number in $\mathbb{D}$ is $d = 25 = 5 \times 5$; since $5 = 2^3 - 3$, $n = 5$ is a nontrivial fixed point of $S_{25}$ (in fact, the only one). Similarly, the only nontrivial fixed points of $S_{35}$ and $S_{55}$ are $n = 7$ and $n = 11$, respectively. More generally, if $d = 5p$ with $p$ a prime not congruent to 5 mod 8, the only nontrivial fixed point of $S_d$ is $p$.

**Example 3.3.** Let $d = 65 = 5 \times 13$; both divisors are of the form $2^k - 3$, so $S_{65}$ has two nontrivial fixed points, 5 and 13. Similarly, the only nontrivial fixed points of $S_{325}$ ($325 = 5 \times 5 \times 13$) are 25 and 65. If $d = 65p$ with $p$ a prime number not congruent to 5 mod 8, the only nontrivial fixed points of $S_d$ are $5p$ and $13p$.

**Example 3.4.** For $m \geq 3$, the number $d = \prod_{l=3}^{m}(2^l - 3)$ has at least $m - 2$ fixed points. Thus, the number of fixed points can be arbitrarily large.

Now we consider cycles. As a matter of terminology, we say that $\{n, S_d(n), S_d^2(n), \ldots, S_d^{k-1}(n)\}$ is a *$k$-loop* if $S_d^k(n) = n$, and that it is a *$k$-cycle* if, in addition, $S_d^j(n) \neq n$ for $0 < j < k$. Obviously, if a $k$-loop is not a $k$-cycle, its first $k'$ elements, for some (unique) factor $k'$ of $k$, do form a $k'$-cycle. We gave on the preceding page examples of cycles of length 2, 3, and 7.

Next, for any positive integer $k$ and any sequence $\pi_k = (p_1, \ldots, p_k)$ of positive integers with $0 < p_1 < p_2 < \cdots < p_k$, define

$$\lambda(\pi_k) = (p_2 - p_1,\, p_3 - p_1,\, \ldots,\, p_k - p_1,\, p_k)$$

and

$$\begin{aligned} A_k(\pi_k) &= A_k(p_1, \ldots, p_k) \\ &= 3^{k-1} + 3^{k-2}2^{p_1} + 3^{k-3}2^{p_2} + \cdots + 3\,2^{p_{k-2}} + 2^{p_{k-1}}. \end{aligned}$$

The following simple result will be very useful in the sequel.

**Lemma 3.5.** *For any given positive integer $k$ and any sequence $\pi_k = (p_1, \dots, p_k)$ of positive integers the following properties are satisfied*:

(1) $A_k(\pi_k) \in \mathbb{D}$.
(2) $A_k(\pi_k) = A_k(\pi'_k)$ *if and only if* $p_j = p'_j$ *for* $j < k$.
(3) $3A_k(\pi_k) + 2^{p_k} - 3^k = A_k\big(\lambda(\pi_k)\big)2^{p_1}$.

*Proof.* Property (1) is trivial. Property (2) has been observed many times—for example, in [Lagarias 1990] (in different notation). The verification of (3) is just a matter of calculation and is left to the reader. □

The next two propositions, which are proved after Remark 3.10, give necessary and sufficient conditions for a trajectory to be, respectively, a loop and a cycle; the first of these statements generalizes the results on the $3x+1$ conjecture of [Böhm and Sontacchi 1978] and of [Lagarias 1990, Theorem 2.1].

**Proposition 3.6.** *Let $d \in 6\mathbb{N} \pm 1$ and let $k$ be a positive integer. An integer $n \in \mathbb{D}$ belongs to a $k$-loop under $S_d$ if and only if there exists a sequence $\pi_k$ as above such that*

$$n\big(2^{p_k} - 3^k\big) = d\,A_k(p_1, \dots, p_{k-1}). \tag{3–1}$$

**Proposition 3.7.** *The $k$-loop that occurs in Proposition 3.6 is a $k$-cycle if and only if all sequences $\pi_k$, $\lambda(\pi_k)$, …, $\lambda^{k-1}(\pi_k)$ are different.*

**Example 3.8.** Take $\pi_k = (p_1, 2p_1, 3p_1, \dots, kp_1)$. If (3–1) is true then $n$ is a fixed point of $S_d$.

**Remark 3.9.** Note that $\lambda^k(\pi_k) = \pi_k$ for all $k$.

**Remark 3.10.** A $k$-loop defined by $d$, $k$ and $\pi_k$ is a $k'$-cycle if and only if $k$ is a multiple of $k'$ and $k'$ is the minimal integer such that $\lambda^{k'}(\pi_k) = \pi_k$.

