The Subgroups of M_{24}, or How to Compute the Table of Marks of a Finite Group

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CONTENTS

Let G be a finite group. The table of marks of G arises from a characterization of the permutation representations of G by certain numbers of fixed points. It provides a compact description of the subgroup lattice of G and enables explicit calculations in the Burnside ring of G. In this article we introduce a method for constructing the table of marks of G from tables of marks of proper subgroups of G. An implementation of this method is available in the GAP language. These computer programs are used to construct the table of marks of the sporadic simple Mathieu group M_{24}. The final section describes how to derive information about the structure of G from its table of marks via the investigation of certain Möbius functions and the idempotents of the Burnside ring of G. Tables with detailed information about M_{24} and other groups are included.

The concept of a table of marks of a finite group G was introduced by William Burnside in the second edition of his classic book *Theory of groups of finite order* [1911, chapter XII]. This table provides a means to characterize the permutation representations of G up to equivalence. At the same time the table of marks describes in some detail the poset (partially ordered set) of all conjugacy classes of subgroups of G. It thereby provides a very compact description of the subgroup lattice of G.

Traditionally, the computation of the table of marks of G starts by constructing the complete subgroup lattice of G. The table of marks of G then is derived mainly by counting inclusions between different conjugacy classes of subgroups of G. This method, described in [Felsch and Sandlöbes 1984],

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is implemented in several computer systems and works for groups up to a certain size.

Our purpose here is to introduce a method for the construction of the table of marks that is independent of the knowledge of the complete subgroup lattice of G and therefore can be applied to groups G that are too big to compute their complete subgroup lattice. The main idea of this approach is to use the "known" tables of marks of subgroups of G and to induce them to G in order to determine the table of marks of G. So the input of this method basically consists of the tables of marks of the maximal subgroups of G. In order to combine these tables into the complete table of marks of G we have to determine the fusion maps from the sets of conjugacy classes of subgroups of the maximal subgroups of G to the set of conjugacy classes of subgroups of G. Such a map associates to the M-conjugacy class of a subgroup U of a maximal subgroup M of G its G-conjugacy class.

These fusion maps can to a large extent be recovered from the tables of marks of the maximal subgroups of G by elementary group theory. We will approximate the fusion maps step by step using this information and, now and then, taking advantage of selected bits of additional information about G until the approximation process eventually stops with the correct fusion maps.

Additional information about the group G stems for example from the character table of G. If G has a small permutation representation we can use explicit representatives of the conjugacy classes of subgroups of the maximal subgroups of G to derive additional information. It is, however, not necessary to have explicit embeddings of the maximal subgroups into G. But, any source of information is welcome if it can answer the questions that arise in the approximation process. The point is that the number of open questions is hopefully small in comparison to the size of the group.

Computer programs that perform this method interactively have been implemented in the GAP language [Schönert et al. 1994]. This package of functions is available from the author.

With this method we can in particular determine the number of subgroups of the simple Mathieu group M_{24} of order $244\,823\,040 = 2^{10}\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 23$.

Theorem. M_{24} has $1\,363\,957\,253$ subgroups in $1\,529$ conjugacy classes.

The construction of the table of marks of M_{24} from which the above number of subgroups is derived is described in some detail in Section 6. This group provides an extensive example of how the computer programs can be used. The list of all conjugacy classes of subgroups of M_{24} is not printed here. The complete list of the classes together with their maximal subgroups, their minimal overgroups and their normalizers, has 82 pages [Pfeiffer 1995].

This paper is organized as follows. Section 1 recalls basic properties of finite group actions, introduces the table of marks and describes its relation to the Burnside ring of a finite group. Section 2 describes the concept of induction of marks; the basic induction formula is proved in Theorem 2.2.

The next sections describe the method used to determine the fusion maps from the maximal subgroups into the given group. First, in Section 3, we use the projective special linear group $L_2(7)$ as a rather informal example to illustrate the process and the problems it raises. This is formalized in Section 4, which also provides the theoretical tools that govern the approximation process. Section 5 summarizes the method into a strategy.

In Section 8 the table of marks of G is used to investigate the Möbius function of the subgroup lattice of G and the idempotents of the Burnside ring of G, together with their implications on the structure of G.

Section 6 also contains tables with more detailed information about the subgroups of M_{24} and subgroups of other simple (and almost simple) groups.

1. THE BURNSIDE RING AND THE TABLE OF MARKS

We recall basic facts about finite group actions, define the table of marks, and describe its relation to the subgroup structure and the Burnside ring.

Let G be a finite group. Denote by $\mathfrak{S}_G = \{U : U \leq G\}$ the set of all subgroups of G. Then $\mathfrak{S}_G$ is a partially ordered set (*poset*) with incidence relation $\leq$. The group G acts on $\mathfrak{S}_G$ by conjugation, that is, via $U^g = g^{-1}Ug$ for $U \leq G$ and $g \in G$. This G-action respects incidence: if $U \leq V$ then $U^g \leq V^g$ for all $U, V \leq G$ and $g \in G$. We denote the G-orbit of $U \leq G$ (that is, the conjugacy class of subgroups of G that contains U) by $[U]_G$ and usually omit the subscript as long as no confusion can arise. The set of G-orbits $\mathfrak{S}_G/G = \{[U] : U \leq G\}$ is also a poset, with incidence $[U] \leq [V]$ if $U \leq V^g$ for $U, V \leq G$ and some $g \in G$. We will refer to $\mathfrak{S}_G/G$ as the *poset structure* of G.

A (right) *G-set* X is a set X together with an *action* $(x, g) \mapsto x \cdot g : X \times G \to X$ such that $x \cdot 1 = x$ and $x \cdot (g_1 g_2) = (x \cdot g_1) \cdot g_2$ for all $x \in X$ and all $g_1, g_2 \in G$. Every G-set X decomposes into a disjoint union of orbits $\{x \cdot g : g \in G\}$, each of which is itself a G-set. A G-set is *transitive* if it consists of only one orbit. All G-sets in this article are assumed to be finite.

A homomorphism between two G-sets X and Y is a map $\psi : X \to Y$ such that $\psi(x \cdot g) = \psi(x) \cdot g$ for all $x \in X$ and all $g \in G$. Two G-sets X and Y are *isomorphic* if there exists a bijective homomorphism $\psi : X \to Y$.

Let $G_1 = 1$, G_2, $\ldots$, $G_r = G$ be representatives of the conjugacy classes of subgroups of G. Then $\mathfrak{S}_G/G = \{[G_i] : i = 1, \ldots, r\}$. For each subgroup $U \leq G$, the group G acts transitively on the set $U\backslash G = \{Ug : g \in G\}$ of right cosets of U in G. Conversely, every transitive G-set X is isomorphic to a G-set $U\backslash G$ where U is a point stabilizer of X in G. For every $g \in G$ the G-set $U^g\backslash G$ is isomorphic to $U\backslash G$. Thus every transitive G-set is isomorphic to $G_i\backslash G$ for some $i \leq r$.

Definition 1.1. Let G be a finite group.

(i) Let X be a finite G-set and let $U \leq G$. The *mark* $\beta_X(U)$ of U on X is defined as

$$\beta_X(U) = \left|\mathrm{Fix}_X(U)\right|,$$

where

$$\mathrm{Fix}_X(U) = \{x \in X : x \cdot u = x \text{ for all } u \in U\}$$

is the set of fixed points of the subgroup U in the action of G on X.

(ii) The *table of marks* of G is the square matrix

$$M(G) = (\beta_{G_i\backslash G}(G_j))_{i,j}$$

where both G_i and G_j run through the system of representatives of the conjugacy classes of subgroups of G.

Remark. If X and Y are isomorphic G-sets, we have $\beta_X(U) = \beta_Y(U)$ for all $U \leq G$. Moreover, $\beta_X(U) = \beta_X(U^g)$ for all $U \leq G$ and all $g \in G$. The table of marks thus consists of a "complete" list of marks of transitive G-sets.

Let $U \leq G$ and consider the G-set $G_i\backslash G$. Then U has fixed points in that action if and only if U is contained in a one point stabilizer, that is, in at least one conjugate of G_i. Thus the table of marks describes the poset $\mathfrak{S}_G/G$: the incidence matrix of this poset is obtained from $M(G)$ by replacing every nonzero entry by 1.

But $M(G)$ contains far more information about the subgroup structure of G. This is due to the following recalculation of the value of a mark.

Proposition 1.2. *Let $U, V \leq G$. Then*

$$\beta_{V!G}(U) = \left|\{V^g : g \in G,\, U \leq V\}\right| \left|N_G(V) : V\right|.$$

Proof. By the definition of a mark, $\beta_{V\backslash G}(U)$ is the number of cosets of V in G that are fixed by the subgroup U. That is,

$$\beta_{V\backslash G}(U) = \left|\{Vg : g \in G, Vgu{=}Vg \text{ for all } u \in U\}\right|.$$

Now for any given $g \in G$ we have $Vgu = Vg$ for all $u \in U$ if and only if $U \leq V^g$. And since there are exactly $|V|$ elements g that give the same coset Vg this means

$$\beta_{V\backslash G}(U) = \left|\{g \in G : U \leq V^g\}\right| / |V|.$$

The claim finally follows from the fact that there are exactly $\left|N_G(V)\right|$ elements g that give the same conjugate V^g of V. $\square$

The next lemma collects some easy consequences of the above formula. In particular the numbers of incidences between two conjugacy classes of subgroups of G can be derived from $M(G)$.

The table of marks of the alternating group A_5 of order 60 in Table 1 serves as an example. A_5 has nine conjugacy classes of subgroups. They are distinguished by their orders and have isomorphism types 1, 2, 3, 2^2, 5, S_3, D_{10}, A_4, and A_5. The rows of the table correspond to the transitive G-sets $U\backslash G$.

	1	2	3	2^2	5	S_3	D_{10}	A_4	A_5
$1\backslash G$	60								
$2\backslash G$	30	2							
$3\backslash G$	20	.	2						
$2^2\backslash G$	15	3	.	3					
$5\backslash G$	12	.	.	.	2				
$S_3\backslash G$	10	2	1	.	.	1			
$D_{10}\backslash G$	6	2	.	.	1	.	1		
$A_4\backslash G$	5	1	2	1	.	.	.	1	
$A_5\backslash G$	1	1	1	1	1	1	1	1	1

TABLE 1. The table of marks of A_5.

Lemma 1.3. *Let $U, V \leq G$. Then the following hold.*

(i) *The first entry of every row of $M(G)$ is the index of the corresponding subgroup*

$$\beta_{V!G}(1) = \left|G : V\right|.$$

(ii) *The entry on the diagonal is its index in its normalizer in G,*

$$\beta_{V!G}(V) = \left|N_G(V) : V\right|.$$

(iii) *The length of the conjugacy class $[V]$ of V is given by*

$$\left|[V]\right| = \left|G : N_G(V)\right| = \frac{\beta_{V!G}(1)}{\beta_{V!G}(V)}.$$

(iv) *The number of conjugates of V that contain U is given by*

$$\left|\{V^g : g \in G,\, U \leq V^g\}\right| = \frac{\beta_{V!G}(U)}{\beta_{V!G}(V)}.$$

Denote by $\nu_G(V,U) = \left|\{U^g : g \in G,\, U^g \leq V\}\right|$ the number of conjugates of a subgroup U of G contained in a fixed subgroup V of G. These numbers also are determined by $M(G)$.

