Nucleation Parameters for Discrete Threshold Growth on $\mathbb{Z}^2$

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Threshold Growth is a cellular automaton on an integer lattice in which the occupied set grows according to a simple local rule: a site becomes occupied if and only if it sees at least a threshold number of previously occupied sites in its prescribed neighborhood. We study the minimal number of sites that these dynamics need for persistent growth in two dimensions.

1. INTRODUCTION

One of the simplest imaginable cartoons for the spread of a "droplet" in space posits that a vacant site should join the occupied region if it sees enough occupied sites around it. This rule distills some key features of cellular automaton models for excitable media and crystallization, which were the subject of our previous empirical and theoretical research [Durrett and Griffeath 1993; Fisch et al. 1991; 1993; Gravner and Griffeath 1997]. The resulting dynamics, called *Threshold Growth*, were studied in detail in [Gravner and Griffeath 1993; 1996], where we established asymptotic shape and first passage results in both the discrete and continuous space settings. Here we continue our investigation of Threshold Growth, focusing on the size, geometry, and abundance of minimal seeds needed for persistent nucleation. We feel that this study provides some modest insight into the mechanism for such nucleation in deterministic spatial interactions, a decidedly murky subject. Let us begin, then, by briefly summarizing some basic ingredients of the theory; readers are referred to our earlier work for additional background.

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Discrete Threshold Growth has two parameters: the *neighborhood* $\mathcal{N}$ and *threshold* θ. Here θ is a positive integer, and $\mathcal{N}$ is a finite subset of $\mathbb{Z}^2$ that includes the origin; we say that $x+\mathcal{N}$ is the *neighborhood* of site x. Given $A \subset \mathbb{Z}^2$, define

$$\mathcal{T}(A) = A \cup \{x : |(x+\mathcal{N}) \cap A| \geq \theta\}.$$

Start from an initial $A_0 \subset \mathbb{Z}^2$ and compute $\mathcal{T}^n(A_0)$, for $n = 0, 1, 2, \ldots$, to generate the dynamics. Also, write $\mathcal{T}^\infty(A_0) = \bigcup_{n\geq 0} \mathcal{T}^n(A_0)$.

We say that a finite initial set A_0 *generates persistent growth* if $|\mathcal{T}^\infty(A_0)| = \infty$. The dynamics are *omnivorous* if, for every A_0 that generates persistent growth, $\mathcal{T}^\infty(A_0) = \mathbb{Z}^2$. *Nucleation parameters* γ and ν are defined as follows. Let $\gamma = \gamma(\mathcal{N}, \theta)$ be the smallest i for which there exists an A_0 that generates persistent growth and such that $|A_0| = i$. Also, let $\nu = \nu(\mathcal{N}, \theta)$ be the number of sets A_0 of size γ that generate persistent growth and have the leftmost of their lowest sites at the origin. Call a Threshold Growth model *voracious* if $\mathcal{T}^n(A_0) = \mathbb{Z}^2$ for any of the ν initial sets A_0 described above. We believe that dynamics induced by nice neighborhoods—for example, neighborhoods that are obese in the sense of [Gravner and Griffeath 1993]—are always voracious (and, indeed, omnivorous), but have not been able to show this. Recently, T. Bohman devised a remarkable combinatorial argument to prove [Bohman $\geq$ 1997, Theorem 1] that Threshold Growth dynamics are omnivorous for any threshold in the box neighborhood case, where $\mathcal{N}$ is the $(2\rho+1) \times (2\rho+1)$ box centered at the origin, that is, $\mathcal{N} = \mathcal{N}_\rho = \{x : \|x\|_\infty \leq \rho\}$. For this reason, and for the sake of simplicity, we will restrict our analysis to box neighborhoods throughout this paper. However, many of our techniques can easily be adapted to more general settings.

As already mentioned, our principal aim here is to study the size, geometry and abundance of the smallest initial configurations that generate persistent growth. See [Gravner and Griffeath 1997] on convergence to Poisson–Voronoi tessellations for a concrete instance of the role of γ and ν in the exact asymptotics of some self-organizing random cellular automata. The "engineering" approach we will adopt is somewhat analogous to the SPO recipes of [Durrett and Griffeath 1993], although our task here is rather more difficult because persistent growth involves infinitely many time steps. We conclude this Introduction by briefly summarizing our results.

Section 2 begins by establishing the *threshold–range* convergence of γ: we show that there exists a right continuous function $\gamma_E : (0,2) \to (0,\infty)$ such that, for each $\lambda \in (0,2)$,

$$\gamma(\mathcal{N}_\rho, \lambda(\rho+1)^2) \sim \gamma_E(\lambda)\rho^2 \quad \text{as } \rho \to \infty.$$

It is not difficult to see that there exists a $\gamma_c > 0$ such that $\gamma_E(\lambda) = \lambda$ on $(0, \gamma_c)$ but not on $(\gamma_c, 2)$. We proceed to combine geometric arguments with large–neighborhood experiments in order to determine that $\gamma_c \in (1.61, 1.66)$. The way we obtain these rather accurate upper and lower bounds offers some insight into the geometry of the most efficient growing droplets, at least if size is deemed the ultimate measure of efficiency. The remainder of Section 2 deals with the behavior of γ_E close to 2. It is clear that γ_E goes to ∞, but how fast? The question remains open, though we are able to show that $\gamma_E(\lambda)$ lies between $C_1(2-\lambda)^{-1/2}$ and $C_2(2-\lambda)^{-1}$ for suitably chosen constants C_1 and C_2.

Section 3 continues our investigation of the size of the smallest growing seeds, but in contrast to Section 2 we impose a severe constraint on their geometry, dealing only with *square* seeds. This restriction makes it possible to study experimentally a critical size $r_c = r_c(\mathcal{N}_\rho, \theta)$, the smallest r for which an $r \times r$ square grows. Our main rigorous result in this section concerns Threshold Growth at the largest threshold for which $\gamma < \infty$: $\theta = \rho(2\rho+1)$. For this extremal θ, and for ρ large, we show that r_c is of order ρ^2.

The paper concludes with a section devoted to some computational and asymptotic aspects of the "abundance" parameter ν. Not surprisingly, its analysis is vastly more difficult, so we have many more questions than answers at this point.

The main notational conventions of the paper are as follows. As is customary, define the norms

$$\begin{aligned}\|(x_1,x_2)\|_1 &= |x_1|+|x_2|,\\ \|(x_1,x_2)\|_2 &= \sqrt{|x_1|^2+|x_2|^2},\\ \|(x_1,x_2)\|_\infty &= \max\{|x_1|,|x_2|\},\end{aligned}$$

and the balls $B_p(a,r) = \{x : \|x-a\|_p \le r\}$. Also, let $R(a,b)$ be the $(2a+1)\times(2b+1)$ rectangle centered at the origin. Context will make it clear whether these are subsets of the lattice $\mathbb{Z}^2$ or the plane $\mathbb{R}^2$. The Euclidean version of Threshold Growth (to be introduced in Section 2) is denoted by $\mathcal{T}_E$, and the function γ_E mentioned above turns out to be the smallest area those dynamics need to grow. Finally, r_c is defined above to be the smallest side of a growing square for the discrete Threshold Growth, while $\tilde{r}_c$ is the analogous quantity for the Euclidean dynamics.

2. THE SIZE OF γ FOR LARGE NEIGHBORHOODS

This section provides some estimates on the size of γ and ν for range ρ box neighborhoods. Our results make use of the Euclidean space version of Threshold Growth from [Gravner and Griffeath 1993]. All subsets of $\mathbb{R}^2$ that we introduce will be assumed to be measurable. Fix a $\theta_E > 0$ and $\mathcal{N}_E \subset \mathbb{R}^2$. Given $B \subset \mathbb{R}^2$, define

$$\mathcal{T}_E(B) = B \cup \{x : \mathrm{area}((x+\mathcal{N}_E)\cap B) \ge \theta_E\}.$$

As before, denote $\mathcal{T}_E^\infty(B) = \bigcup_{n=0}^\infty \mathcal{T}_E^n(B)$. A Euclidean set B is *extensible* if it is compact and for every bounded $F \subset \mathbb{R}^2$ there exists an n with $F \subset \mathcal{T}_E^n(B)$. Define $\gamma_E = \gamma_E(\mathcal{N}_E, \theta_E)$ as the infimum of $a \ge 0$ such that there exists an extensible B with $\mathrm{area}(B) = a$; this is, roughly speaking, the smallest area needed for growth.