*Proof of Proposition 3.6.* First we prove that the condition is necessary. We start by a formula for $S_d^j(n)$, for $j \geq 1$. By definition, we get successively,

$$\begin{aligned} S_d(n) &= (3n+d)2^{-l_1}, \\ S_d^2(n) &= \big(3S_d(n)+d\big) = \big(3(3n+d)2^{-l_1}+d\big)2^{-l_2} \\ &= 3^2 n\, 2^{-l_1-l_2} + d\,\big(3\cdot 2^{-l_1} + 2^{-l_1-l_2}\big), \end{aligned}$$

and so on, with

$$l_j = l_j(n) = v_2\big(3S_d^{j-1}(n)+d\big),$$

where $v_2(m)$ is the 2-adic valuation of the positive integer $m$ (the number $e \in \mathbb{N}$ such that $m/2^e$ is an odd integer).

Define $p_j = p_j(n) = l_1 + \cdots + l_j$. Then

$$S_d^j(n) = \big(3^j\, n + d\,A_j(p_1, \dots, p_{j-1})\big)2^{-p_j}.$$

Now let $n \in \mathbb{D}$ be such that $S_d^k(n) = n$; the result follows easily.

To prove that the condition is sufficient, let $d$, $k$ and $\pi_k$ be defined as above and satisfying (3–1). Then, according to Lemma 3.5,

$$\begin{aligned} S_d(n) &= S_d\left(\frac{d}{2^{p_k}-3^k}A_k(\pi_k)\right) \\ &= \frac{d}{2^{p_k}-3^k}\big(3A_k(\pi_k)+2^{p_k}-3^k\big)2^{-l_1} \\ &= \frac{d}{2^{p_k}-3^k}\,A_k\big(\lambda(\pi_k)\big)\,2^{p_1-l_1}. \end{aligned}$$

Since the denominator is odd, as are $d$ and $A_k$, we have $l_1 = p_1$.

Similarly,

$$S_d^2(n) = \frac{d}{2^{p_k}-3^k}\,A_k\big(\lambda^2(\pi_k)\big).$$

And the proof goes on by induction with

$$S_d^k(n) = \frac{d}{2^{p_k}-3^k}\,A_k\big(\lambda^k(\pi_k)\big) = n$$

(see Remark 3.9). □

*Proof of Proposition 3.7.* According to the above definitions of cycles and loops, this follows from the fact that

$$S_d^j(n) = \frac{d}{2^{p_k} - 3^k} A_k\big(\lambda^j(\pi_k)\big) \qquad (3\text{–}2)$$

for all $j \geq 1$; this equality was established during the proof of Proposition 3.6. □

Now we prove that the above constructions can be applied to show that, for suitable $d$'s, there exist cycles of any type.

**Proposition 3.11.** *Let $k$ be a positive integer and consider a sequence $\pi_k$ of $k$ positive integers $0 < p_1 < p_2 < \cdots < p_k$ such that $2^{p_k} > 3^k$ and such that the $\lambda^i(\pi_k)$ are all distinct, for $0 \leq i < k$. Then there exist $d$ and $n$ in $\mathbb{D}$ such that $n$ belongs to a cycle of length $k$ for $S_d$ such that, for $j = 1, \ldots, k$, we have*

$$S_d^j(n) = (3S_d^{j-1}(n) + d)\, 2^{-p_j + p_{j-1}}$$

(*with the conventions $S_d^0(n) = n$ and $p_0 = 0$*).

*Proof.* Proposition 3.11 immediately follows from relation (3–2), if we choose $d = 2^{p_k} - 3^k$ and $n = A_k(p_1, \ldots, p_{k-1})$. □

**Example 3.12.** The smallest cycle of length 4 given by Theorem 2 is $\{65 \to 121 \to 205 \to 331 \to 65\}$, coming from $d = 47$, $(p_1, p_2, p_3, p_4) = (1, 2, 3, 7)$.