Proposition 1.4. *Let $U, V \leq G$. Then the number of conjugates of U that are contained in V is*

$$\nu_G(V,U) = \frac{|V|}{\left|N_G(U)\right|}\,\beta_{V!G}(U) = \frac{\beta_{V!G}(U)\,\beta_{U!G}(1)}{\beta_{U!G}(U)\,\beta_{V!G}(1)}.$$

Proof. In the subgroup lattice of G the number of edges joining the class $[V]$ of V and the class $[U]$ of U can be expressed in two different ways as length of the class times the number of edges leaving one member of the class, that is

$$\left|[U]\right| \cdot \left|\{V^g : g \in G,\, U \leq V^g\}\right| = \left|[V]\right| \cdot \left|\{U^g : g \in G,\, U^g \leq V\}\right|.$$

Thus $\nu_G(V,U)$ can be expressed in terms of marks by Lemma 1.3. □

On the other hand $M(G)$ is determined by the numbers $\nu_G(V,U)$ for all $V, U \leq G$ and the additional knowledge of the index $\left|G : V\right|$ for every $V \leq G$.

Lemma 1.5. *Let $V, U \leq G$. Then*

$$\beta_{V!G}(U) = \left|G : V\right|\, \nu_G(V,U)/\nu_G(G,U).$$

Denote for any G-set X its isomorphism class by $[X]$. The *Burnside ring* $\Omega(G)$ of G is the free abelian group

$$\Omega(G) = \Big\{\sum_{i=1}^{r} a_i[G_i\backslash G] : a_i \in \mathbb{Z}\Big\}$$

generated by the isomorphism classes of transitive G-sets $[G_i\backslash G]$, $i = 1, \dots, r$. Here the sum $[X]+[Y]$ of the isomorphism classes of G-sets X and Y is the isomorphism class $[X \,\dot\cup\, Y]$ of the disjoint union of X and Y. Moreover, their product $[X] \cdot [Y]$ is the isomorphism class $[X \times Y]$ of the Cartesian product of X and Y. This turns $\Omega(G)$ into a commutative ring with identity $[G\backslash G]$.

Let X and Y be G-sets and let $U \leq G$. Then

$$\mathrm{Fix}_{X\dot\cup Y}(U) = \mathrm{Fix}_X(U) \,\dot\cup\, \mathrm{Fix}_Y(U),$$
$$\mathrm{Fix}_{X\times Y}(U) = \mathrm{Fix}_X(U) \times \mathrm{Fix}_Y(U).$$

Therefore

$$\beta_{X\dot\cup Y}(U) = \beta_X(U) + \beta_Y(U),$$
$$\beta_{X\times Y}(U) = \beta_X(U) \cdot \beta_Y(U).$$

Thus, if we define β_X for each $X \in \Omega(G)$ to be the r-tuple $\beta_X = (\beta_X(G_1), \dots, \beta_X(G_r))$, the map

$$\beta\colon X \mapsto \beta_X$$

is a ring homomorphism from $\Omega(G)$ to $\mathbb{Z}^r$.

Let $X = \sum a_i[G_i\backslash G] \in \Omega(G)$. Then β_X can be expressed in terms of the table of marks $M(G)$ as

$$\beta_X = (a_1, \dots, a_r) \cdot M(G).$$

Moreover, the G-set X is characterized up to isomorphism by β_X.

Theorem 1.6 (Burnside). *The homomorphism*

$$\beta\colon \Omega(G) \to \mathbb{Z}^r$$

is injective. In other words, two G-sets X and Y are isomorphic if and only if $\beta_X = \beta_Y$.

Proof. We may assume that the representatives G_i are sorted in such a way that $[G_i] \leq [G_j]$ implies $i \leq j$. Then $\beta_{G_i\backslash G}(G_j) = 0$ unless $j \leq i$ and $\beta_{U\backslash G}(U) \geq 1$ for all $U \leq G$. Thus the matrix $M(G)$ is lower triangular with nonzero diagonal entries, hence invertible. □

Let X be a G-set. The *permutation character* π_X of G on X is defined as $\pi_X(g) = |\mathrm{Fix}_X(g)|$ for any element $g \in G$. This number, of course, coincides with the mark $\beta_X(\langle g\rangle)$ of the cyclic subgroup generated by g on X. Therefore, the table of marks $M(G)$ contains in the columns corresponding to cyclic subgroups a complete list of transitive permutation characters 1_V^G of G corresponding to the transitive G-sets $V\backslash G$.

The following proposition [Kerber 1991, 3.2.18] provides a way to determine the columns of $M(G)$ that correspond to cyclic subgroups.

Let $e_i = (e_i(G_1), \dots, e_i(G_r)) \in \mathbb{Z}^r$ be the primitive idempotent of $\mathbb{Z}^r$ corresponding to the conjugacy class of G_i, that is $e_i(G_j) = \delta_{ij}$, and write

$$e_i = \sum_j e_{ij}\beta_{G_j\backslash G}$$

with rational coefficients e_{ij}. (The matrix (e_{ij}) then is the inverse of $M(G)$.)

Proposition 1.7. *Let $i, j \leq r$ and let e_{ij} and G_i be as above. Then*

$$\sum_j e_{ij} = \begin{cases} \varphi(|G_i|)/|N_G(G_i)| & \text{if } G_i \text{ is cyclic,} \\ 0 & \text{otherwise,} \end{cases}$$

where φ denotes the Euler function.

Proof. Let $z = (z_1, \dots, z_r) \in \mathbb{Z}^r$. For any subgroup $U \leq G$ denote $z(U) = z_i$ if U is a conjugate of G_i. Then z defines a class function π_z of G via $\pi_z(g) = z(\langle g\rangle)$. Note that $\beta_{G_j\backslash G}$ this way yields the permutation character $1_{G_j}^G$ and that $(1_G, 1_{G_j}^G) = 1$ for each $1 \leq j \leq r$. Hence $\pi_{e_i} = \sum_j e_{ij}1_{G_j}^G$ and

$$\sum_j e_{ij} = \sum_j e_{ij}(1_G, 1_{G_j}^G) = \Big(1_G, \sum_j e_{ij}1_{G_j}^G\Big)$$
$$= (1_G, \pi_{e_i}),$$

where $\pi_{e_i} = 0$ and therefore $(1_G, \pi_{e_i}) = 0$ unless G_i is cyclic.

In that case the result follows from the fact that the $\varphi(|G_i|)$ different generators of G_i lie in

$$\varphi(|G_i|)/|N_G(G_i) : C_G(G_i)|$$

different conjugacy classes of elements of G that contribute $1/|C_G(G_i)|$ each to $(1_G, \pi_{e_i})$. □

2. INDUCTION OF MARKS

Let M be a subgroup of G. Then the marks of M can be induced to marks of G by means of the following trivial observation. A subgroup V of M is also a subgroup of G, and V has the same subgroups regardless of whether it is viewed as a subgroup of M or as a subgroup of G. But in general not every subgroup of V that is conjugate in G to a given subgroup U of V is conjugate to U in M. More precisely, a G-conjugacy class of subgroups of V is a disjoint union of M-conjugacy classes of subgroups of V. In terms of numbers of subgroups this means

Lemma 2.1. *Let $V \leq M \leq G$ and $U \leq G$. Then the number of G-conjugates of U that are contained in V is*

$$\nu_G(V,U) = \sum_{U' \sim U} \nu_M(V,U'),$$

where the sum ranges over all representatives U' of conjugacy classes of M that are conjugate to U in G.

This leads to the following induction formula for tables of marks.

Theorem 2.2 (Induction of marks). *Let $V \leq M \leq G$ and $U \leq G$. Then the mark $\beta_{V!G}(U)$ is given by*

$$\beta_{V!G}(U) = \left|N_G(U)\right| \sum_{U' \sim U} \frac{1}{\left|N_M(U')\right|} \beta_{V!M}(U')$$

where the sum ranges over all representatives U' of conjugacy classes of M that are conjugate to U in G.

Proof. By Proposition 1.4 the mark $\beta_{V\backslash G}(U)$ is expressed in terms of numbers of subgroups as

$$\beta_{G\backslash V}(U) = \frac{\left|N_G(U)\right|}{\left|V\right|} \nu_G(V,U).$$

By the preceding Lemma, this equals

$$\frac{\left|N_G(U)\right|}{\left|V\right|} \sum_{U' \sim U} \nu_M(V,U').$$

And by Proposition 1.4 this translates into

$$\beta_{V\backslash G}(U) = \frac{\left|N_G(U)\right|}{\left|V\right|} \sum_{U' \sim U} \frac{\left|V\right|}{N_M(U')} \beta_{M\backslash V}(U'),$$

which completes the proof. □

Remark. Replacing subgroups by group elements and normalizers by centralizers in the above induction formula yields the induction formula for ordinary characters of finite groups: if φ is a character of $M \leq G$ and $g \in G$ then the value at g of the induced character φ^G is

$$\varphi^G(g) = \left|C_G(g)\right| \sum_{g' \sim g} \frac{1}{\left|C_M(g')\right|} \varphi(g'),$$

where the sum ranges over all representatives g' of conjugacy classes of M that are conjugate to g in G [Isaacs 1994, p. 64].

In order to compute the table of marks of G by induction of marks it is sufficient to know the tables of marks of representatives of the conjugacy classes of maximal subgroups of G since every proper subgroup of G is contained in a maximal subgroup M of G. The remaining problem is to determine the *fusion* from the maximals to G in order to know which representatives of subgroups U' of M are conjugate to a representative U of subgroups of G. This means to determine for each maximal subgroup a map from its conjugacy classes of subgroups to the conjugacy classes of subgroups of G. The next sections provide tools to determine the conjugacy classes of subgroups of G together with the fusion maps from its maximals.

