Periodic Gaussian Moats

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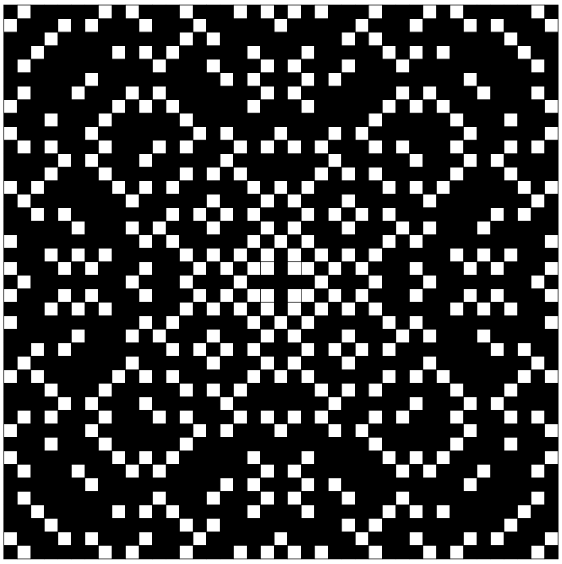

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A question of Gordon, mistakenly attributed to Erdős, asks if one can start at the origin and walk from there to infinity on Gaussian primes in steps of bounded length. We conjecture that one can start anywhere and the answer is still no. We introduce the concept of periodic Gaussian moats to prove our conjecture for step sizes of $\sqrt{2}$ and 2.

1. MOATIVATION

Consider an elementary question: starting at the origin, and thereafter stepping only on the rational primes, can one walk to infinity in steps of bounded length? The answer it no, because for any $N \in \mathbb{Z}^+$ we can produce N consecutive composites: $(N+1)!+2$, $(N+1)!+3$, $\ldots$, $(N+1)!+N+1$.

At the 1962 International Congress of Mathematicians in Stockholm, Basil Gordon proposed an analogous journey on the Gaussian primes: starting at the origin, and thereafter stepping only on the Gaussian primes, can one walk to infinity in steps of bounded length? (The Gaussian primes are the primes in $\mathbb{Z}[i]$; shown at the left in white are all the Gaussian primes $x+iy$ with $-20 \leq x, y \leq 20$.)

The same problem was also posed in [Guy 1994; Montgomery 1994]. Still open, it has seemingly joined the ranks of the many number theoretic questions that are easy to state but deeply difficult to solve. As often happens in such cases, one turns to computational techniques for insight.

To our knowledge three such computational papers have appeared in the literature. Jordan and Rabung [1970] first showed that at least one step of size 4 is necessary on a walk to infinity. More recently, Gethner, Wagon, and Wick [Gethner et al. 1998] showed that at least one step of size $4\sqrt{2}$ is implicit. In both cases, one shows the existence of regions of composite Gaussian integers, called *Gaussian moats*, which surround the origin. In [Wagon 1996] the computational techniques used in [Gethner et al. 1998] are described for some of the small moats.

If one believes, as we do, that one *cannot* walk to infinity on the primes in steps of bounded length, one way of showing this would be to prove the existence of Gaussian moats of arbitrary minimum width. We call a Gaussian moat of minimum width k a *Gaussian k-moat*. For complete background and history, see [Gethner et al. 1998].

We offer the following conjecture in answer to Gordon's question:

Conjecture. *Given any $k > 0$, there is an M_k such that taking steps of at most size k and starting on any Gaussian prime one can take at most M_k steps on distinct Gaussian primes before being forced to step on a composite Gaussian integer.*

The conjecture implies that, on any walk to infinity along Gaussian primes, one necessarily takes steps of size at least k infinitely often, and within predictable bounds. Interestingly, Jordan and Rabung [1976] proved that the conjecture is true for $k = \sqrt{2}$, with $M_k = 48$, but they make reference neither to the moat problem nor to their earlier paper [Jordan and Rabung 1970].