**Remark 3.13.** The theorem is also true if we choose $d = Q\,(2^{p_k} - 3^k)$ and $n = Q\,A_k(p_1, \ldots, p_{k-1})$ for any $Q \in \mathbb{D}$.

**Theorem 3.14.** *For a given $d \in \mathbb{D} \cup \{-1\}$ and a given positive integer $k$, the number of $k$-periodic points for $S_d$ is finite.*

*Proof.* To simplify the notation, put $n' = S_d(n) = (3n + d)/2^p$, $n'' = S_d^2(n) = (3n' + d)/2^q$, and so on. For any cycle $\{n, n', n'', \ldots\}$, we choose $n$ minimal. And we want to find an upper bound for $n$. The condition $n' > n$ implies $3n + d \geq 2^p(n + 1)$, so $n \leq d - 4$ for $p \geq 2$. Thus, we assume that $p = 1$ and then

$$n'' = \frac{3\big((3n + d)/2\big) + d}{2^q} = \frac{9n + 5d}{2^{q+1}}.$$

In the 2-periodic and 3-periodic cases, the proof is more effective and instructive, so we will first consider these cases. When $d = -1$ there are no cycles of length 2 or 3 (the easy proof is left to the reader), so we assume $d > 0$ in the following proof.

For $k = 2$, we have $n'' = n$ and the above formula gives $q \geq 3$ and $n \leq \frac{5}{7}d$. Thus, in all cases,

$$n \leq \max\{d - 4,\ \tfrac{5}{7}d\}.$$

For $k = 3$, we have to consider two cases: $n' < n''$ or $n' > n''$. In the first case, by the remark above, we can assume that $q = 1$. Then $n'' = (9n + 5d)/4$,

$$n''' = \frac{27n + 19d}{2^{r+2}},$$

and the condition $n''' = n$ implies $r \geq 3$ and

$$n \leq \frac{19d}{5}.$$

In the case $n'' < n'$ we have $q \geq 2$. For $q \geq 3$, the inequality $n'' > n$ implies $n \leq 5d/7$. Whereas, for $q = 2$,

$$n''' = \frac{27n + 23d}{2^{r+3}},$$

and the condition $n''' = n$ implies $r \geq 2$ and $n \leq 23d/5$. Thus, in all cases

$$n \leq \frac{23d}{5}.$$

The general periodic pattern is a little harder to analyse. The upper bound given below uses Proposition 3.6 and concerns, in fact, points on $k$-loops.

Let $d$ and $k$ be given. For any $(k + 1)$-tuple $\{n, p_1, p_2, \ldots, p_k\}$, with $0 < n$ and $0 < p_1 < \cdots < p_k$, we have

$$n = |d|\, \frac{A_k(p_1, \ldots, p_{k-1}) 2^{-p_k}}{|1 - 3^k\, 2^{-p_k}|}.$$

To deduce from this an upper bound, one has to bound $|1-3^k\,2^{-p_k}|$ from below and $A_k(p_1,\dots,p_{k-1})$ from above. Our bounds are relatively efficient in the first case and relatively rough in the second one. Still, even in the $3x+1$ case, the lower bound we use improves on the known results of [Böhm and Sontacchi 1978; Steiner 1978].

It follows immediately from the deep result of [Baker and Wüstholz 1993] that there is an effectively computable constant $C$ such that

$$|1-3^k\,2^{-l}| > k^{-C}$$

for all $l$ and $k$; this is the desired lower bound.

We also have to study the magnitude of $A_k$. The worst case occurs when $p_{k-j}=p_k-j$ for $0\le j<k$. To simplify the notation, put $p=p_k$. Then

$$\begin{aligned}
A_k(p_1,\dots,p_{k-1})2^{-p} &= 2^{-p}\bigl(3^{k-1}+3^{k-2}\,2^{p_1}+\dots+3\cdot 2^{p_{k-2}}+2^{p_{k-1}}\bigr)\\
&\le 2^{-p}\bigl(3^{k-1}+3^{k-2}\,2^{p-k+1}+\dots+2^{p-1}\bigr)\\
&= 3^{k-1}\,2^{-p}+2^{-k+1}\bigl(3^{k-1}-2^{k-1}\bigr)\\
&< 1+(3/2)^{k-1}-1=(3/2)^{k-1},
\end{aligned}$$