(A technical remark is in order here. The requirement that an extensible set be compact may seem unnecessarily strict; boundedness might be a more natural assumption. In fact, these variants give the same value of γ_E. The proof is left to the reader.)

Since our analysis will focus on the square neighborhood with radius 1, that is, $\mathcal{N}_E = B_\infty(0,1)$, let us abbreviate $\gamma_E(\theta_E) = \gamma_E(B_\infty(0,1),\theta_E)$.

Lemma 2.1. *γ_E is a strictly increasing function on (0,2). In fact, for $\lambda_1 < \lambda_2$, we have $\gamma_E(\lambda_2) \ge \gamma_E(\lambda_1)+\lambda_2-\lambda_1$. Moreover, γ_E is right continuous.*

Proof. To prove the first assertion, assume that it is not true, i.e., there exists a set B_0, extensible for $\mathcal{T}_E(\mathcal{N}_E,\lambda_2)$, with $\mathrm{area}(B_0) < \gamma_E(\lambda_1)+\lambda_2-\lambda_1$. Construct a set B_0' by removing *any* subset of area $\lambda_2-\lambda_1$ from B_0. Then B_0' must be extensible for $\mathcal{T}_E(\mathcal{N}_E,\lambda_1)$, but $\mathrm{area}(B_0) < \gamma_E(\lambda_1)$, a contradiction.

We now proceed to check right continuity. To this end, fix $\lambda < 2$, $a > \gamma_E(\lambda)$, and choose an extensible B_0 with $\mathrm{area}(B_0) < a$. Pick a positive $\varepsilon < 0.01$ small enough so that $\lambda+\varepsilon < 2$ and take an integer R so large that the square $B_\infty(0,R)$ is an extensible set for $\mathcal{T}_E(\mathcal{N}_E,\lambda+\varepsilon)$, and that $B_0 \subset B_\infty(0,R)$. (The choice of 0.01 will become clear in the next paragraph.) Then there exists a finite n such that $B_\infty(0,R) \subset \mathcal{T}_E(\mathcal{N}_E,\lambda)^n(B_0) \subset B_\infty(0,R+n)$.

Now construct a slightly larger set $B_0' \supset B_0$ as follows. Divide $B_\infty(0,R+n+2)$ into $\frac12\times\frac12$ squares. Let S_j be one of those squares, and k_j the smallest k for which $\mathrm{area}(\mathcal{T}_E(\mathcal{N}_E,\lambda)^k(B_0)\cap S_j) \ge 0.23$. If $k_j = 0$ make B_0' agree with B_0 on S_j. Otherwise, pick a set $S_j' \subset S_j$ such that $S_j' \cap \mathcal{T}_E^{k_j-1}(B_0) = \varnothing$ and $\mathrm{area}(S_j') = \varepsilon$, and then choose B_0' so that $B_0'\cap S_j = (S_j'\cup B_0)\cap S_j$. Since, for every $x \in B_\infty(0,R+n)$, $x+\mathcal{N}_E$ includes at least 9 such squares, $\mathrm{area}((x+\mathcal{N}_E)\cap\mathcal{T}_E^i(\mathcal{N}_E,\lambda)(B_0')) \ge \max\{2, \mathrm{area}((x+\mathcal{N}_E)\cap\mathcal{T}_E^i(\mathcal{N}_E,\lambda)(B_0))+\varepsilon\}$ for any $i \le n$. This clearly implies that

$$\mathcal{T}_E(\mathcal{N}_E,\lambda+\varepsilon)^n(B_0') \supset B_\infty(0,R).$$

Finally, note that $\mathrm{area}(B_0') \le \mathrm{area}(B_0) + 4\varepsilon\times(R+n+2)^2$, which finishes the proof. □

For noninteger ρ and θ, let us interpret $\mathcal{T}(\mathcal{N}_\rho, \theta) = \mathcal{T}(\mathcal{N}_{\lfloor \rho \rfloor}, \lceil \theta \rceil)$. Before proving a proposition that establishes γ_E as the threshold–range limit of γ, we need two observations that help connect discrete and Euclidean versions of the dynamics. The first lemma below assumes that the two versions use the same threshold, that the radius of the discrete growth is slightly larger, and that the initial set of the Euclidean growth dominates (in a natural way) the initial set of the discrete dynamics. The conclusion is that this last property holds for the iterates of the two dynamics as well. The second lemma then reverses the roles of the two versions and proves a similar statement.

Lemma 2.2. *Pick an integer ρ, a $\lambda \in (0,2)$ and let $\theta = \lceil (\rho + \frac{1}{2})^2 \lambda \rceil$. Assume that $B_0 \subset \mathbb{R}^2$ and $A_0 \subset \mathbb{Z}^2$ have the property that $B_0 \subset A_0 + B_\infty(0, \frac{1}{2})$. Then, for every $n \geq 0$,*

$$\begin{aligned}
&\mathcal{T}_E((\rho + \tfrac{1}{2})\mathcal{N}_E, (\rho + \tfrac{1}{2})^2 \lambda)^n (B_0) \\
&\quad \subset \mathcal{T}(\mathcal{N}_{\rho+1}, \theta)^n (A_0) + B_\infty(0, \tfrac{1}{2}) \\
&\quad \subset \mathcal{T}(\mathcal{N}_\rho, \theta - 8(\rho+1))^n (A_0) + B_\infty(0, \tfrac{1}{2}). \qquad (2\text{–}1)
\end{aligned}$$

Proof. The second inclusion follows from the fact that $|\mathcal{N}_{\rho+1} \setminus \mathcal{N}_\rho| = 8(\rho+1)$. To prove the first inclusion, assume that $y \in \mathcal{T}_E((\rho+\frac{1}{2})\mathcal{N}_E, (\rho+\frac{1}{2})^2\lambda)(B_0)$ and let $x \in \mathbb{Z}^2$ be such that $\|y - x\|_\infty \leq \frac{1}{2}$. Since $\operatorname{area}((y + (\rho+\frac{1}{2})\mathcal{N}_E) \cap B_0) \geq (\rho+\frac{1}{2})^2\lambda$, there must exist at least θ sites $x_i \in \mathbb{Z}^2$ such that the intersection: $\operatorname{interior}(B_\infty(x_i, \frac{1}{2})) \cap (y + (\rho + \frac{1}{2})\mathcal{N}_E)) \cap B_0$ is nonempty; assume this set contains a point $y_i \in \mathbb{R}^2$. The triangle inequality, and the fact that $\|x - x_i\|_\infty$ is an integer, imply that $\|x - x_i\|_\infty \leq \rho + 1$. Moreover, $\operatorname{interior}(B_\infty(x_i, \frac{1}{2})) \cap B_0 \neq \varnothing$ implies that $x_i \in A_0$. Hence $x \in \mathcal{T}(\mathcal{N}_{\rho+1}, \theta)(A_0)$, and the statement of the lemma holds for $n = 1$. Now iterate to finish the proof. □

Lemma 2.3. *Pick integers ρ and θ, and sets $A_0 \subset \mathbb{Z}^2$ and $B_0 \subset \mathbb{R}^2$ such that $A_0 + B_\infty(0, \frac{1}{2}) \subset B_0$. Then, for every $n \geq 0$,*

$$\mathcal{T}(\mathcal{N}_\rho, \theta)^n (A_0) + B_\infty(0, \tfrac{1}{2}) \subset \mathcal{T}_E((\rho+1)\mathcal{N}_E, \theta)^n (B_0).$$