2. PERIODICITY

Our conjecture arose after extensive experiments by computer. Though finding Gaussian k-moats for arbitrary k would certainly solve Gordon's original problem, the reality seems to be that, for each k, there is a single connected maximal Gaussian moat that extends throughout the complex plane and whose complement may consist of infinitely many nontrivial and nonoverlapping compact subsets of uniformly bounded size. We have proved that such is the case for $k = \sqrt{2}$ and for $k = 2$.

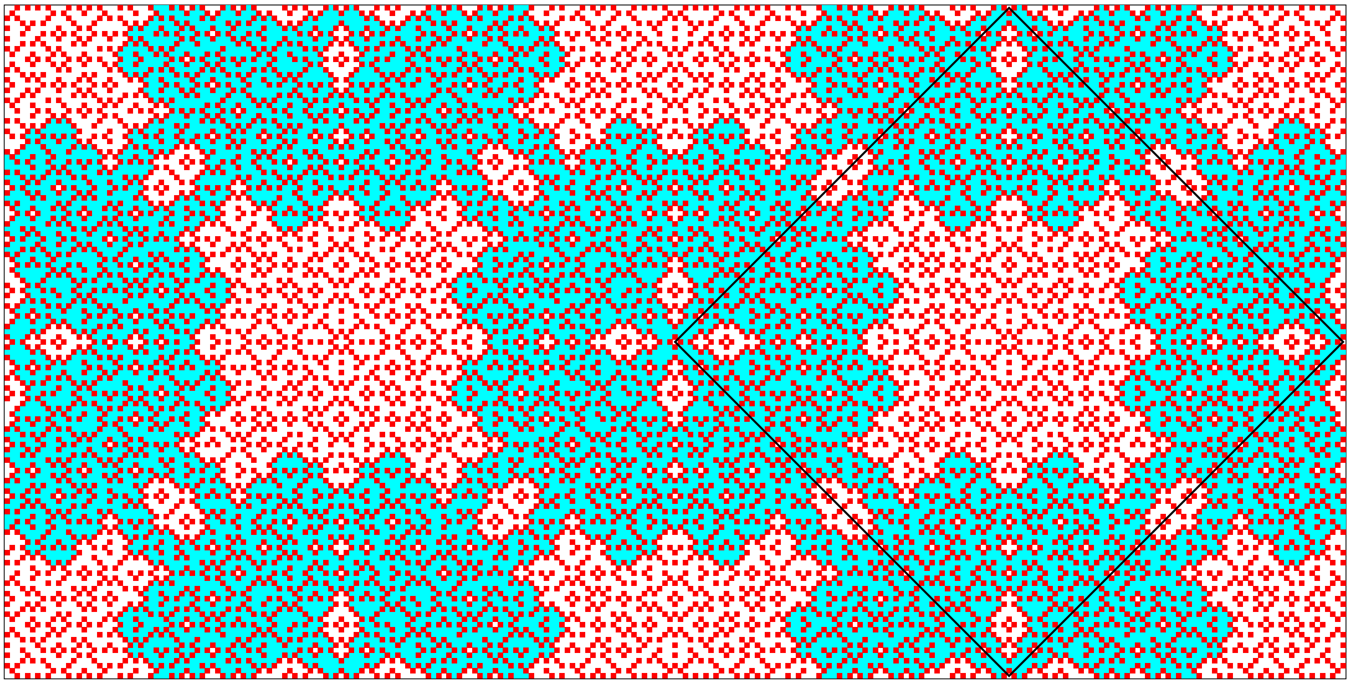

FIGURE 1. The origin is at the center. The set B defined on page 291 is shown in dark gray (red in the online version). The connected region shown in light gray (blue in the online version) is a periodic set contained in the maximal Gaussian $\sqrt{2}$-moat. The black lines bound a period parallelogram S for this set.