3. $L_2(7)$, A SMALL EXAMPLE

Consider the projective special linear group $G = L_2(7)$ of order 168. It has three conjugacy classes of maximal subgroups, two classes of type S_4 and one class of type $7:3$; see, for example, the *Atlas of finite groups* [Conway et al. 1985, p. 3]. Assume that we already know the tables of marks of these groups, that is the corresponding posets of conjugacy classes of subgroups together with additional incidence information. The poset structure of S_4 is given in Figure 1 and the underlying set of elements is

$$\{1, 2a, 2b, 3, 4, 2^2a, 2^2b, S_3, D_8, A_4, S_4\},$$

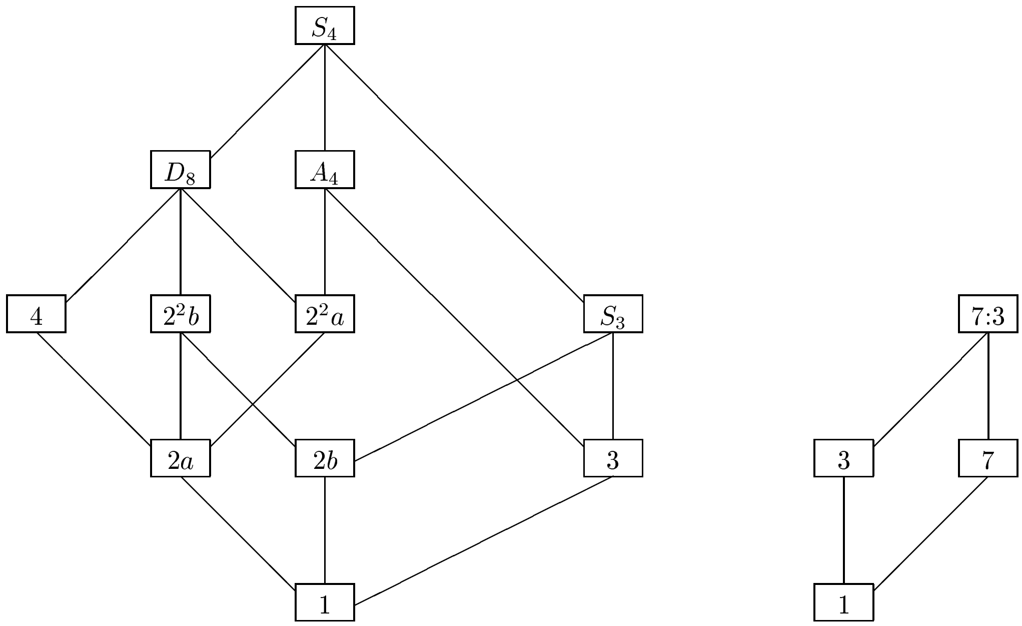

FIGURE 1. The poset structures of S_4 and 7:3.

where $2a$ and 2^2a are contained in A_4. The poset structure of $7:3$ is also given in Figure 1 with underlying set of elements

$$\{1, 3, 7, 7:3\}.$$

In order to construct the poset structure of $L_2(7)$ we start with the disjoint union of two copies of the poset structure of S_4, a yellow (Y) and a red (R) one, say, and a blue (B) copy of the poset structure of $7:3$. They form one colored diagram with vertex set

$$\{1(Y),\ 2a(Y),\ 2b(Y),\ 3(Y),\ 4(Y),\ 2^2a(Y),\ 2^2b(Y),\ S_3(Y),\ D_8(Y),\ A_4(Y),\ S_4(Y),\ 1(R),\ 2a(R),\ 2b(R),\ 3(R),\ 4(R),\ 2^2a(R),\ 2^2b(R),\ S_3(R),\ D_8(R),\ A_4(R),\ S_4(R),\ 1(B),\ 3(B),\ 7(B),\ 7:3(B)\}.$$

Here a symbol like $2^2a(Y)$ is just a name for a vertex in the yellow part of the disjoint union of diagrams. The fact that it denotes an elementary abelian group of order 4 follows from the information that is stored in the table of marks, rather than from its name. Of course, the names we are working with here were carefully chosen as to indicate the type of object they denote.

According to Lemma 1.3(i) the table of marks contains information about the size of each subgroup and conjugate subgroups of $L_2(7)$ have the same size. Therefore, if we split the whole set of vertices into subsets according to the orders of the corresponding subgroups then only vertices in the same subset can be conjugate in $L_2(7)$. This yields the following partition on the set of vertices.

$$\{1(Y), 1(R), 1(B)\},\ \{2a(Y), 2b(Y), 2a(R), 2b(R)\},\ \{3(Y), 3(R), 3(B)\},\ \{4(Y), 2^2a(Y), 2^2b(Y), 4(R), 2^2a(R), 2^2b(R)\},\ \{S_3(Y), S_3(R)\},\ \{7(B)\},\ \{D_8(Y), D_8(R)\},\ \{A_4(Y), A_4(R)\},\ \{7:3(B)\},\ \{S_4(Y), S_4(R)\}.$$

Our aim now is to manipulate this colored diagram with the partition of the vertices step by step until it represents the poset structure of $L_2(7)$. This is achieved by two sorts of manipulations:

(i) We will *split* a part of the partition of the vertices into subsets whenever we can ensure that only vertices lying in the same subset correspond to subgroups of $L_2(7)$ that can possibly be conjugate in $L_2(7)$.

(ii) We will *fuse* (identify) two vertices whenever we find out that they correspond to subgroups of $L_2(7)$ that are conjugate.

Eventually each subset will consist of only a single element: then we are done!The interested reader is encouraged to illustrate the progress we make by drawing updated versions of the colored diagram after every single manipulation.

The subsets $\{7(B)\}$ and $\{7:3(B)\}$ both contain only one element, hence each of them already determines a unique conjugacy class of subgroups of $L_2(7)$.

There is, of course, only one trivial subgroup in $L_2(7)$, so we are allowed to fuse the vertices $1(R)$, $1(Y)$ and $1(B)$ corresponding to the trivial subgroups of the three maximal subgroups into a single vertex, which we simply denote by 1.

From the table of marks of S_4 we can derive that subgroups named 4 contain one cyclic subgroup of order 2, in contrast to the subgroups named 2^2, which contain three cyclic subgroups of order 2 each. Hence these subgroups can not be conjugate in $L_2(7)$ and we are allowed to split the set of groups of order 4 into two subsets accordingly. Now the situation is as follows.

$$\begin{aligned}&\{1\},\ \{2a(Y), 2b(Y), 2a(R), 2b(R)\},\\ &\{3(Y), 3(R), 3(B)\},\ \{4(Y), 4(R)\},\\ &\{2^2a(Y), 2^2b(Y), 2^2a(R), 2^2b(R)\},\\ &\{S_3(Y), S_3(R)\},\ \{7\},\ \{D_8(Y), D_8(R)\},\\ &\{A_4(Y), A_4(R)\},\ \{7:3\},\ \{S_4(Y), S_4(R)\}.\end{aligned}$$

By Sylow's theorem, G acts transitively on the set of Sylow p-subgroups of G for any prime p. Hence we can fuse all the subgroups of type D_8 (for $p = 2$) and all the subgroups of type 3 (for $p = 3$).

The normalizer in G of a group of type 3 is a group of type S_3. (Note that this can also be decided from the tables of marks, since a cyclic 3 has index two in S_3 and the order of its normalizer in S_4 is 6.) Together with the subgroups of type 3 their normalizers are conjugate in G, hence we can fuse the subset corresponding to these groups into a single vertex. (See Corollary 4.10 for how knowledge about normalizers can be exploited in the general case.)

Within the Sylow 2-subgroup of type D_8 of S_4 there is only one S_4-class of subgroups of type 4. Hence, by Sylow's theorem, all subgroups of this type are conjugate in G and we can fuse the yellow and the red group named 4. (See Corollary 4.8 for how knowledge about the Sylow subgroups can be exploited in the general case.)

From the character table of $L_2(7)$ we can read off that there is only one class of elements (and hence of subgroups) of order 2, so we can fuse all vertices named $2a$ or $2b$ into a single vertex 2.

We are left with three subsets containing more than one element: those corresponding to the subgroups of type 2^2, A_4 or S_4. In order to show that subgroups of type 2^2 that are not conjugate in a maximal S_4 are not conjugate in G either, we examine the permutation character of G on S_4. The character table of $L_2(7)$ admits only one character π of degree 7 that satisfies $\pi(g) \geq 0$ for all $g \in G$, therefore π is the permutation character of G on S_4. The restriction of π to S_4 admits two different decompositions into transitive components, corresponding to the two classes of maximal subgroups of type S_4 in G. The corresponding sums of rows of the table of marks of S_4 reveal different values of fixed points for the two different conjugacy classes of subgroups of type 2^2 inside S_4.

Hence the set containing the groups of this type must split in two subsets. One of them contains the red $2^2a(R)$ and the yellow $2^2b(Y)$ (remember that the labeling was chosen in such a way that groups of type 2^2a are normal in S_4). And both subsets now correspond to one conjugacy class of subgroups of G.

Now the red $A_4(R)$ and the yellow $A_4(Y)$ lie above different classes of groups of type 2^2, whence they must correspond to different classes of subgroups of G. So we split the subset $\{A_4(Y), A_4(R)\}$ in two parts. The same holds for S_4 (we knew right from the start that there are two classes of them in $L_2(7)$) and we split the subset $\{S_4(Y), S_4(R)\}$ into

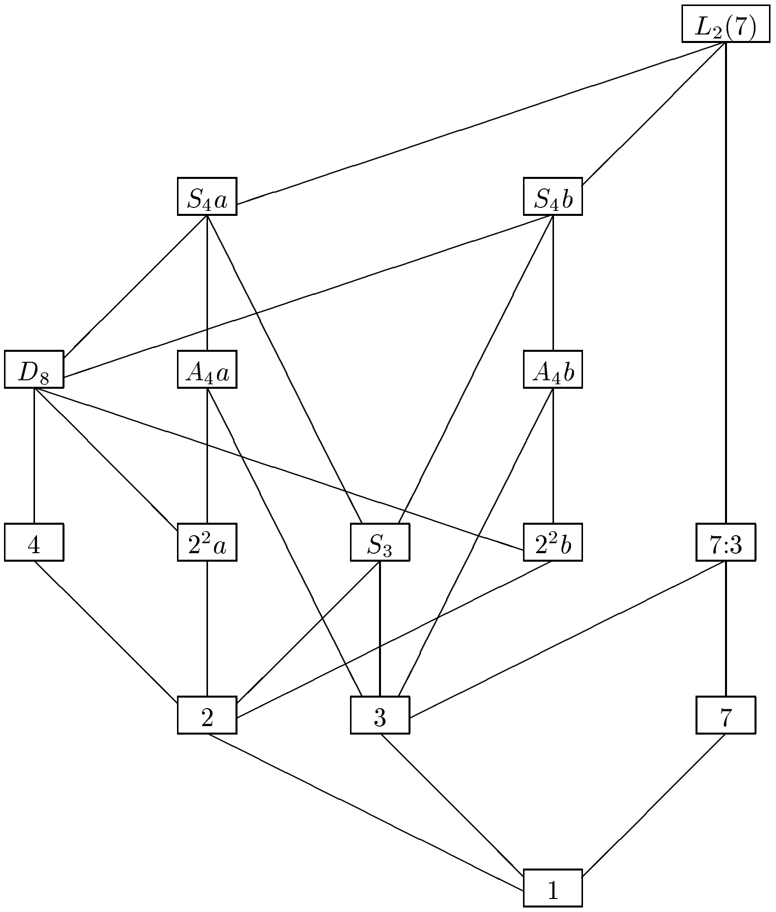

FIGURE 2. The poset structure of $L_2(7)$.

two parts. (See Corollary 4.5 for how knowledge about numbers of subgroups can be exploited in the general case.)