Proof. Assume that $x \in \mathcal{T}(\mathcal{N}_\rho, \theta)^n(A_0)$ and $y \in \mathbb{R}^2$ are such that $\|y - x\|_\infty \leq \frac{1}{2}$. If $x_i \in x + \mathcal{N}_\rho$, then by the triangle inequality, $B_\infty(x_i, \frac{1}{2}) \subset y + (\rho+1)\mathcal{N}_E$. Therefore, $\operatorname{area}(B_0 \cap (y + (\rho+1)\mathcal{N}_E)) \geq \theta$ and $y \in \mathcal{T}_E((\rho+1)\mathcal{N}_E, \theta)^n(B_0)$. This verifies the claim for $n = 1$, which, again, is clearly enough. □

Proposition 2.4. *For $\lambda \in (0,2)$,*

$$\lim_{\rho \to \infty} \frac{\gamma(\mathcal{N}_\rho, \lambda(\rho+1)^2)}{\rho^2} = \gamma_E(\lambda). \qquad (2\text{–}2)$$

Proof. It is useful to note that the scale invariance of the Euclidean dynamics yields, for every $r > 0$,

$$\gamma_E(r\mathcal{N}_E, \theta_E) = r^2 \gamma_E(\theta_E / r^2). \qquad (2\text{–}3)$$

First fix integers ρ and θ, and take an $A_0 \subset \mathbb{Z}^2$ that generates persistent growth, with $|A_0| = i$. Put $B_0 = A_0 + B_\infty(0, \frac{1}{2}) \subset \mathbb{R}^2$. Then $\operatorname{area}(B_0) = i$, and B_0 is extensible for $\mathcal{T}_E((\rho+1)\mathcal{N}_E, \theta)$ by Lemma 2.3. (Note that we have used Theorem 1 from [Bohman ≥ 1997] here.) It follows from (2–3) that

$$\gamma_E\left(\frac{\theta}{(\rho+1)^2}\right) \leq \frac{\gamma(\mathcal{N}_\rho, \theta)}{(\rho+1)^2}. \qquad (2\text{–}4)$$

Now fix $\lambda' > \lambda$, $a > \gamma_E(\lambda')$, choose an extensible B_0 for $\mathcal{T}_E(\mathcal{N}_E, \lambda')$ with $\operatorname{area}(B_0) < a$, and define $A_0 = \{x \in \mathbb{Z}^2 : B_\infty(x, \frac{1}{2}) \cap (\rho + \frac{1}{2})B_0 \neq \varnothing\}$. If ρ is large enough that $(\rho + \frac{1}{2})^2\lambda' - 8(\rho+1) > \lambda\rho^2 + 1$, then, by Lemma 2.2, A_0 generates persistent growth for $\mathcal{T}(\mathcal{N}_\rho, \rho^2\lambda)$, and therefore

$$\begin{aligned}
\gamma(\mathcal{N}_\rho, \rho^2\lambda) &\leq |A_0| = \operatorname{area}(A_0 + B_\infty(0, \tfrac{1}{2})) \\
&\leq (\rho+\tfrac{1}{2})^2 \operatorname{area}(B_0 + B_\infty(0, 1/(\rho+\tfrac{1}{2}))) \\
&\leq (\rho + \tfrac{1}{2})^2 a, \qquad (2\text{–}5)
\end{aligned}$$

as soon as ρ is large enough.

Abbreviate the limit in (2–2) as lim, and similarly for lim inf and lim sup. It follows from (2–4) and (2–5) that for any $\lambda < \lambda'$,

$$\gamma_E(\lambda) \leq \liminf \leq \limsup \leq \gamma_E(\lambda'),$$

and hence $\lim = \gamma_E(\lambda)$ by Lemma 2.1. □

We make two remarks before proceeding. First, whether γ_E is continuous remains an open problem. We suspect that the regularity of Threshold Growth should ensure continuity, but have no compelling argument. A second observation concerns voracity of the Euclidean dynamics. Namely, call a compact set B *weakly extensible* if $\mathcal{T}_E^\infty(B)$ is unbounded. Assume for a moment that an extensible set is replaced by a weakly extensible set in the definition of γ_E. As the proof of Lemma 2.3 shows, convergence result (2–2) still holds, and hence the two definitions are equivalent.

Having established the existence of the threshold-range limit γ_E, let us turn to the question of its size. One easily verifies that $\gamma_E(\lambda) = \lambda$ for relatively small λ, e.g., for $\lambda < 1$, in which case a $\sqrt{\lambda} \times \sqrt{\lambda}$ square is extensible. It is natural, then, to ask which thresholds admit an omnivorous "droplet" of the smallest possible size, so we define

$$\gamma_c = \sup\{\lambda > 0 : \gamma_E(\lambda) = \lambda\}.$$

One of our main objectives in this paper is to determine γ_c within two percent.

Theorem 2.5. $1.61 < \gamma_c < 1.66$.

Proof. We start with the upper bound, for which the following estimates will be applied.

Proposition 2.6. (i) *Fix an* $\alpha \in [0, 1)$. *If* $\mathcal{T}_E(\mathcal{N}_E, \lambda)^\infty$ *of a* $2 \times (2 - \alpha)$ *Euclidean box is bounded*, *then* $\gamma_c \le \lambda + 4\alpha^2$.

(ii) *Fix an integer* $a \in [0, \rho]$. *If* $\mathcal{T}(\mathcal{N}_{\rho+1}, \theta)^\infty$ *of a* $(2\rho + 1) \times (2\rho + 1 - a)$ *box is bounded*, *then* $\gamma_c \le 4(\theta + 4a^2)/(2\rho + 1)^2$.

Proof of of Proposition 2.6. Let us denote the Euclidean box from (i) by $\tilde{B}_\alpha$ and the discrete one from (ii) by B_a. Let $\lambda + \varepsilon < \gamma_c$, for some $\varepsilon > 0$. We start by proving that there exists an extensible $B_0' \subset \mathcal{N}_E$ for $\mathcal{T}_E(\mathcal{N}_E, \lambda)$ with $\text{area}(B_0') < \lambda + \varepsilon$. To see this, take $\lambda' \in (\lambda + \varepsilon/2, \lambda + \varepsilon)$. Then there exists an extensible B_0 for $\mathcal{T}_E(\mathcal{N}_E, \lambda')$ such that $\text{area}(B_0) < \lambda + \varepsilon$. Find an x so that $\text{area}((x + \mathcal{N}_E) \cap B_0) \ge \lambda'$, and take $B_0' = ((x + \mathcal{N}_E) \cap B_0) - x$. Then $B_0' \subset \mathcal{N}_E$ is extensible for $\mathcal{T}_E(\mathcal{N}_E, \lambda)$ and $\text{area}(B_0') < \lambda + \varepsilon$.

Denote by $F_1, \dots, F_4$ the four $2 \times \alpha$ rectangles that can be removed from $\mathcal{N}_E$ to leave a $2 \times (2 - \alpha)$ box. We claim that $\text{area}(B_0' \cap F_i) \le 4\alpha^2 + \varepsilon$ for at least one i. If this were not true, then no site outside the $2\alpha \times 2\alpha$ Euclidean box around 0 would be added by the dynamics since the area such a site could see would be then less than $4\alpha^2 + \lambda + \varepsilon - (4\alpha^2 + \varepsilon) = \lambda$; thus, B_0' would not be extensible.

Therefore, if $\lambda < \gamma_c$, then for any $\varepsilon > 0$ there exists a set $\tilde{B}_\alpha'$, obtained by adding to the full $\tilde{B}_\alpha$ a set of area at most $4\alpha^2 + \varepsilon$, which is extensible for $\mathcal{T}_E(\mathcal{N}_E, \lambda)$. Consequently, $\tilde{B}_\alpha$ itself is extensible if the threshold is lowered to $\lambda - 4\alpha^2 - \varepsilon$. This is equivalent to statement (i) of the Proposition.