We begin by describing our method for showing the existence of the plane-filling Gaussian $\sqrt{2}$-moat. Let S be the set of lattice points in the square with vertices 0, $65+65i$, 130, and $65-65i$, and let B be the set of Gaussian integers relatively prime to $65+65i$. In other words, sieve out all multiples of the Gaussian prime divisors of $65+65i$. The set B is periodic with period S. Color the points in B red (see Figure 2). Now connect two points in B with a line if and only if they are at most $\sqrt{2}$ apart. With the network of lines acting as boundaries, color the connected region in the plane containing the point $2+2i$ blue.

S is a period parallelogram generated by $65+65i$ in $\mathbb{Z}[i]$. The blue region extends to all sides of S and contains congruent points on opposite sides, the congruence being modulo S, and therefore by periodicity extends to the entire complex plane in a single connected component. Except for primes dividing 130 and their associates, all points in the extended blue region are Gaussian composites by construction; thus, the maximal Gaussian $\sqrt{2}$-moat contains the periodic blue region as a submoat. In particular, the number of lattice points in S, namely $(65\sqrt{2})^2 = 8450$, gives an upper bound for the number of steps of size at most $\sqrt{2}$ that one can take before being forced to step on a Gaussian composite.

We used the same method to show that the conjecture is true for $k = 2$. As expected, the computations were longer and more complicated than for $k = \sqrt{2}$. To find a 2-moat, we used the lattice in $\mathbb{Z}[i]$ generated by 7113990 and the period parallelogram with vertices

$$0,\ 7113990,\ 7113990+7113990i,\ 7113990i.$$

S is the set of lattice points in this region and B is the set of all Gaussian integers relatively prime to 7113990; that is, it is obtained by sieving out from S all multiples of 2, 3, $2\pm i$, $3\pm 2i$, $4\pm i$, $5\pm 2i$, and $6\pm i$. As before, B is periodic with period S. One can use a smaller lattice to generate a periodic Gaussian 2-moat (and we ultimately did so), but we chose S for ease in programming.

We took advantage of the symmetry about the line $x = \frac{1}{2}\cdot 7113990 = 3556995$. More specifically, starting at the point $18i$, we produced a connected portion of a Gaussian 2-moat in the region

$$\{(x,y) : 0 \le x \le 3556995,\ 0 \le y \le 400\}.$$

Importantly, the 2-moat includes points on the y-axis and the line $x = 3556995$ (to insure that the moat truly connects at both vertical sides of the period parallelogram). Symmetry about the line $y = x$ guarantees a vertical component, a priori connected to the horizontal component, that connects the two horizontal sides of the period parallelogram. This periodic submoat of the maximal Gaussian 2-moat shows that the latter extends throughout the complex plane in a single connected component, whose complement may contain infinitely many nontrivial and nonoverlapping compact subsets. The number of lattice points in S, 7113990^2, gives an upper bound for the number of steps one can take on distinct Gaussian primes before being forced to take a step on a Gaussian composite. The computation took approximately 9.5 hours on an Atari 1040ST.

3. MOAT'S END?

The next natural step would be to show that the maximal Gaussian $\sqrt{10}$-moat has a connected periodic subset. We did some random experiments to make a best guess at a set of Gaussian primes to sieve out from $\mathbb{Z}[i]$, and decided upon all primes of norm less than 200. We followed what we believe to be a portion of the Gaussian $\sqrt{10}$-moat out to $x = 3336000$ and were able to stay within $0 \le y \le 500$. The computation took about 12 hours. Our intention was simply to gain further evidence in support of our conjecture.

The distribution of rational primes, mysterious at best, is linked to the distribution of Gaussian primes, and the Gaussian primes have the added difficulty of being 2-dimensional. The method of sieving out small primes to find periodic subsets of a k-moat, for arbitrary k seems hopeful for finding a more general solution to the conjecture that all k-moats fill the entire complex plane.

ACKNOWLEDGEMENT

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