Now every subset corresponds to exactly one conjugacy class of subgroups of G. We finally add the group itself. Thus we have constructed the complete poset structure of $L_2(7)$ (see Figure 2), with vertex set

$$\{1\}, \{2\}, \{3\}, \{4\}, \{2^2a\}, \{2^2b\}, \{S_3\}, \{7\}, \{D_8\}, \{A_4a\}, \{A_4b\}, \{7:3\}, \{S_4a\}, \{S_4b\}, \{L_2(7)\}.$$

This small example illustrates several aspects of the general procedure. We have seen essentially two types of steps in the development of the poset of $L_2(7)$. Most of the conclusions, like those using Sylow's theorem or the conjugacy of normalizers, were based on general facts about the structure of finite groups. Other conclusions, like the existence of exactly one conjugacy class of subgroups of order 2 or the fusion of the subgroups of type 2^2, arose from additional knowledge about the particular group $L_2(7)$. The next section formalizes the general setting.

4. APPROXIMATING THE FUSION MAP

Let $M_1, \dots, M_r$ be a complete set of representatives of conjugacy classes of maximal subgroups of G and, for $i = 1, \dots, r$, denote by j_i the inclusion map from M_i into G, given by $j_i(m) = m$ for each $m \in M_i$. For each $i = 1, \dots, r$, let $\mathfrak{M}_i$ be the poset of conjugacy classes of subgroups of M_i and let $\mathfrak{G} = \mathfrak{S}_G/G$ be the poset of conjugacy classes of subgroups of G. Then each inclusion map j_i

induces a map $j_i: \mathfrak{M}_i \to \mathfrak{G}$ mapping the conjugacy class $[U]_{M_i}$ of subgroups of M_i to the conjugacy class $[U]_G$ of subgroups of G. Denote by $\mathfrak{M}$ the *disjoint* union

$$\mathfrak{M} = \dot{\bigcup_{i=1}^{r}} \mathfrak{M}_i$$

and let $j: \mathfrak{M} \to \mathfrak{G}$ be the union of the maps j_i given by

$$j(m) = j_i(m) \quad \text{if } m \in \mathfrak{M}_i, \quad i = 1, \dots, r.$$

Let $m \in \mathfrak{M}$. Then m is of the form $m = [U]_{M_i}$ for some $i = 1, \dots, r$. Denote $n(m) = |N_{M_i}(U)|$ and $\beta_{V\backslash M_i}(m) = \beta_{V\backslash M_i}(U)$. Then the induction formula 2.2 can, for any subgroups $U, V \le M_i \le G$, be written as

$$\beta_{V\backslash G}(U) = |N_G(U)| \sum_{j(m)=[U]_G} \frac{1}{n(m)} \beta_{V\backslash M_i}(m)$$

where the sum ranges over all $m \in \mathfrak{M}_i$ such that $j(m) = [U]_G$.

In this section we discuss how to determine this fusion map j.

Fusion Maps and Prefusion Maps

Let $\mathfrak{M}$ and $\mathfrak{G}$ be as above. Moreover, let $\mathfrak{G}' = \mathfrak{G} \setminus \{[G]_G\}$. In order to approximate the map $j: \mathfrak{M} \to \mathfrak{G}'$ we will work with idempotent maps on $\mathfrak{M}$. An approximation of j will be called a prefusion map. Moreover, any map on $\mathfrak{M}$ that describes which elements of $\mathfrak{M}$ map into the same conjugacy class of subgroups of G will be called a fusion map. If we can determine one such fusion map f we can identify $\mathfrak{G}'$ and $f(\mathfrak{M})$ and thus have found $j = f$.

Definition 4.1. A *prefusion map* on $\mathfrak{M}$ is a pair $(f, \equiv)$ such that

(i) f is an idempotent map from $\mathfrak{M}$ to $\mathfrak{M}$,
(ii) $\equiv$ is an equivalence relation on the image $f(\mathfrak{M})$, and
(iii) there is a map $g: f(\mathfrak{M}) \to \mathfrak{G}'$ with $g \circ f = j$ and $m_1 \equiv m_2$ whenever $g(m_1) = g(m_2)$ for any $m_1, m_2 \in f(\mathfrak{M})$.

A *fusion map* on $\mathfrak{M}$ is an idempotent map $f: \mathfrak{M} \to \mathfrak{M}$ such that there exists a bijection $g: f(\mathfrak{M}) \to \mathfrak{G}'$ with $g \circ f = j$.

For any prefusion map $(f, \equiv)$ on $\mathfrak{M}$ we denote the set of all $\equiv$-classes on $f(\mathfrak{M})$ by $f(\mathfrak{M})/\equiv$ and denote by $/\equiv$ the canonical map from $f(\mathfrak{M})$ to $f(\mathfrak{M})/\equiv$. Thus, for any $m \in f(\mathfrak{M})$, its $\equiv$-class is $m/\equiv$. The relation between the pair $(f, \equiv)$ and the sets $\mathfrak{M}$ and $\mathfrak{G}$ is illustrated by the following diagram.

$$\mathfrak{M} \xrightarrow{f} f(\mathfrak{M}) \xrightarrow{g} \mathfrak{G}' \longrightarrow f(\mathfrak{M})/\equiv$$

Due to (iii), we may regard $\equiv$ as an equivalence relation on the set $\mathfrak{S}_G$ of all subgroups of G, with the property that $U \equiv V$ whenever U and V are conjugate subgroups of G. Moreover, we may regard f as a function from the disjoint union of the sets of subgroups $\mathfrak{S}_{M_i}$ of the groups M_i to $\mathfrak{M}$ via $f(U) = f([U]_{M_i})$ for any subgroup $U \le M_i$.

Note that, since $j: \mathfrak{M} \to \mathfrak{G}'$ is surjective, and $j = g \circ f$, also g must be surjective.

If, for example, $f = \mathrm{id}_{\mathfrak{M}}$ is the identity map on $\mathfrak{M}$ and $\Leftrightarrow$ is the global equivalence on $\mathfrak{M}$ defined by $m_1 \Leftrightarrow m_2$ for all $m_1, m_2 \in \mathfrak{M}$ then $\varphi_1 = (\mathrm{id}_{\mathfrak{M}}, \Leftrightarrow)$ is a prefusion map since $j = j \circ \mathrm{id}_{\mathfrak{M}}$ and, vacuously, $j(m_1) = j(m_2)$ implies $m_1 \Leftrightarrow m_2$.

The following lemma provides the termination condition of the approximation process.

Lemma 4.2. *Let $f: \mathfrak{M} \to \mathfrak{M}$ be such that $(f, =)$ is a prefusion map on $\mathfrak{M}$. Then f is a fusion map.*

Proof. Since $(f, =)$ is a prefusion map, there is a map $g: f(\mathfrak{M}) \to \mathfrak{G}'$ such that $j = g \circ f$ and $g(m_1) = g(m_2)$ implies $m_1 = m_2$ for all $m_1, m_2 \in \mathfrak{M}$. Hence g is injective. □

The approximation process is guided by a partial order on the set of prefusion maps on $\mathfrak{M}$.

Let $(f_1, \equiv_1)$ and $(f_2, \equiv_2)$ be prefusion maps on $\mathfrak{M}$. We say that $(f_2, \equiv_2)$ is *stronger* than $(f_1, \equiv_1)$, and write $(f_1, \equiv_1) \ge (f_2, \equiv_2)$, if there exist maps $h_1: f_1(\mathfrak{M}) \to f_2(\mathfrak{M})$ and $h_2: f_2(\mathfrak{M})/\equiv_2 \to f_1(\mathfrak{M})/\equiv_1$

such that $f_2 = h_1 \circ f_1$ and $/{\equiv_1} = h_2 \circ /{\equiv_2} \circ h_1$, that is, if the following diagram is commutative.

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, "f_1"] \arrow[d, equal] & f_1(\mathfrak{M}) \arrow[r] \arrow[d, "h_1"'] & \mathfrak{G}' \arrow[r] \arrow[d, equal] & f_1(\mathfrak{M})/{\equiv_1} \\
\mathfrak{M} \arrow[r, "f_2"] & f_2(\mathfrak{M}) \arrow[r] & \mathfrak{G}' \arrow[r] & f_2(\mathfrak{M})/{\equiv_2} \arrow[u, "h_2"']
\end{tikzcd}$$

Let U_1 and U_2 be conjugate subgroups of G. Then $|U_1| = |U_2|$. This allows us to define $|m| = |U|$ for $m = [U]_{M_i} \in \mathfrak{M}$. Thus, if we define an equivalence $\|$ on $\mathfrak{M}$ by $m_1 \parallel m_2$ if $|m_1| = |m_2|$, then $\varphi_2 = (\mathrm{id}_{\mathfrak{M}}, \|)$ is a prefusion map. (Because then, $j = j \circ \mathrm{id}_{\mathfrak{M}}$ and $j(m_1) = j(m_2)$ implies $m_1 \parallel m_2$.) Moreover, φ_2 is stronger that φ_1, since, with $h_2(m/\|) = \mathfrak{M}$ for all $m \in \mathfrak{M}$ and h_1 the identity map on $\mathfrak{M}$ we have $/{\Leftrightarrow} = h_2 \circ /{\|} \circ h_1$.

Every group contains one trivial subgroup. There are r elements $t_1, \dots, t_r$ in $\mathfrak{M}$ corresponding to the trivial subgroups of the r maximal subgroups of G. Define

$$f'(m) = \begin{cases} t_1 & \text{if } m = t_i \text{ for some } i = 1, \dots, r, \\ m & \text{otherwise,} \end{cases}$$

then $\varphi_3 = (f', \|)$ is a prefusion map and φ_3 is stronger than φ_2.

Lemma 4.3. *Let $(f, \equiv)$ be a prefusion map that is minimal with respect to the partial order $\leq$ on the set of all prefusion maps on $\mathfrak{M}$. Then $\equiv$ is $=$, and f is a fusion map.*

Proof. Suppose there are $m_1, m_2 \in f(\mathfrak{M})$ such that $m_1 \neq m_2$ and $m_1 \equiv m_2$. Define

$$f'(m) = \begin{cases} m_1 & \text{if } m = m_1 \text{ or } m_2, \\ f(m) & \text{otherwise.} \end{cases}$$

Then, either $(f', \equiv)$ is a prefusion map, in which case it is strictly stronger than $(f, \equiv)$. Otherwise, there is no g such that $j = g \circ f'$, whence there is an equivalence $\equiv'$ on $f(\mathfrak{M})$ such that $m_1 \not\equiv' m_2$ and $(f, \equiv')$ is a prefusion map. But then $(f, \equiv')$ is strictly stronger than $(f, \equiv)$. So the claim follows from Lemma 4.2. $\square$

Subgroups of Subgroups

Let $(f, \equiv)$ be a prefusion map on $\mathfrak{M}$.

Define, for subgroups U and A of G, the set $S(U, A) = \{B \leq U : B \equiv A\}$ as the set of all subgroups B of U that lie in the $\equiv$-class of A. Denote by $s(U, A) = |S(U, A)|$ its size. This number is determined as follows. If U is a subgroup of the maximal subgroup M_i of G, then (by Proposition 1.4)

$$\begin{aligned} s(U, A) &= \sum_{f(B) \equiv f(A)} \nu_{M_i}(U, B) \\ &= \sum_{f(B) \equiv f(A)} \frac{|U|}{|N_{M_i}(B)|} \beta_{U \backslash M_i}(B) \end{aligned}$$

where the sums range over all representatives B of conjugacy classes of M_i with $f(B) \equiv f(A)$.