To prove (ii), fix a and set $\alpha = a/(\rho + \frac{1}{2})$. Assume that $\theta \le (\rho + \frac{1}{2})^2 \lambda$. Under the hypothesis, we claim that $\mathcal{T}_E((\rho + \frac{1}{2})\mathcal{N}_E, (\rho + \frac{1}{2})^2 \lambda)^\infty((\rho + \frac{1}{2})\tilde{B}_\alpha)$ is bounded, and therefore so is $\mathcal{T}_E(\mathcal{N}_E, \lambda)^\infty(\tilde{B}_\alpha)$. To this end, observe that $B_a + B_\infty(0, \frac{1}{2})$ is a Euclidean $(2\rho + 1) \times (2\rho + 1 - a)$ box, and $(\rho + \frac{1}{2})\tilde{B}_\alpha$ is a box of the same dimensions. Therefore, Lemma 2.2 shows that if B_a does not generate persistent growth for $\mathcal{T}(\mathcal{N}_{\rho+1}, \theta)$, then $(\rho + \frac{1}{2})\tilde{B}_\alpha$ cannot be extensible for $\mathcal{T}_E((\rho + \frac{1}{2})\mathcal{N}_E, (\rho + \frac{1}{2})^2 \lambda)$, and hence $\gamma_c \le \lambda + 4\alpha^2$. This completes the proof. $\square$

Our bounds for γ_c in Theorem 2.5 are virtually impossible to derive by hand. To obtain them we relied heavily on the the Windows-based CA simulation program *WinCA* [Fisch and Griffeath 1996]. To demonstrate that experiments on finite boxes, such as the one displayed in Figure 1, prove (or disprove) persistent growth, we rely on the following simple proposition. Denote $R(a, b) = \{(x, y) : |x| \le a, |y| \le b\}$, and let $\mathcal{T}|S$ be the dynamics restricted to set $S \subset \mathbb{Z}^2$, with zero boundary conditions. That is, $(\mathcal{T}|S)(A) = \mathcal{T}(A) \cap S$.

Proposition 2.7. *Assume that* $A_0 \subset R(a, b)$.

(1) *If* $R(a+1, b+1) \subset (\mathcal{T}|R(a', b'))^\infty(A_0)$, *for some* $a' > a$, $b' > b$, *then* $\mathcal{T}^\infty(A_0) = \mathbb{Z}^2$.

(2) *If* $R(a,b)^c \cap (\mathcal{T}|R(a',b'))^\infty(A_0) = \varnothing$, *for some* $a' > a$, $b' > b$, *then* $\mathcal{T}^\infty(A_0) \subset R(a,b)$.

Proof. To prove (1), note that $R(a,b) \subset \mathcal{T}^\infty(A_0)$, and that $R(a+1,b+1) \subset \mathcal{T}^{n_0}(R(a,b))$, for some finite n_0. Hence, by translation invariance of $\mathcal{T}$, $R(a{+}2,b{+}2) \subset \mathcal{T}^{n_0}(R(a{+}1,b{+}1)) \subset \mathcal{T}^{2n_0}(R(a,b))$. Continuing in this manner, we conclude that arbitrarily large rectangles are contained in $\mathcal{T}^\infty(A_0)$.

To prove (2), note first that if n_0 is the first time n at which $\mathcal{T}^n(A_0) \cap R(a,b)^c \neq \varnothing$, then

$$\mathcal{T}^{n_0}(A_0) \cap (R(a+1,b+1) \setminus R(a,b)) \neq \varnothing,$$

and thus

$$(\mathcal{T}|R(a',b'))^{n_0}(A_0) \cap (R(a+1,b+1) \setminus R(a,b)) \neq \varnothing.$$

Under the assumptions in (2), then, the restricted and unrestricted dynamics agree on A_0, that is,

$$\mathcal{T}^n(A_0) = (\mathcal{T}|R(a',b'))^n(A_0),$$

hence $\mathcal{T}^n(A_0) \subset R(a,b)$ for all n. □

Propositions 2.6 and 2.7 now reduce the upper bound to a large, judiciously chosen computation. Namely, *WinCA* shows that $\mathcal{T}(\mathcal{N}_{161}, 42736)^\infty$ of a 321×319 box is bounded, and so

$$\gamma_c < 1.6597. \tag{2–6}$$

For a lower bound we use (2–4). That inequality implies that if a set A_0 with $|A_0| = \theta$ generates persistent growth for $\mathcal{T}(\mathcal{N}_\rho, \theta)$, then $\gamma_c \geq \theta/(\rho{+}1)^2$. Hence the challenge is to find the most efficient shape for such an A_0 that grows, with the largest possible θ. An example of our best design to date, and its first 500 iterates, is shown in Figure 1.

The initial configuration (shaded dark gray) fits snugly into $\mathcal{N}_\rho$. If the octagonal hole in the center were filled the set would consist of the square $\mathcal{N}_{\rho'}$, where $\rho' \approx \lfloor(1+\sqrt{5})\rho/4\rfloor \approx 0.809\rho$, to which protruding triangles with slopes $\approx \pm 2^{\pm 3/2}$ are added at the corners. The octagonal hole is cut so that the ratio of the length of its 90° sides to its height is approximately $\sqrt{2}-1$. Finally, the size of the hole is chosen so as to make the number of sites in A_0 exactly θ.

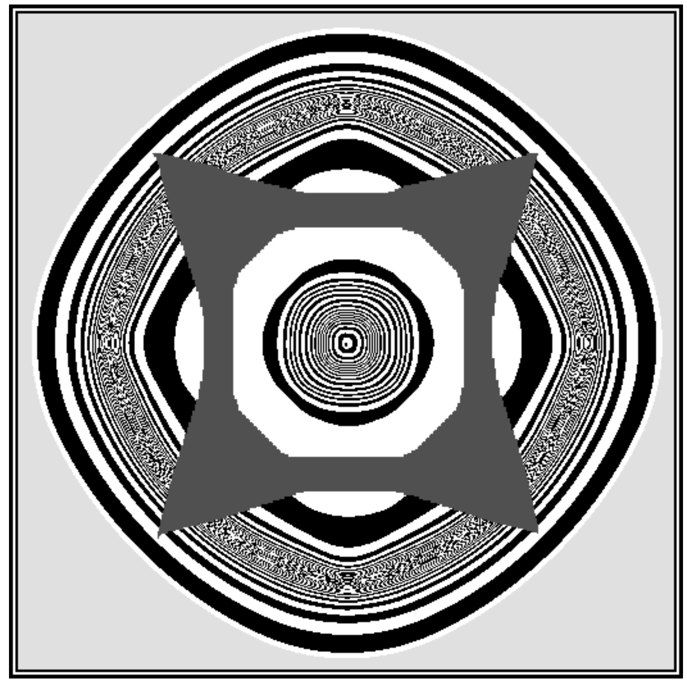

FIGURE 1. A growing "hole" for $\rho = 150$, $\theta = 36760$.

A few words are in order about how we determined these characteristics. Start from a "square annulus" $(1-\alpha)\mathcal{N}_E \setminus (1-\beta)\mathcal{N}_E$, where $0 < \alpha < \beta$, with area λ. Let $x < 1-\beta$ and add $x\mathcal{N}_E$ to the hole. The dynamics then fills in at least the square $(\alpha + \Delta(x))\mathcal{N}_E$ inside the hole, where $\Delta = \Delta(x) = 2(1-\alpha) - 2\sqrt{(1-\alpha)^2 - x^2}$. The iteration given by $x_0 = \alpha$, $x_{n+1} = \alpha + \Delta(x_n)$, generates an $x_n > 1-\alpha$ if $\alpha > (3-\sqrt{5})/4$, suggesting $\rho' \approx (1-\alpha)\rho$. At this point, we know that if θ is such that $\mathcal{N}_{\rho'}$ grows, then there is a set A_0 of θ sites (in fact, a square annulus), such that $\mathcal{T}(\mathcal{N}_\rho, \theta)^\infty(A_0) = \mathbb{Z}^2$.