since $2^p>3^k$. This gives the inequality

$$n < d\,k^C\left(\tfrac{3}{2}\right)^{k-1}.$$

Hence the result. □

## 4. A NORMALIZED VERSION OF T

We now consider a reformulation of the $3x+1$ problem that has certain formal advantages. When we passed form $T$ to $S$, we lost the "periodically linear" character of $T$ [Conway 1972; Lagarias 1985]; yet, the previously mentioned undecidability result of [Conway 1972] concerns just such functions. Another formal difficulty is that $S$ is defined as a composition of the functions $T_0 : n\mapsto n/2$ and $T_1 : n\mapsto (3n+1)/2$ acting outside the domain $\mathbb{D}$ of definition of $S$. Both considerations remain valid for the general definition $S_d$ as well.

On the other hand, as observed in Section 2, the function $T$ is for many reasons not a very convenient formal tool; in particular, it has the unpleasant property of $\#T^{-1}(n)$ being either 1 or 2. To overcome these shortcomings of both $T$ and $S$, we define here a function $W$ on the domain $\mathbb{D}$, which would replace $T$ in the iterative definition of $S$, retaining at the same time the periodically linear character of $T$. Set

$$W_0(n)=\begin{cases}(n-1)/4 & \text{if } n\equiv 5 \bmod 24,\\ (n-5)/16 & \text{if } n\equiv 85 \bmod 96,\end{cases}$$

$$W_1(n)=\begin{cases}(3n+1)/2 & \text{if } n\equiv 7,11 \bmod 12,\\ (3n+1)/4 & \text{if } n\equiv 1,17 \bmod 24,\\ (3n+1)/8 & \text{if } n\equiv 13 \bmod 48,\\ (3n+1)/16 & \text{if } n\equiv 37 \bmod 96.\end{cases}$$

One checks easily that $\mathrm{Def}(W_0)$ and $\mathrm{Def}(W_1)$, the domains of definition of $W_0$ and $W_1$, are disjoint, and that their union is $\mathbb{D}$. In fact, $W_0$ and $W_1$ are bijections from their respective domains onto $\mathbb{D}$. Now, for $n\in\mathbb{D}$, set

$$W(n)=\begin{cases}W_0(n) & \text{if } n\in\mathrm{Def}(W_0),\\ W_1(n) & \text{if } n\in\mathrm{Def}(W_1);\end{cases}$$

then $W$ is a two-to-one function from $\mathbb{D}$ to $\mathbb{D}$.

The function $W$ can replace $T$ as a "primitive" periodically linear "skeleton" of $S$, in the following sense. Recall that

$$S(n)=\mathrm{odd}(3n+1)=T_0^{v_2(T_1(n))}T_1(n).$$

Thus

$$S(n)=W(n)=W_1(n)$$

if $n\in\mathrm{Def}(W_1)$, and

$$S(n)=W^{k+1}(n)=W_1\bigl(W_0^k(n)\bigr)$$

if $n\in\mathrm{Def}(W_0)$, $W_0(n)\in\mathrm{Def}(W_0)$, ..., $W_0^{k-1}\in\mathrm{Def}(W_0)$ but $W_0^k\in\mathrm{Def}(W_1)$.

We hope that the fact that $W$ is 2-to-1 will prove to be a significant advantage over $T$; the 2-to-1 property is crucial if one wants to study the problem within the framework of any theory related to discrete dynamics, both theoretical (in the spirit of Furstenberg) and computational.

Clearly, for any $n \in \mathbb{D}$, the set $\#S^{-1}(n)$ is infinite, and it can be described explicitly as

$$S^{-1}(n) = \{W_1^{-1}(n), W_0^{-1}(W_1^{-1}(n)), W_0^{-2}(W_1^{-1}(n)), W_0^{-3}(W_1^{-1}(n)), \ldots\}.$$

An explicit arithmetic formula for any set $S_d^{-1}(n)$ is given in [Belaga 1995].

**Remark 4.1.** The definition of $W$ can be extended to the $3x+d$ context, so that functions can be defined on $\mathbb{D}$ that are periodically linear and that can be used to iteratively define the corresponding functions $S_d$. This construction is left to the interested reader as a straightforward technical exercice.

**Remark 4.2.** To be precise, the result of [Conway 1972] concerns transformations on $\mathbb{Z}$. Still, with minor technical adjustments, the transformation $W$ can be extended to $\mathbb{Z}$ preserving its periodically linear features.

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