Proposition 4.4. *Let $U, A \leq G$ and $g \in G$. Then the map $B \mapsto B^g$ is a bijection between $S(U, A)$ and $S(U^g, A)$. In particular, $s(U, A) = s(U^g, A)$.*

Proof. Let $B \in S(U, A)$, that is $B \leq U$ and $B \equiv A$. Then $B^g \leq U^g$ and $B^g \equiv A$ whence $B^g \in S(U^g, A)$. The inverse map is given by conjugation with g^{-1}. $\square$

This property can be used to detect elements of $f(\mathfrak{M})$ that are not conjugate in G.

Corollary 4.5. *Let $\mathfrak{R}$ be a complete system of representatives of $\equiv$-classes on $f(\mathfrak{M})$. Define an equivalence relation $\equiv'$ on $f(\mathfrak{M})$ by saying that $U \equiv' V$ if and only if $U \equiv V$ and $s(U, A) = s(V, A)$ for all $A \in \mathfrak{R}$. Then $(f, \equiv')$ is a prefusion map, and is stronger than $(f, \equiv)$.*

Proof. Since $(f, \equiv)$ is a prefusion map, there is a map $g \colon f(\mathfrak{M}) \to \mathfrak{G}'$ such that $j = g \circ f$ and $j(m_1) = j(m_2)$ implies $m_1 \equiv m_2$ for all $m_1, m_2 \in f(\mathfrak{M})$. By Proposition 4.4 and the definition of $\equiv'$ then $j(m_1) = j(m_2)$ also implies $m_1 \equiv' m_2$ for all $m_1, m_2 \in f(\mathfrak{M})$. Hence $(f, \equiv')$ is a prefusion map.

Let h_1 be the identity map on $f(\mathfrak{M})$ and let $h_2(m/{\equiv'}) = m/{\equiv}$ for any $m \in f(\mathfrak{M})$. Then $/{\equiv} = h_2 \circ /{\equiv'} \circ h_1$ and therefore $(f, \equiv')$ is stronger than $(f, \equiv)$. $\square$

A procedure that systematically applies this corollary to $f(\mathfrak{M})$ in order to produce the strongest possible prefusion map from $(f,\equiv)$ will be called RefineClassesFrame.

Comparing the sets $S(U,A)$ for conjugate subgroups U leads to even more detailed insights.

Let $U \leq M_i$ and let $A \leq G$. Let

$$F(U,A) = f(S(U,A)) = \{s_1,\ldots,s_q\} \subseteq f(\mathfrak{M})$$

be the image of $S(U,A)$ under f. Note that by the definition of $S(U,A)$ all elements $s_j \in F(U,A)$ lie in the same $\equiv$-class. To each $s_j \in F(U,A)$ we associate a number $\mu_U(s_j)$ defined by

$$\mu_U(s_j) = \sum_{f(B)=s_j} \nu_{M_i}(U,B),$$

where the sum runs over the conjugacy classes of subgroups $[B]$ of M_i with $f(B) = s_j$. Conjugation in G partitions the set $F(U,A)$ in such a way that the sum of the $\mu_U(s_j)$ corresponding to one part gives the number of G-conjugate subgroups of U of a certain type.

Corollary 4.6. *Let $U \leq M_i$ and $U' \leq M_{i'}$ be such that $f(U) = f(U')$, and let $A \leq G$. Then there are an integer k, partitions $F(U,A) = \pi_1 \dot\cup \cdots \dot\cup \pi_k$ and $F(U',A) = \pi'_1 \dot\cup \cdots \dot\cup \pi'_k$, elements $s_j \in \pi_j$ for $j = 1,\ldots,k$, and a map $f'\colon \mathfrak{M} \to \mathfrak{M}$ defined, for $m \in \mathfrak{M}$, by*

$$f'(m) = \begin{cases} s_j & \text{if } f(m) \in \pi_j \text{ or } f(m) \in \pi'_j, \\ f(m) & \text{otherwise,} \end{cases}$$

such that

$$\sum_{s\in\pi_j} \mu_U(s) = \sum_{s\in\pi'_j} \mu_{U'}(s)$$

and $(f',\equiv)$ is a stronger prefusion map than $(f,\equiv)$.

This last result appears to be not as explicit as one might wish. In many cases, however, it is possible to explicitly determine the partitions and thus a new map f'.

If, for example, $F(U,A) = \{s\}$ contains only one element, then, regardless of the size of $F(U',A)$, the partition will be trivial and f' can be defined as $f'(s') = s$ for each s' such that $f(s') \in F(U',A)$.

A procedure that systematically searches $f(\mathfrak{M})$ for places where Corollary 4.6 can be applied in order to produce the strongest possible prefusion map from $(f,\equiv)$ will be called ConcludeFrame.

Sylow's Theorem

All Sylow p-subgroups of G are conjugate, and every p-subgroup of G lies inside a Sylow p-subgroup. Here this can be used as follows.

Proposition 4.7. *Let $(f,\equiv)$ be a prefusion map on $\mathfrak{M}$ that is stronger than φ_2. Let p be a prime and $S \in \mathrm{Syl}_p(G)$ a Sylow p-subgroup of G. Let $P \leq S$ and assume that $P \equiv Q$ implies $P \sim Q$ for all $Q \leq S$. Then $P \sim Q$ for all $Q \leq G$ with $P \equiv Q$.*

Proof. Let $Q \leq G$ with $Q \equiv P$. Then Q is a p-subgroup of G and by Sylow's theorem there exists a $g \in G$ with $Q^g \leq S$. Since $(f,\equiv)$ is a prefusion map we have $Q^g \equiv P$ and the hypothesis implies $Q^g \sim P$ whence $Q \sim P$. □

This result allows us to fuse certain $\equiv$-classes of p-subgroups into singletons.

Corollary 4.8. *Let $(f,\equiv)$ be a prefusion map on $\mathfrak{M}$ that is stronger than φ_2. Let $P \in \mathfrak{M}$ with $P \leq S \leq M_i$ for $S \in \mathrm{Syl}_p(G)$ and some maximal $M_i < G$. Suppose $f(P)/{\equiv} \cap f(\mathfrak{M}_i)$ contains exactly one element m_P and define a map $f'\colon \mathfrak{M} \to \mathfrak{M}$ by*

$$f'(m) = \begin{cases} m_P & \text{if } f(m) \equiv f(P), \\ f(m) & \text{otherwise.} \end{cases}$$

Then $(f',\equiv)$ is a prefusion map and $(f',\equiv) \leq (f,\equiv)$.

Proof. Since $(f,\equiv)$ is a prefusion map, there is a map $g\colon f(\mathfrak{M}) \to \mathfrak{G}$ such that $j = g \circ f$ and $g(m_1) = g(m_2)$ implies $m_1 \equiv m_2$ for all $m_1, m_2 \in f(\mathfrak{M})$. Let g' be the restriction of g to $f'(\mathfrak{M}) \subseteq f(\mathfrak{M})$. Then, by Proposition 4.7, also $j = g' \circ f'$ and, of course, $g'(m_1) = g'(m_2)$ implies $m_1 \equiv m_2$ for all $m_1, m_2 \in f'(\mathfrak{M})$. Hence $(f',\equiv)$ is a prefusion map.

Let $h_1(m) = f'(m)$ for all $m \in f(\mathfrak{M})$ and let h_2 map the $\equiv$-class of each $m \in f'(\mathfrak{M})$ to its $\equiv$-class in $f(\mathfrak{M})$. Let $/{\equiv'}$ be the restriction of $/{\equiv}$ to

$f'(\mathfrak{M})$. Then $/\!\equiv\, = h_2 \circ /\!\equiv' \circ h_1$, whence $(f', \equiv')$ is stronger than $(f, \equiv)$. □

A procedure that systematically searches $f(\mathfrak{M})$ for places where Corollary 4.8 can be applied in order to produce a stronger prefusion map will be called CheckSylowFrame.

Normalizers

If two subgroups are conjugate in G, so are their normalizers in G. This translates into the present context as follows.

Proposition 4.9. *Let $U, V \leq G$ with $U \trianglelefteq V$ and assume that $V \equiv A$ implies $V \sim A$ for all $A \leq N_G(U)$. Then $V \sim A$ for all $A \leq G$ with $V \equiv A$ and $U \trianglelefteq A^g$ for some $g \in G$.*

Proof. Let $A \leq G$ and $g \in G$ with $A \equiv V$ and $U \trianglelefteq A^g$. Then $A^g \leq N_G(U)$ and $A^g \equiv V$ and the hypothesis implies $A^g \sim V$ whence $A \sim V$. □

This result allows us to fuse certain classes of subgroups into singletons.

Corollary 4.10. *Let $(f, \equiv)$ be a prefusion map. Let $H \in \mathfrak{M}$ with $U \leq H \leq N_G(U) \leq M_i$ for a subgroup $U \leq G$ and some maximal $M_i < G$. Suppose that each $h \equiv f(H)$ contains a conjugate of U as a normal subgroup and that $f(H)/\!\equiv \cap\, f(\mathfrak{M}_i) = \{m_H\}$. Define a map $f' \colon \mathfrak{M} \to \mathfrak{M}$ as*

$$f'(m) = \begin{cases} m_H & \text{if } f(m) \equiv f(H)\ , \\ f(m) & \text{otherwise.} \end{cases}$$

Then $(f', \equiv)$ is a prefusion map and $(f', \equiv) \leq (f, \equiv)$.

A procedure that systematically searches $f(\mathfrak{M})$ for places where Corollary 4.10 can be applied in order to produce a stronger prefusion map will be called CheckNormalizerFrame.

5. A STRATEGY

The complete program of how to apply the procedures described in the previous section and when to supplement them with additional information about the group is given by the following strategy.

A rough guideline is given by Lemma 4.3: Start with a trivial prefusion map, then keep producing stronger prefusion maps by splitting and fusing until a fusion map is reached.

1. Start with $\varphi_2 = (\mathrm{id}_{\mathfrak{M}}, \|)$ (see above) as the initial prefusion map.
2. Refine the equivalence relation with RefineClassesFrame (see 4.5).
3. Apply CheckSylowFrame (see 4.8) and CheckNormalizerFrame (see 4.10) in appropriate places.
4. Apply ConcludeFrame (see 4.6).
5. Stop if all classes are singletons. Otherwise find a legitimation for an action of one of the two following kinds.
   a. Split an $\equiv$-class into two (or more) parts and goto 2., or
   b. Fuse two (or more) images $f(m)$ and goto 4.

Suppose, for example, that it is known (from the list of conjugacy classes of elements of G) that G has exactly two conjugacy classes of elements of order 2. Then G has two conjugacy classes of subgroups of order two. Suppose further that we know a prefusion map $(f, \equiv)$ such that there exactly two images $f(m)$ of size two and these two form one $\equiv$-class. Then, if the equivalence $\equiv'$ is derived from $\equiv$ by splitting that class in two, then $(f, \equiv')$ is a stronger prefusion map than $(f, \equiv)$. Moreover, this splitting will, via RefineClassesFrame, have an effect on all classes of subgroups that contain subgroups of order 2, depending on the distribution of the two types of subgroups of order two within them.