Using a similar argument, one can add sites (and simultaneously increase θ) outside $\mathcal{N}_{\rho'}$. Assume that their number if sites in the n'th vertical strip of $\mathcal{N}_\rho$, starting with the leftmost one, increases linearly as βn. Consider an occupied set of octagonal shape in the middle, with the ratio of the length of its 90° sides to its height given by α, that is, the convex hull of $\{(\pm n, \pm\alpha n), (\pm\alpha n, \pm n)\}$. This set will grow if its area $2(2-(1-\alpha)^2)n^2$ is larger than the maximal area a point on its boundary does *not* see, which is $\frac{\beta}{2}(1+\alpha^2)n^2$. This determines the largest possible β to be $4\sqrt{2}$, and the corresponding α then is $\sqrt{2}-1$. A little experimentation shows that the proper placement of the extra sites is at the outside corners; in this case there is room for some sites to be added inside the hole. We have chosen the final shape of the hole to be octagonal supposing that the greatest number of sites can be added if the resulting shape matches the described growing octagon. (An octagonal shape is merely one that works, and most likely is not optimal.) Using this architecture, and a substantial amount of interactive optimization at the pixel level, we have designed a growing set of size θ for $\rho = 150$ and $\theta = 36760$ (the one in Figure 1), hence

$$\gamma_c > 1.612. \tag{2–7}$$

(Actually this configuration was obtained by first finding analogous shapes for range 50, and then range 100, optimizing interactively in each case, and then rescaling with a paint program to boost the range.) Since the upper bound (2–6) and the lower bound (2–7) are so close, we conclude that the suggested design is not too far from being an optimal one. This completes the proof of Theorem 2.5. □

We now turn to the study of $\gamma_E(\lambda)$ near $\lambda = 2$. Let us begin by determining how large R should be so that $B_2(0, R)$ is extensible. An easy computation shows this as soon as $R\cdot(2-\lambda) > 2$. But then $B_2(0,R)\setminus B_2(0,R-\sqrt{2})$ grows, proving the upper bound

$$\gamma_E(\lambda) \le C\cdot\frac{1}{2-\lambda}, \tag{2–8}$$

for some constant $C > 0$.

The issue of lower bounds is much trickier. We start with a relatively simple argument that shows that

$$\gamma_E(\lambda) \ge -C\log(2-\lambda). \tag{2–9}$$

The argument begins with the elementary observation that there exists a large enough R_0 so that $\text{area}((x+\mathcal{N}_E)\cap B_2(0,R)) < 2-1/(4R)$ for any $x \notin B_2(0,R)$ and $R \ge R_0$. Next, start with an extensible set A_0, tessellate the plane $\mathbb{R}^2$ into 2×2 squares, and call such a square S *R–occupied* if $\text{area}(S\cap A_0) \ge 0.1/R$. Assume now that $100\cdot R_0 \le R \le 1/(4(2-\lambda))$. Then we claim that the number of R–occupied squares is at least $0.01\cdot R$. Given this, the logarithmic lower bound (2–9) follows easily.

To prove the claim above, assume that it is not true for an R in the specified range. This implies that there exist a finite set of balls $B_1, B_2, \ldots$ with radii $r_1, r_2, \ldots$, which are at least distance 4 apart, cover all R–occupied squares, and are such that $r_1 + r_2 + \ldots \le 0.04\cdot R$. The final observation is that the dynamics cannot add even one new point outside the union $V = B_1\cup B_2\cup\ldots$. To see this, note that any point $x\notin V$ has $\text{area}((x+\mathcal{N}_E)\cap V) \le 2-1/(0.04\cdot R)$ and moreover $x+\mathcal{N}_E$ has nonempty intersection with at most 4 squares from the tessellation, so that $\text{area}((x+\mathcal{N}_E)\cap(V\cup A_0)) \le 2-1/(0.04\cdot R)+0.4/R < \lambda$. This contradicts the assumption that A_0 is extensible.

Which bound is closer to the correct order, (2–8) or (2–9)? In fact, our next proposition indicates that $\gamma_E(\lambda)$ obeys a power law close to $\lambda = 2$. A detailed proof of this result would be exceedingly technical, so we will merely sketch the argument. We suspect, but are presently unable to prove, that (2–8) gives the correct exponent.

Proposition 2.8. *For some constant $C > 0$,*

$$\gamma_E(\lambda) \ge C\cdot\frac{1}{\sqrt{2-\lambda}}. \tag{2–10}$$

Proof. In this proof, C will be a "generic constant," possibly changing value from one appearance to the next.

We start by introducing the comparison dynamics $\mathcal{T}_{E_l}$, a "local" version of $\mathcal{T}_E$. In this part of the argument, a set $B_0 \subset \mathbb{R}^2$ will remain fixed, and is *not* the starting state for $\mathcal{T}_{E_l}$, but rather consists of "helpful" points. Let's define $\mathcal{T}_{E_l}(B)$ as the union of B with the set of $x \in \mathbb{R}^2$ such that $(x+\mathcal{N}_E)\cap B \neq \varnothing$ and $\operatorname{area}((x+\mathcal{N}_E)\cap(B\cup B_0)) \geq \lambda$. The following estimate gives a lower bound on the additional area $\mathcal{T}_{E_l}$ needs to see in order to add a shell of width 2 to a ball of radius R. We believe that the order $1/\sqrt{R}$ is optimal; integration then yields (2–10). This suggests that substantial improvement on (2–10) is impossible without introducing variable shapes into the argument.

Claim. *There exists a small enough constant $C > 0$ so that ${\mathcal{T}_{E_l}}^{\infty}(B_2(0,R)) \subset B_2(0,R+2)$ as soon as $\operatorname{area}(B_0 \setminus B_2(0,R)) < C/\sqrt{R}$, and*

$$5 \leq R \leq C(2-\lambda)^{-1}.$$

To prove the claim, pick an $M > 0$ and divide the annulus $A(R) = B_2(0,R+2) \setminus B_2(0,R)$ into M small radial sectors $S_0, S_1, \dots, S_M = S_0$ of equal shape (see Figure 2). M should be chosen so that $S_i \cap B_2(0,R+\frac{1}{2}) \subset x + \mathcal{N}_E$ for every $x \in S_i \cap B_2(0,R+\frac{1}{2})$ (hence it is of the order $O(R)$). Associate every sector S_i with an angle α_i defined so that the following holds. Let $W(x_i,\alpha_i)$ be any wedge with opening α_i based at a point $x_i \in S_i \cap B_2(0,R+\frac{1}{2})$ so that $B_2(0,R) \subset W(x_i,\alpha_i)$. Then, for every $x \in (S_i \cap B_2(0,R+\frac{1}{2})) \setminus W(x_i,\alpha_i)$, we have $\operatorname{area}((x+\mathcal{N}_E)\cap(B_2(0,R)\cup B_0) < \lambda$. A short computation shows that that $\pi - \alpha_i$ can be chosen to be

$$C \cdot \sum_{j=i-m}^{i+m} \operatorname{area}(S_j \cap B_0),$$

where m is some fixed number. Another observation is that the two tangents to $B_2(0,R)$ at a point z with $\|z\|_2 = R+1$ form an angle φ such that $\pi - \varphi \sim \sqrt{2/R}$, when R is large.

With some work one can show that if $\sum_i(\pi - \alpha_i) < C/\sqrt{R}$ for some small enough C, then there exists a piecewise linear closed curve ℓ, such as that partly shown in Figure 2, which encloses the boundary of $\mathcal{T}_{E_l}^{\infty}(B_2(0,R))$. To illustrate how the curve is constructed, imagine that only one of the sectors, say S_0, has nonempty intersection with A_0 (we believe this is the worst case). Start with the described wedge (with angle opening on the order $1/\sqrt{R}$) inside S_0 and "bend" it at angles of order $1/R$ in the other sectors. The general case is considerably messier, since it is then necessary to combine many bends of different magnitudes. We omit the remaining details of the proof of the Claim from our sketch.