Suppose that the intersections of two classes of maximal subgroups are known, that one can find two classes $[U_1]_{M_{i_1}}$ and $[U_2]_{M_{i_2}}$ where U_1 and U_2 are in fact identical as subgroups of G, that is, where $j(U_1) = j(U_2)$. Suppose further that we know a prefusion map $(f, \equiv)$ such that $f(U_1) \neq f(U_2)$. Then, if f' is derived from f by fusing the images of U_1 and U_2, then $(f', \equiv)$ is a stronger prefusion map than $(f, \equiv)$. Moreover, this fusion will, via ConcludeFrame, have an effect on all classes that contain subgroups of U_1 or U_2, because they also must fuse in some way. For examples of intersections of

```
gap> m24:= AllPrimitiveGroups(Size, 48 * Product([20..24]))[1];;
gap> m24.name:= "m24";
gap> m24.generators;
[ ( 1, 7,12,16,19,21, 6)( 2, 8,13,17,20, 5,11)( 3, 9,14,18, 4,10,15),
  ( 2,14,18,20, 8)( 3, 7,12,13,19)( 4,21,17,15,10)( 5,11,16, 6, 9),
  ( 1,22)( 2,10)( 3,14)( 4,17)( 8,15)( 9,11)(13,20)(19,21),
  ( 3,19)( 4,14)( 5,20)( 6,10)( 8,15)(11,18)(17,21)(22,23),
  ( 2,10)( 3,13)( 4,11)( 5,18)( 8,15)( 9,17)(14,20)(23,24) ]
```

GAP commands giving the permutation representation of M_{24}.

maximal subgroups, and methods for determining them, see [Komissartschik and Tsaranov 1986], for example.

6. THE TABLE OF MARKS OF M_{24}

In this section the table of marks of the sporadic simple Mathieu group M_{24} is determined. We assume that we know the maximal subgroups of M_{24}, and that we have already computed their tables of marks. An implementation in GAP of the procedures described in Sections 4 and 5 is used to determine the fusion map from the maximal subgroups into M_{24}. We give only a short account of the development. This should give an impression of the amount and the type of work involved.

The permutation representation of M_{24} we will work with is taken from the GAP library of primitive groups, being obtained as shown at the top of this page.

A complete list of conjugacy classes of maximal subgroups is given in Table 2 as it is found in the *Atlas* [Conway et al. 1985, p. 94]. This is based on [Todd 1966; Choi 1972a; 1972b]. For a detailed combinatorial description see also [Conway 1971; Curtis 1976; 1977].

Table 2 lists for each maximal subgroup of M_{24} its name, its index in M_{24} and its order. The specification describes the subgroup by the kind of object it stabilizes. We furthermore list the name that is used in the GAP session and the number of conjugacy classes of subgroups of each maximal subgroup.

In GAP, the maximal subgroups of M_{24} are then constructed as stabilizers according to the above table or by explicit generating permutations, as shown in the sidebar of the next page.

The computation of all but two of the tables of marks of the maximal subgroups via the lattice of subgroups is almost automatic. It requires up to 80 MB of main memory and some hours of cpu time.

Name	Index	Order	Specification	GAP	Classes
M_{23}	24	10 200 960	point	`m23`	204
$M_{22}:2$	276	887 040	duad	`n22`	490
$2^4:A_8$	759	322 560	octad	`ea8`	1 766
$M_{12}:2$	1 288	190 080	duum	`n12`	213
$2^6:3.S_6$	1 771	138 240	sextet	`e3s6`	2 261
$L_3(4):S_3$	2 024	120 960	triad	`n21`	226
$2^6:(L_3(2)\times S_3)$	3 795	64 512	trio	`nea`	2 156
$L_2(23)$	40 320	6 072	projective line	`l23`	23
$L_2(7)$	1 457 280	168	octern	`l7`	15

TABLE 2. The maximal subgroups of M_{24}.

```
m23:= Stabilizer(m24, 24);  m23.name:= "m23";

n22:= Stabilizer(m24, [23, 24], OnSets);  n22.name:= "n22";

trio:=[[1,2,3,4,5,6,12,16],[8,10,13,20,21,22,23,24],[7,9,11,14,15,17,18,19]];
ea8:= Stabilizer(m24, trio[2], OnSets);  ea8.name:= "ea8";

duum:= [[1,5,8,10,11,14,17,18,20,21,23,24], [2,3,4,6,7,9,12,13,15,16,19,22]];
m12:= Stabilizer(m24, duum[1], OnSets);  m12.name:= "m12";
n12:= Normalizer(m24, m12);  n12.name:= "n12";

sextet:= [[1,12,16,18],[2,15,21,23],[3,14,22,24],[4,7,8,20],[5,9,17,19]];
grp:= m24;  for four in sextet do grp:= Stabilizer(grp, four, OnSets); od;
e3s6:= Normalizer(m24, grp);  e3s6.name:= "e3s6";

n21:= Stabilizer(m24, [22..24], OnSets);  n21.name:= "n21";

trio:=[[1,6,10,11,12,13,16,18],[2,3,7,8,9,15,19,24],[4,5,14,17,20,21,22,23]];
grp:= Stabilizer(Stabilizer(m24, trio[1], OnSets), trio[2], OnSets);
nea:= Normalizer(m24, grp);  nea.name:= "nea";

l23:= Subgroup(m24,
[(1,7)(2,16)(3,14)(4,9)(5,6)(8,21)(10,11)(12,19)(13,15)(17,24)(18,20)(22,23),
 (1,23,2)(3,10,16)(4,19,20)(5,8,22)(6,24,17)(7,18,11)(9,21,13)(12,14,15)]);
l23.name:= "l23";

l7:= Subgroup(m24,
[(1,14)(2,20)(3,9)(4,16)(5,12)(6,24)(7,10)(8,21)(11,23)(13,19)(15,18)(17,22),
 (1,11,18)(2,7,5)(3,21,16)(4,20,15)(6,24,8)(9,14,10)(12,13,19)(17,23,22)]);
l7.name:= "l7";

m24.max:= [m23, n22, ea8, n12, e3s6, n21, nea, l23, l7];
```

Construction of the stabilizers.

In addition to the permutation representation of a maximal subgroup M one has to supply a complete list of representatives of perfect subgroups of M as input for the lattice program.

For the maximal subgroups M_{23} and $M_{22}:2$ (and for M_{22} inside $M_{22}:2$) the method described here has been applied to obtain their table of marks together with a list of representatives of the conjugacy classes of subgroups.

We will skip quickly through the automatic part of the strategy, and just report some interesting figures. The disjoint union $\mathfrak{M}$ of the conjugacy classes of maximal subgroups of M_{24} consists of 7354 subgroups that initially fall into 116 $\equiv$-classes, i.e., that have 116 different orders. The refinement via RefineClassesFrame yields 445 classes. This tells us that there are at least 445 different isomorphism types of subgroups of M_{24}, besides M_{24} itself.

The inspection of the Sylow subgroups via CheckSylowFrame yields the fusion of the 6 $\equiv$-classes of Sylow subgroups. The $\equiv$-class of the trivial subgroups is detected and fused. Moreover, 31 $\equiv$-classes of 2-subgroups inside the Sylow 2-subgroup are fused. The Sylow p-subgroups for $p = 5$, 7, 11, and 23 are cyclic, so here is nothing left for CheckSylowFrame to do. The Sylow 3-subgroup has order 3^3, here are some $\equiv$-classes with more than one element. The routine CheckSylowFrame will prove most powerful inside the Sylow 2-subgroup of order 2^{10} that contains lots of classes of subgroups.

The application of CheckNormalizerFrame yields the fusion of 36 more classes. Then 302 fusions of single subgroups are caused by ConcludeFrame. We are left with 320 unfinished classes.

Since M_{24} has this small permutation representation on 24 points, and all maximal subgroups are given explicitly as subgroups of this permutation group on 24 points, we have access to a representative for each conjugacy class of subgroups of each maximal subgroup of M_{24}. Thus, we can use these permutation groups in order to distinguish groups that are not conjugate in M_{24}.

Let U be a subgroup of M_{24}. Then U acts on 24 points and the lengths of the orbits of U give a partition π_U of 24. If U' is a subgroup of M_{24} that is conjugate to U then $\pi_{U'} = \pi_U$. If we use this criterion on the $\equiv$-classes then 269 of them split into two or more classes. A further refinement with RefineClassesFrame yields a total of 1453 classes. Then CheckSylowFrame can fuse 523 classes inside the Sylow 2-subgroup and three classes inside the Sylow 3-subgroup. Moreover, CheckNormalizerFrame can fuse 479 classes. Then ConcludeFrame causes 1722 fusions of elements of $f(\mathfrak{M})$. We are left with 93 unfinished classes.

The representatives of the conjugacy classes of subgroups of the maximal subgroups of M_{24} can also be used to determine the size of their normalizers in M_{24}. Of course, the normalizers of conjugate subgroups have the same size. This criterion can be used to split one of the $\equiv$-classes of subgroups of order 4 into three classes. RefineClassesFrame then yields a total of 1510 classes. CheckSylowFrame fuses 4 classes inside the Sylow 2-subgroup. CheckNormalizerFrame fuses 20 further classes. ConcludeFrame causes one additional fusion in $f(\mathfrak{M})$.

Checking the sizes of normalizers of the $\equiv$-classes of small size (4, 8, 12, and 16) together with the application of RefineClassesFrame leads to a total of 1527 classes.

The remaining problem is quite small; we are faced with a pre-fusion map $(f, \equiv)$ where only 28 of the $\equiv$-classes have more than one element. These classes contain subgroups of order 16, 60 [2 classes], 120 [3 classes], 168, 240 [3 classes], 336, 360 [2 classes], 480, 660, 720 [4 classes], 1320, 1344, 1440, 2520, 7920, 20160 [2 classes], 40320, and 443520.

Five of these classes can be solved by looking at intersections of the maximal subgroups.

The point stabilizer `m23` and the duad stabilizer `n22` intersect in a group of size 443520,

```
gap> Size(Intersection(m23, n22));
443520
```

(which is, of course, the 2 point stabilizer M_{22}) and, since the $\equiv$-class of groups of this size consists of a conjugacy class of subgroups of M_{23} and a conjugacy class of subgroups of $M_{22} : 2$, we can fuse this class.

The point stabilizer `m23` and the triad stabilizer `n21` intersect in a group of size 40320 (which is $M_{21} : 2$, the duad stabilizer in M_{23}), so do the duad stabilizer `n22` and `n21`.

```
gap> Size(Intersection(m23, n21));
40320
gap> Size(Intersection(n22, n21));
40320
```

Thus the three elements in the $\equiv$-class of groups of that size belong to one single conjugacy class of subgroups of M_{24}.

We look at the size of the intersection of `ea8` and the stabilizer of the point 1 in M_{24},

```
gap> Size(Intersection(ea8,
                Stabilizer(m24, 1)));
20160
```

and can decide for one of the $\equiv$-classes of groups of size 20160 that they indicate a single conjugacy class.

Similarly we find a group of order 7920 (M_{11}) as the intersection of `n12` and `m23` and a group of order 1440 ($A_6.2^2$) as the intersection of `n22` and `n12`. This allows us to fuse the $\equiv$-classes of groups of these orders. After the fusion of these five classes ConcludeFrame causes 23 further fusions. In particular, the class of groups of order 660 is fused into a singleton, and consequently the class

```
gap> o1:= Orbits(der1, [1..24]);
[[1,23,17,24,20,5,21,10,18,8,14,11], [2,15,13,7,19,9,16,4,22,6,12,3]]
gap> o2:= Orbits(der2, [1..24]);
[[1,11,9,4,13,22,24,7,21,2,18,5], [3,23,8,16,15,14,6,17,12,20,10,19]]
gap> g:= RepresentativeOperation(m24, Set(o2[1]), Set(o1[1]), OnSets);
( 2, 5,10,16, 4, 8,19,12, 3, 7,11,23,22,17,13,14, 6,15, 9,24,21,18,20)
gap> h:= RepresentativeOperation(n12, rep1, rep2^g);
( 2,19, 6,15)( 3,22,16, 7)( 4,13)( 9,12)(10,24,14,17)(18,20,23,21)
gap> rep1^h = rep2^g;
true
```

Calculation of a conjugating element.

of groups of order 1320 can be identified as the class of their normalizers, so it also can be fused.

The ≡-class of groups of order 480 consists of two elements, a subgroup of `n12` and a subgroup of `e3s6`. The duums stabilized by each of these groups is exhibited by passing to the derived subgroups. Let `rep1` and `rep2` be representatives of the two elements in the ≡-class and let `der1` and `der2` be their respective derived subgroups. As shown at the top of the page, we determine their orbits on the 24 points and an element `g` of M_{24} that maps the duum of `rep2` to that of `rep1` (and thereby conjugates `rep2` into `n12`). The index of the subgroups in `n12` is relatively small (396), and we can find a conjugating element inside `n12`. We thus have explicitely shown that `rep1` and `rep2` are conjugate in M_{24} and can now fuse the corresponding ≡-class. ConcludeFrame then fuses all but one of the remaining classes.

The remaining ≡-class of groups of order 16 consists of two elements. Each of them is the image under f of five elements of $\mathfrak{M}$, where two are conjugacy classes of subgroups of `nea`, two of `e3s6` and one of `ea8`.

Let `rep3` and `rep4` be representatives of the two elements in that ≡-class. Again we look at the orbits of these groups on the 24 points; see code at the bottom of the page.

The groups `rep3` and `rep4` both stabilize the same octad and they are not conjugate in the octad stabilizer `ea8`. Therefore they are not conjugate in M_{24} and this class splits.

Finally the table of marks of M_{24} can be computed via `InducedFrame`, a program that implements the induction formula 2.2. The number of subgroups of M_{24} now can be derived from that table by means of Lemma 1.3.

```
gap> tom:= InducedFrame(frame);;
gap> Length(tom.subs);
1529
```

Out of the 704 divisors of the order of M_{24} only 117 occur as orders of subgroups of M_{24}. Table 3 lists for each such order the number of conjugacy classes of subgroups with that order and the total number of subgroups in these classes. The most popular orders among the conjugacy classes are 32 (212 classes) and 64 (209 classes). 32 is also the most

```
gap> Orbits(rep3, [1..24]);
[[1,6,10,11,13,18,16,12], [2,3,8,7,14,9,19,23,15,5,20,4,24,17,21,22]]
gap> Orbits(rep4, [1..24]);
[[1,6,10,11,13,18,16,12], [2,3,7,4,8,19,5,9,21,24,23,22,15,14,17,20]]
```

Orbits of `rep3` and `rep4`.

Order	Cl.	Subgroups	Order	Cl.	Subgroups	Order	Cl.	Subgroups
1	1	1	112	2	1457280	1728	1	17710
2	2	43263	120	9	11221056	1920	5	414414
3	2	356224	126	1	1943040	2160	1	113344
4	12	5668971	128	120	57323475	2304	6	398475
5	1	1020096	144	10	10838520	2520	1	97152
6	6	18944640	160	4	701316	2688	8	667920
7	1	1943040	168	7	6072000	2880	2	85008
8	50	62606115	180	2	1360128	3072	4	318780
9	2	2833600	192	86	37940386	3456	3	88550
10	3	7140672	216	2	1700160	3840	5	148764
11	1	2225664	240	9	4250400	4032	1	30360
12	24	69026496	253	1	967680	4608	9	239085
14	1	5829120	256	47	13149675	5760	3	106260
15	1	4080384	288	13	5738040	6072	1	40320
16	129	145731795	320	6	1976436	6912	3	53130
18	5	23235520	336	6	4371840	7680	2	63756
20	5	19381824	360	4	1870176	7920	1	30912
21	3	7772160	384	66	22929390	8064	1	30360
22	1	2225664	432	1	566720	9216	3	79695
23	1	967680	448	1	30360	10752	1	3795
24	51	153864480	480	4	1530144	11520	3	42504
27	1	1133440	504	1	242880	13824	1	17710
30	1	4080384	512	15	2948715	20160	2	14168
32	212	186438483	576	11	2514820	21504	2	22770
36	7	18701760	640	3	446292	23040	2	21252
40	3	7140672	660	2	370944	32256	1	3795
42	2	7772160	720	5	1020096	40320	2	12144
48	74	125578068	768	29	3665970	60480	1	2024
54	3	5667200	896	2	182160	64512	1	3795
55	1	2225664	960	6	754446	69120	1	1771
56	3	2914560	1008	1	242880	95040	1	1288
60	7	10540992	1024	1	239085	120960	1	2024
63	1	1943040	1080	1	113344	138240	1	1771
64	209	137745091	1152	12	1452220	190080	1	1288
72	15	24368960	1280	1	191268	322560	1	759
80	3	3379068	1320	2	370944	443520	1	276
96	71	75444600	1344	6	394680	887040	1	276
108	3	3400320	1440	2	340032	10200960	1	24
110	1	2225664	1536	23	1753290	244823040	1	1

TABLE 3. For each number that occurs as the order of a subgroup of M_{24}, the table shows the number of conjugacy classes of subgroups (Cl.) and the complete number of subgroups of that order.

popular order among the subgroups (186 438 483 subgroups). On the other hand, there are 44 conjugacy classes that are uniquely determined by the order of the subgroups they contain.

7. RESULTS FOR OTHER GROUPS

The methods described above have been developed with a particular interest in the tables of marks of simple and almost simple groups. It should be possible to adapt them to larger classes of groups, but we note that arguments involving Sylow subgroups (and in particular Corollary 4.8) lose their meaning when the table of marks of a p-group is to be determined.

In Table 4 we list the number of subgroups and the number of conjugacy classes of subgroups for these groups: all projective special linear groups $L_2(p^e)$ of order less than 10^6; the projective special linear groups $L_3(n)$ for $n = 3, 4, 5$; the alternating groups A_n for $n = 5, \ldots, 11$; the symmetric groups S_n for $n = 5, \ldots, 10$; the unitary groups $U_3(n)$ for $n = 3, 4, 5$ and $U_4(2)$; the Suzuki group $Sz(8)$; the symplectic group $S_4(4)$; the Mathieu groups M_{11}, M_{12}, M_{22} and M_{23} plus their automorphism groups $M_{12} : 2$ and $M_{22} : 2$; the Janko groups J_1, J_2 and J_3; and the McLaughlin group McL.

The complete tables of marks of these groups have been determined by the methods described above. These tables form a library that is part of GAP.

Buekenhout and Rees [1988] have determined the poset structure of the Mathieu group M_{12}. The poset structure of the sporadic simple Janko group J_2 was determined in [Pahlings 1987]. The poset structure of the sporadic simple Janko group J_3 is determined in [Pfeiffer 1991]. Informations about parts of the subgroup lattice of the sporadic simple McLaughlin group McL can be found in [Diawara 1987].

A rather incomplete set of facts about subgroups of the Mathieu groups is given in [Greenberg 1973]. P. Fong (*Math Reviews* 50:4731) points out that "this can not remain the last work on the subject."

8. APPLICATIONS

In this final section we extract more information about the group G from its table of marks. This is done in two different ways: via the Möbius function of the subgroup lattice of G and via the idempotents of the Burnside ring of G.

8.1. Möbius Functions

For any finite poset X, a Möbius function μ can be defined as inverse of the incidence relation $\leq$ via

$$\sum_{x \leq y \leq z} \mu(x, y) = \delta_{xz}$$

for all $x, z \in X$, where δ denotes the Kronecker delta. In particular, we have $\mu(x, x) = 1$ for all $x \in X$. This is a natural generalization of the well known Möbius function in number theory, where the partial order is given as divisibility (compare [Rota 1964]).

Two posets are in a natural way associated to a given finite group G: one is the lattice $\mathfrak{S}_G = \{U : U \leq G\}$ of all subgroups of G with inclusion as incidence. Denote its Möbius function by μ_G. The investigation of this case goes back to [Hall 1936], where it was intensely studied. For every $U \leq G$ the value $\mu_G(U, G)$ can be derived from the table of marks of G (compare [Pahlings 1993, Proposition 1]).

Lemma 8.1. *Let $U \leq G$. Then $\mu_G(U, G)$ is determined by*

$$\sum_{[A]} \beta_{A!G}(U)/\beta_{A!G}(A)\, \mu_G(A, G) = \delta_{AG},$$

where the sum ranges over all conjugacy classes $[A]$ of subgroups of G.

In other words, $\mu_G(U, G)$ is the entry in the column corresponding to U of the final row of the inverse of the *unweighted* table of marks, i.e., the matrix that is derived from the table of marks by dividing each row by its diagonal value.