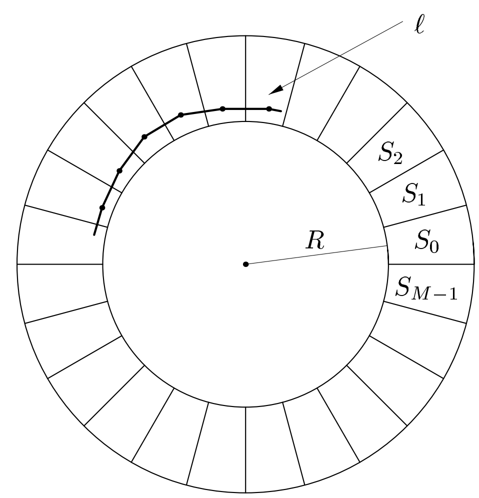

FIGURE 2. Proof of the Claim.

For the second part of the argument, fix an extensible set A_0. We will use the Claim to find a lower bound on $\operatorname{area}(A_0)$ for λ close to 2. Sites in A_0 will be "helpful" for the local dynamics comparison. The initial step is to cover all "nucleation centers" (sets that are able to start growing immediately) with well separated balls.

To accomplish this task, choose any x_1 such that $\operatorname{area}((x_1+\mathcal{N}_E)\cap A_0) \geq \lambda$. Let $\bar{B}_1^0 = B_2(x_1,8)$. If there is an $x_2 \notin B_2(x_1,4)$ with $\operatorname{area}((x_2+\mathcal{N}_E)\cap A_0) \geq \lambda$, let $\bar{B}_2^0 = B_2(x_2,8)$. If there is an $x_3 \notin B_2(x_1,4)\cup B_2(x_2,4)$ with $\operatorname{area}((x_3+\mathcal{N}_E)\cap A_0) \geq \lambda$, let $\bar{B}_3^0 = B_2(x_3,8)$. Continue until no more steps are possible. After this is done, find two sets $\bar{B}_i^0$ and $\bar{B}_j^0$ at distance less than 4, and replace them by the smallest ℓ^2 ball containing both. Continue this procedure until all the balls are at distance at least 4. The final outcome is a set of balls $B_1^0, B_2^0, \dots$ with radii $r_1^0 \geq r_2^0 \geq \cdots$ and $r^0 = r_1^0 + r_2^0 + \cdots$.

(Note how this procedure is quite similar to part of the argument that establishes (2–9).)

We now present an iterative scheme that first enlarges any ball that has the potential to grow, and then forms a well-separated covering of balls as in the previous paragraph. (Enlargement mimics growth, while the covering mimics coalescence, of nucleating droplets in the underlying dynamics.) More precisely, assume that there is a collection of balls $B_1^k, B_2^k, \dots$ at distance at least 4 from one another, with radii $r_1^k \geq r_2^k \geq \cdots$ and $r^k = r_1^k + r_2^k + \cdots$. If A_0 has area at least $C/\sqrt{r_i^k}$ (where C is the constant in the statement of the Claim) in the annulus $(B_i^k + B_2(0,2)) \setminus B_i^k)$, then $\bar{B}_i^{k+1} = B_i^k + B_2(0,2)$. Otherwise $\bar{B}_i^{k+1} = B_i^k$. Then use the same procedure as in the initial step to get new balls separated by distance at least 4, and call them B_i^{k+1}.

Stop this procedure when it either leads to a collection of balls that are unaffected by the first step described in the previous paragraph, or else $r^k \geq C(2-\lambda)^{-1}$. In the first case, the Claim guarantees that A_0 cannot be extensible. Therefore, to finish the proof, we have to find a lower bound on area(A_0) in the second case.

In the $k \to k+1$ step described above, any i such that $\bar{B}_i^{k+1} \neq B_i^k$ requires at least area $C/\sqrt{r_i^k} \geq C/\sqrt{r^k}$ in A_0. It follows that the entire step requires A_0 to have area at least $C(r^{k+1} - r^k)/\sqrt{r^k}$ in $\bigcup_i B_i^{k+1} \setminus \bigcup_i B_i^k$. Another observation is that $r^{k+1} \leq r^k + 3\cdot(\text{number of balls } B_i^k) \leq 4r^k$. The proof is finished immediately if $r^0 \geq C/(2-\lambda)$. Otherwise,

$$\begin{aligned} \text{area}(A_0) &\geq Cr^0 + C \sum_{k \geq 0, r^k \leq C/(2-\lambda)} \frac{r^{k+1} - r^k}{\sqrt{r^k}} \\ &\geq Cr^0 + C \int_{r_0}^{C/(2-\lambda)} \frac{dr}{\sqrt{r}} \\ &\geq \frac{C}{\sqrt{2-\lambda}}, \end{aligned}$$

as required. □

3. THE SMALLEST GROWING SQUARES

Thus far, while studying the smallest sets that grow forever, we have imposed no restrictions on their geometry. However, when doing computer experiments one typically starts with simple shapes for the configuration of occupied sites, since it is tedious to initialize the dynamics with sets such as that shown in Figure 1. In this vein T. Bohman recently posed a question, which arose in [Bohman $\geq$ 1997], concerning the size of the smallest square box that grows forever if θ is the largest supercritical threshold, i.e., $\theta = \rho(2\rho + 1)$.

Bohman's problem has motivated us to study the size of the smallest $r \times r$ squares that generate persistent growth. We start this analysis by proving that the iterates must have a high degree of regularity in this case. To this end, we use the concept of an *obese* set, a strong form of convexity introduced in our paper [Gravner and Griffeath 1993]. Let us briefly review the definitions here.

A set $A \subset \mathbb{Z}^2$ is called *completely symmetric* if it is symmetric with respect to switching sign of either coordinate and switching the coordinates. Moreover, A is *obese* if the two–part cone condition is satisfied:

(i) If $x \in A$ and the first coordinate $x_1 > 0$, then $x - e_1 \in A$.
(ii) If $x \in A$ and $x_1 > x_2 > 0$, then $x - e_1 + e_2 \in A$.

If A is obese, then Proposition A1 of [Gravner and Griffeath 1993] guarantees that so is $\mathcal{T}(A)$; obese initial sets thus eliminate nightmares associated with irregular growth. In addition, obesity enables us to prove the following simple proposition, useful in deciding whether a square grows. In its statement, the set S_0 consists of the four points: $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ (note that by adding S_0 to a square of odd side length one obtains a square of even side length).

Proposition 3.1. *Fix an integer $k \geq 0$.*

(1) *Either $\mathcal{T}^\infty(B_\infty(0,k)) \subset B_1(0,2k)$ or else*

$$\mathcal{T}^\infty(B_\infty(0,k)) = \mathbb{Z}^2.$$

(2) *Either* $\mathcal{T}^\infty(S_0 + B_\infty(0,k)) \subset S_0 + B_1(0,2k)$ *or else* $\mathcal{T}^\infty(S_0 + B_\infty(0,k)) = \mathbb{Z}^2$.

Proof. We will only prove the first statement, the proof of the second being quite similar. If

$$\mathcal{T}^{n_0}(B_\infty(0,k)) \cap B_1(0,2k) \neq \varnothing,$$

then, by obesity, $\mathcal{T}^{n_0}(B_\infty(0,k))$ must include the points $(k+1,k)$ and $(k+1,-k)$. For the same reason, $\mathcal{T}^{n_0}(B_\infty(0,k))$ includes all of $e_1+B_\infty(0,k)$, and hence by symmetry $B_1(0,1) + B_\infty(0,k)$. Thus, for every $n \geq 1$, $B_1(0,n)+B_\infty(0,k) \subset \mathcal{T}^{n\cdot n_0}(B_\infty(0,k))$. □

Define $r_c = r_c(\mathcal{N}, \theta)$ as the least value of r such that an $r \times r$ square generates persistent growth, and define $\tilde{r}_c(\lambda)$ as the infimum of $R \geq 0$ such that $B_\infty(0, R/2)$ square is extensible for $\mathcal{T}_E(\mathcal{N}_E, \lambda)$. Then the same argument as in the proof of Proposition 2.4 yields that

$$\lim_{\rho\to\infty} \frac{r_c(\mathcal{N}_\rho, \lambda\rho^2)}{\rho} = \tilde{r}_c(\lambda)$$

at all points $\lambda \in (0,2)$ for which $\tilde{r}_c(\lambda)$ is continuous. Unfortunately, it turns out that continuity of $\tilde{r}_c$ is even more difficult to establish than continuity of γ_E. So far, we have been able to prove only that $\tilde{r}_c$ is continuous on the interval $(0, \gamma_c')$, where γ_c' is the largest threshold λ for which $\mathcal{N}_E$ is extensible (or, equivalently, for which $\tilde{r}_c(\lambda) < 2$). Methods developed in Section 2, together with the fact that $\mathcal{T}^\infty(\mathcal{N}_{161}, 42837)(\mathcal{N}_{161}) = \mathbb{Z}^2$, imply that $\gamma_c' \geq 42837/162^2 > 1.63$. We will not present the complete argument for continuity below γ_c' (which uses the Euclidean version of obesity quite heavily), but merely indicate the main idea. Fix an $R < 2$ and $\varepsilon < 2 - R$. Then every point $z \in B_1(0,R)$ sees a protruding corner of the ε–enlarged box, i.e., $\text{area}((z+\mathcal{N}_E) \cap (B_\infty(0,(R+\varepsilon)/2) \setminus B_1(0,R)) \geq \varepsilon^2/2$. This property (which fails to hold for $R > 2$), allows one to show that there exists a $\delta = \delta(\varepsilon) > 0$ so that if $B_\infty(0,R/2)$ is extensible for $\mathcal{T}(\mathcal{N}_E, \lambda)$, then $B_\infty(0,(R+\varepsilon)/2)$ is extensible for $\mathcal{T}(\mathcal{N}_E, \lambda+\delta)$.

Exploiting Proposition 3.1, $r_c(\mathcal{N}_\rho, \theta)$ is easy to compute for rather large ranges. We have used a computer program and some *WinCA* experimentation to determine $r_c(\mathcal{N}_\rho, \theta)/\rho$ for $\rho = 20$ and $\theta = 1, \ldots, 820$: Figure 3 shows a plot of this function vs. $\sqrt{\theta}/\rho$. The graph should approximate well that of the function $\tilde{r}_c(\lambda)$ versus $\sqrt{\lambda}$; the emerging message seems to be that this graph is not only continuous, but even convex. What is clear from the arguments in Section 2 is that $\tilde{r}_c(\lambda) = \sqrt{\lambda}$ if and only if $\lambda \in [0,1]$ and that $\tilde{r}_c(\lambda)$ diverges like $(2-\lambda)^{-1}$ as $\lambda \uparrow 2$. Note that Figure 3 also suggests discontinuous derivatives at the two values $\sqrt{\lambda} = 1$ and $\sqrt{\lambda} = \sqrt{\gamma_c'} \approx 1.28$.

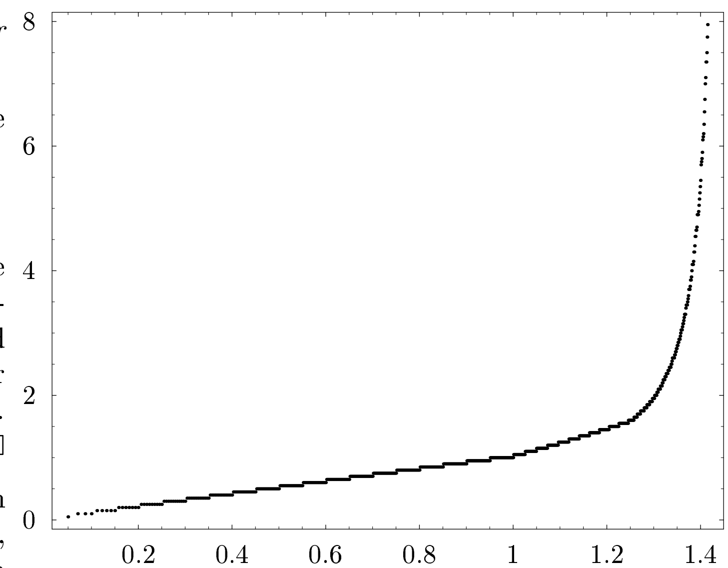

FIGURE 3. Graph of $r_c(\mathcal{N}_\rho, \theta)/\rho$ versus $\sqrt{\theta}/\rho$, for $\rho = 20$.

To conclude this section, we provide an answer to Bohman's question mentioned earlier. As a consequence of the argument below,

$$r_c(\mathcal{N}_\rho,\ \rho(2\rho+1))$$

lies between $(\sqrt{2}/16)\rho^2$ and $2\sqrt{2}\rho^2$, ignoring lower order corrections in ρ. Figure 4 shows the largest box that stops, together with its iterates, in the case $\rho = 20$, $\theta = 820$. Its side has 322 sites; thus $r_c/\rho = 16.1$ and $r_c/\rho^2 = 0.8075$ in this case.

Proposition 3.2. *There exist constants* C_1 *and* C_2 *such that for every* ρ,

$$r_c(\mathcal{N}_\rho, \rho(2\rho+1)) \in [C_1\rho^2, C_2\rho^2].$$

It seems difficult to prove that $r_c(\mathcal{N}_\rho, \rho(2\rho+1)) \cdot \rho^{-2}$ converges, as there is no threshold–range limit in this regime and so a straightforward comparison with Euclidean dynamics is unavailable. Instead, we make use of the transformation $\bar{\mathcal{T}}$, which operates on subsets of $\mathbb{R}^2$ and is conjugate to $\mathcal{T}$ [Gravner and Griffeath 1996]. We will also use the standard notation for half–spaces: for a unit vector $u \in S^1$, $H_u^- = \{x \in \mathbb{R}^2 : \langle x, u\rangle \le 0\}$.

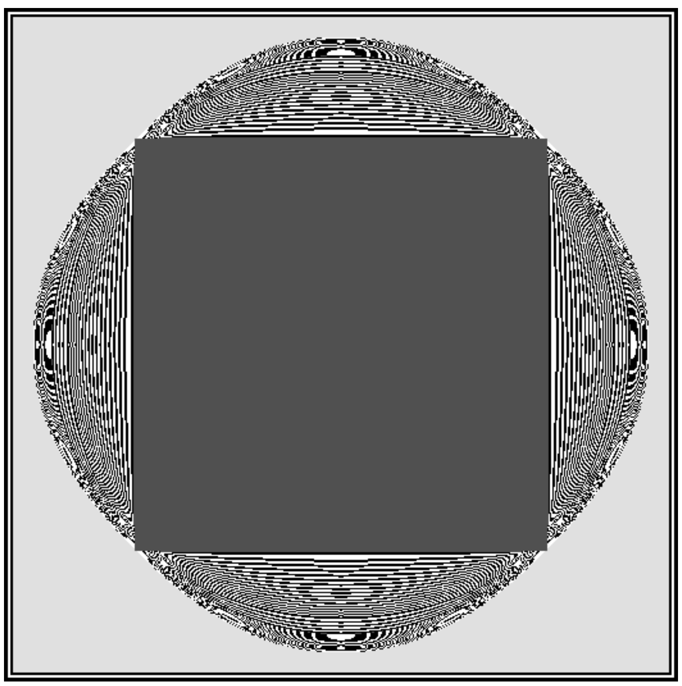

FIGURE 4. Growth from a square of side 322 for $\rho = 20$, $\theta = 820$.