Name	Order	Cl.	Subgroups	Name	Order	Cl.	Subgroups
Linear Groups				$L_3(3)$	5616	51	6374
$L_2(7)$	168	15	179	$L_3(4)$	20160	95	44877
$L_2(8)$	504	12	386	$L_3(5)$	372000	140	345809
$L_2(11)$	660	16	620	Alternating Groups			
$L_2(13)$	1092	16	942	A_5	60	9	59
$L_2(17)$	2448	22	2420	A_6	360	22	501
$L_2(19)$	3420	19	2912	A_7	2520	40	3786
$L_2(16)$	4080	21	3455	A_8	20160	137	48337
$L_2(23)$	6072	23	5915	A_9	181440	223	508402
$L_2(25)$	7800	37	9559	A_{10}	1814400	428	6469142
$L_2(27)$	9828	16	5286	A_{11}	19958400	788	81711572
$L_2(29)$	12180	22	10040	Symmetric Groups			
$L_2(31)$	14880	29	15413	S_5	120	19	156
$L_2(37)$	25308	23	17731	S_6	720	56	1455
$L_2(32)$	32736	24	22328	S_7	5040	96	11300
$L_2(41)$	34440	33	36129	S_8	40320	296	151221
$L_2(43)$	39732	20	25462	S_9	362880	554	1694723
$L_2(47)$	51888	29	48837	S_{10}	3628800	1593	29594446
$L_2(49)$	58800	51	73945	Unitary Groups			
$L_2(53)$	74412	20	43254	$U_3(3)$	6048	36	5150
$L_2(59)$	102660	26	82368	$U_4(2)$	25920	116	45649
$L_2(61)$	113460	32	91144	$U_3(4)$	62400	34	31373
$L_2(67)$	150348	20	79602	$U_3(5)$	126000	80	179308
$L_2(71)$	178920	39	203705	Suzuki Groups			
$L_2(73)$	194472	38	176087	$Sz(8)$	29120	22	17295
$L_2(79)$	246480	37	247355	Symplectic Groups			
$L_2(64)$	262080	76	360787	$S_4(4)$	979200	496	4045873
$L_2(81)$	265680	69	433087	Sporadic Groups			
$L_2(83)$	285852	24	190904	M_{11}	7920	39	8651
$L_2(89)$	352440	37	341323	M_{12}	95040	147	214871
$L_2(97)$	456288	45	451547	$M_{12}:2$	190080	213	538243
$L_2(101)$	515100	29	343307	M_{22}	443520	156	941627
$L_2(103)$	546312	29	396865	$M_{22}:2$	887040	490	3396237
$L_2(107)$	612468	24	374718	M_{23}	10200960	204	17318406
$L_2(109)$	647460	36	523864	J_1	175560	40	158485
$L_2(113)$	721392	39	622753	J_2	604800	146	1104344
$L_2(121)$	885720	63	976309	J_3	50232960	137	71564248
$L_2(125)$	976500	29	708273	McL	898128000	373	1719739392

TABLE 4. For each of the groups given in Section 7, the table lists its name, its order, the number of conjugacy classes of subgroups and the number of subgroups.

Proof. We have $\mu_G(G,G) = 1$ and for $U < G$ by the definition of the Möbius function

$$0 = \sum_{U \leq A} \mu_G(A,G),$$

where the sum runs over all subgroups A of G that contain U. Since the Möbius function is invariant under conjugation in G this can be written as

$$0 = \sum_{[A]} \left|\{A^g : g \in G,\, U \leq A^g\}\right| \, \mu_G(A,G),$$

where the sum ranges over all conjugacy classes $[A]$ of subgroups of G. Finally the number of conjugates of A that contain U equals $\beta_{A\backslash G}(U)/\beta_{A\backslash G}(A)$ by Lemma 1.3(iv). □

The knowledge of these values of the Möbius function is sufficient in many applications. They determine, for example, the number of essentially different ways in which G can be generated by m elements. Denote this number by $e_m(G)$. Then

$$\left|\mathrm{Aut}(G)\right| \, e_m(G) = g_m(G),$$

where $g_m(G)$ is the number of m-tuples of elements of G that generate G and we have (compare [Hall 1936, (3.3)])

Lemma 8.2. *For each $m > 0$ the number $e_m(G)$ is determined by the order of* $\mathrm{Aut}(G)$ *and the table of marks of G as*

$$e_m(G) = \frac{1}{\left|\mathrm{Aut}(G)\right|} \sum_{H \leq G} \mu_G(H,G)\, \left|H\right|^m .$$

Proof. Let $H \leq G$. Each of the $\left|H\right|^m$ m-tuples of elements of H certainly generates a subgroup U of H, hence

$$\left|H\right|^m = \sum_{U \leq H} g_m(U).$$

Möbius inversion then yields

$$g_m(H) = \sum_{U \leq H} \mu_G(U,H)\, \left|U\right|^m ,$$

and the lemma follows for $H = G$ from 8.1. □

From this we get $e_2(A_5) = \frac{1}{120}(-60 + 4{\cdot}15{\cdot}2^2 + 2{\cdot}10{\cdot}3^2 - 10{\cdot}6^2 - 6{\cdot}10^2 - 5{\cdot}12^2 + 60^2) = 19$ essentially different ways to generate A_5 by 2 elements.

Even more detailed questions can be answered from the table of marks of G. We can, for example, fix one element $x \in G$ and ask for the number of elements $y \in G$ with $\langle x,y\rangle = G$. For any $U \leq G$ let

$$n_x(U) = \left|\{y \in G : \langle x,y\rangle = U\}\right|,$$
$$m_x(U) = \left|\{y \in G : \langle x,y\rangle \leq U\}\right|.$$

We have

$$m_x(U) = \left|\{y \in U : \langle x,y\rangle \leq U\}\right| = \begin{cases} |U|, & \text{if } x \in U, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$m_x(U) = \sum_{H \leq U} n_x(H).$$

By Möbius inversion we get

$$n_x(U) = \sum_{H \leq U} \mu_G(H,U)\, m_x(H)$$

where μ_G denotes the Möbius function of the subgroup lattice of G. In particular,

$$n_x(G) = \sum_{U \leq G} \mu_G(U,G)\, m_x(U) = \sum_{[U]} \mu_G(U,G)\, |\{\langle x\rangle \leq U^g | g \in G\}|\, |U|,$$

where the last summation is over all representatives U of conjugacy classes $[U]$ of subgroups of G. Note that by Lemma 1.3(iv)

$$\left|\{\langle x\rangle \leq U^g : g \in G\}\right| = \frac{\beta_{U\backslash G}(\langle x\rangle)}{\beta_{U\backslash G}(U)}.$$

We finally get

$$n_x(G) = \sum_{[U]} \mu_G(U,G) \frac{\beta_{U\backslash G}(\langle x\rangle)}{\beta_{U\backslash G}(U)} |U|.$$

Hence the numbers $n_x(G)$ can be computed from the table of marks for every $x \in G$ (or even for any subgroup $U \leq G$ instead of $\langle x\rangle$).

U	1	2	3	2^2	5	S_3	D_{10}	A_4	A_5
$\lvert N_G(U):U\rvert$	60	2	2	3	2	1	1	1	1
$\mu_G(U,G)$	-60	4	2	0	0	-1	-1	-1	1
$\lambda_G(U,G)$	-1	2	1	0	0	-1	-1	-1	1

TABLE 5. Calculation of the Möbius function for $G = A_5$.

For the alternating group $G = A_5$ one obtains thus the following values $n_x(G)$.

order of x	1	2	3	5
number of such x	1	15	20	24
$n_x(G)$	0	24	36	50

(This result again yields $e_2(A_5) = (15{\cdot}24 + 20{\cdot}36 + 50{\cdot}24)/120 = 19$.)

The second relevant poset in this context is the poset $\mathfrak{S}_G/G$ of conjugacy classes of subgroups, in which the class of a subgroup U is incident to the class of V if there is a $g \in G$ such that $U^g \leq V$.

Denote by λ_G the Möbius function of $\mathfrak{S}_G/G$.

Again, the values $\lambda_G(U,G)$ for $U \leq G$ can be computed from the table of marks. The incidence matrix of the poset of conjugacy classes of G is obtained from the table of marks by replacing every nonzero entry by 1. The inverse of this matrix contains the values of the Möbius function of the poset $\mathfrak{S}_G/G$.

In the case of $G = A_5$ we get the values shown in Table 5.

The values of the functions μ_G and λ_G are related in the following way.

Theorem 8.3. *Let G be a solvable group and let $U \leq G$. Then*

$$\mu_G(U,G) = \left|N_{G'}(U) : G' \cap U\right| \lambda_G(U,G).$$

This is obviously true for abelian groups G, since in that case the two posets $\mathfrak{S}_G$ and $\mathfrak{S}_G/G$ coincide. For $U = 1$ the theorem has been proved in [Hawkes et al. 1989]. The generalization stated above is proved in [Pahlings 1993]. The theorem also holds for many non-solvable groups, (see [Bianchi et al. 1990; Pahlings 1993], and, in particular, the above table for A_5). In [Pahlings 1993] it is shown that the theorem holds for the projective special linear group $L_2(p)$, p a prime. There are, however, counterexamples:

In [Bianchi et al. 1990] it has been observed that, for $G = M_{12}$, $\mu_G(1,G) = \left|M_{12}\right|$, while $\lambda_G(1,G) = 2$. Thus M_{12} provides a counterexample for $U = 1$. In its general form, the formula of theorem does not hold for the simple groups $U_3(3)$, J_2, and M_{23} [Pahlings 1993].

Using the tables of marks of the groups in Table 4 one finds that the theorem also fails for the simple groups A_9, A_{10}, A_{11}, M_{24} and McL, in the latter also for $U = 1$.

8.2. Idempotents in the Burnside Ring

We identify the Burnside ring $\Omega(G)$ with its image under the map β in $\mathbb{Z}^r$.

The only idempotents in $\mathbb{Z}$ are 0 and 1. Each idempotent in $\mathbb{Z}^r$ is of the form $e = (e_1, \dots, e_r)$ where $e_i = 0$ or 1 for all $i = 1, \dots, r$. Let $e^{(i)} = (0, \dots, 0, 1, 0, \dots 0) \in \mathbb{Z}^r$ be such that its entry in th i-th position equals 1 and all other entries are 0 and, for any $I \subseteq \{1, \dots, r\}$, let $e^I = \sum_{i \in I} e^{(i)}$.

The table of marks $M(G)$ provides an efficient means to determine those subsets $I \subseteq \{1, \dots, r\}$ for which the idempotent e^I is an element of $\Omega(G)$.

Let X be a G-set and suppose $H \leq K \leq G$ such that H is a normal subgroup of index p of K for some prime p. Then K acts on the fixed point set $\mathrm{Fix}_X(H)$ with kernel containing H. Since $|K:H| = p$ is prime, the orbits of K on $\mathrm{Fix}_X(H)$ have either length 1 (corresponding to $\mathrm{Fix}_X(K)$) or length p. It follows that

$$\beta_X(H) \equiv \beta_X(K) \pmod p \quad \text{for all } G\text{-sets } X,$$

in other words, if H lies in the conjugacy class $[G_i]$ of subgroups of G and if K lies in class $[G_j]$ then the i-th column and the j-th column of the table of marks $M(G)$ are equal modulo p.

In order to describe the idempotents of $\Omega(G)$ we define two relations on the set $\{1, \dots, r\}$. For any prime p let

$i \to_p j$ if there are $H \le K \le G$ such that H is normal of index p in K, H lies in $[G_j]$ and K lies in $[G_i]$;

$i \equiv_p j$ if the i-th and the j-th column of $M(G)$ are equal modulo p;

The idempotents of $\Omega(G)$ are then completely described by the following proposition [Dress 1969, Proposition 1].

Proposition 8.4. *Let $\equiv$ be the transitive closure of the union of all $\equiv_p$.*

(i) *The relation $\equiv_p$ is the transitive, reflexive, and symmetric closure of the relation $\to_p$.*

(ii) *Let $I \subseteq \{1, \dots, r\}$. The idempotent e^I lies in $\Omega(G)$ if and only if I is a union of $\equiv$-classes.*

In particular, it is possible to decide from the table of marks of G whether G is solvable or not:

Since solvable groups are characterized by the fact that every nontrivial subgroup has a normal subgroup of index p for some prime p we get the following characterization of solvable groups.

Theorem 8.5 [Dress 1969]. *Let G be a finite group. Then G is solvable if and only if 0 and 1 are the only idempotents in the Burnside ring $\Omega(G)$.*

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