Proof. It is enough to show that $\bar{\mathcal{T}}^\infty(B_2(0,R))$ is unbounded if $R \ge C_2\rho^2$ and bounded if $R \le C_1\rho^2$, for suitably chosen constants C_1 and C_2.

To prove the first assertion, assume R is so large that, for every $u \in S^1$,

$$|(Ru + \mathcal{N}) \cap (Ru + H_u^-) \cap (\text{interior}(B_2(0,R)))^c| \le 2\rho + 1.$$

Then there is an ε such that $B_2(0, R+\varepsilon)$ is contained in $\bar{\mathcal{T}}(B_2(0,R))$ and thus $\bar{\mathcal{T}}^\infty(B_2(0,R)) = \mathbb{R}^2$.

How large must R be so that the preceding condition is true? A simple geometric argument shows that the worst case is when u is a 45 degree vector, in which case the lower bound for R (up to order ρ^2) is given by the following equation:

$$\sqrt{R^2 - (\rho\sqrt{2})^2} = R - \sqrt{2}/2.$$

Therefore, if $R > \sqrt{2}\rho^2 + o(\rho^2)$, then the ball will grow forever.

To prove the second assertion, assume R is small enough so that, for every $u \in S^1$, we have

$$|(Ru + \mathcal{N}) \cap B_2(0,R)| < \theta, \tag{3–1}$$

in which case $\mathcal{T}(B_2(0,R)) = B_2(0,R)$.

To see when (3–1) is true, assume without loss of generality that the direction of u is between 45 and 90 degrees. Then for any integer $i \in [-\rho, \rho]$, the vertical line $\{x + ie_1 + ae_2, a \in \mathbb{R}\}$ (where e_1, e_2 are the standard basis vectors) intersects

$$(Ru + \mathcal{N}) \cap (Ru + H_u^-) \cap (Ru - 2e_2 + H_{-u}^-) \tag{3–2}$$

in at least two integer points. Hence the set (3–2) intersects $\{x + be_1 + ae_2 : a \in \mathbb{R}, |b| \ge \rho/2\}$ in at least ρ points. We are done once we make sure that $B_2(0,R)$ leaves out these points. The worst case situation is now when u is the 90 degree vector, which, up to order ρ^2, requires R to be smaller than the one given by

$$\sqrt{R^2 - (\rho/2)^2} = R - 2.$$

Thus we conclude that $B_2(0,R)$ does not grow for $R \le \rho^2/16 + o(\rho^2)$. □

4. COMPUTATION OF ν FOR SMALL AND LARGE NEIGHBORHOODS

Recall that ν measures the abundance of persistent seeds of minimal size. Thus, in order to compute ν, we need to determine those A_0 with $|A_0| = \gamma$ for which $\mathcal{T}^\infty(A_0)$ is bounded, and those for which $\mathcal{T}^\infty(A_0) = \mathbb{Z}^2$. Bohman's Theorem ensures that these are the only two possibilities, although our approach will end up checking voracity anyway. At first glance, this task seems formidable, even in small cases, since $\mathcal{T}^\infty(A_0)$ depends on arbitrarily large times (and we cannot assume that iterates become obese). Fortunately, Threshold Growth is so well-behaved that Proposition 2.7 allows us to compute ν and check voracity in all cases we have tried. In fact, if we chose a' and b' about 2ρ larger than a and b, respectively, then either (1) or (2) of that Proposition always occurred. In any case, a computer program can set a flag whenever neither (1) nor (2) happens, and those cases can then be checked separately.

We remark that Proposition 2.7 is formulated in terms of rectangles to facilitate the computation of ν. Suppose $\gamma = \theta$, so that all A_0 with γ sites that generate persistent growth are included in a translate of $\mathcal{N}$. To determine ν one does not count translations of a set A_0 as distinct. An easy implementation translates any prospective A_0 so that its leftmost lowest site is at the origin. Such an A_0 has all its other sites in $([-2\rho, 2\rho] \times [0, 2\rho]) \setminus ([-2\rho, 0] \times \{0\})$. Thus one must check all subsets of size $\theta - 1$ within a set of $4\rho(2\rho+1)$ sites, a considerable but easily automated task. Interested readers can download the program `c4.c`, which was used to generate the Table below, by anonymous ftp from cam8.math.wisc.edu. Only the $\theta = 2$ case of this algorithm is easy to enumerate.

Proposition 4.1. *If $\theta = 2$, then $\gamma = 2$ and $\nu = 4\rho(2\rho + 1)$.*

Proof. We only need to show that *every* pair of sites inside $\mathcal{N}$ fills $\mathbb{Z}^2$. This is clearly true for a horizontally adjacent pair. However, such a pair must exist at time 2. $\square$

Let us conclude the paper with a brief and rather speculative discussion of the behavior of ν for large range. One expects $\frac{1}{\theta}\log\nu$ to have a threshold–range limit, i.e.,

$$\lim_{\rho\to\infty} \frac{\log \nu(\mathcal{N}_\rho, \lambda\rho^2)}{\lambda\rho^2} = \eta_E(\lambda) \qquad (4\text{–}1)$$

should exist and be finite for $\lambda \in (0, 2)$. At present, however, we do not know techniques to show convergence, or even to give reasonable estimates on $\eta_E(\lambda)$ for general λ. This section contains a couple of simple preliminary results indicating only that minimal seeds are effectively *random* when λ is small.

Proposition 4.2. *Let λ be such that*

$$\gamma(\mathcal{N}_\rho, \theta) = \theta \text{ for any } \theta \le \lambda\rho^2. \qquad (4\text{–}2)$$

Then $\eta_E(\lambda') \in (0, \infty)$ for any $\lambda' < \lambda$ (meaning that $\liminf$ *of the expression in* (4–1) *is strictly positive, and* $\limsup$ *is finite).*

Proof. If λ satisfies (4–2), then there exists a set A_0 of size $\theta = \lfloor \lambda\rho^2 \rfloor$ that generates persistent growth. Let $\theta' < \theta$ and $A_0' \subset A_0$ with $|A_0'| = \theta'$. Then A_0' grows indefinitely for the dynamics with threshold θ', and therefore $\nu(\mathcal{N}_\rho, \theta') \ge \frac{1}{|\mathcal{N}_\rho|}\binom{\theta}{\theta'}$. This gives

$$\eta_E(\lambda') \ge \Bigl(\frac{\lambda}{\lambda'} - 1\Bigr) \log\Bigl(\frac{\lambda}{\lambda'} \Big/ \Bigl(\frac{\lambda}{\lambda'} - 1\Bigr)\Bigr).$$

On the other hand, $\nu(\mathcal{N}_\rho, \theta) \le \binom{(2\rho+1)^2}{\theta}$, which gives

$$\eta_E(\lambda') \le \Bigl(\frac{4}{\lambda'} - 1\Bigr) \log\Bigl(\frac{4}{\lambda'} \Big/ \Bigl(\frac{4}{\lambda'} - 1\Bigr)\Bigr).$$

$\square$

Proposition 4.3. *Assume that $\rho \to \infty$, $\theta \to \infty$, $\frac{1}{\theta}\log\rho \to 0$, and $\lambda = \theta/\rho^2 \to 0$. Then*

$$\frac{1}{\theta}\log\nu - \log\frac{4e}{\lambda} \to 0.$$

Moreover $\eta_E(\lambda) \sim \log\frac{1}{\lambda}$ as $\lambda \to 0$ (meaning that both $\liminf$ *and* $\limsup$ *of the expression in* (4–1) *satisfy this asymptotic formula).*

Proof. Fix an $\varepsilon > 0$. Then, for large enough ρ, $4\varepsilon^2\rho^2 > \theta$, so $\mathcal{N}_{\varepsilon\rho}$ grows. Therefore any subset of $\mathcal{N}_\rho$ of size θ will grow as soon as it has no sites outside $\mathcal{N}_{(1-\varepsilon)\rho}$. However, if $n = (2\rho + 1)^2$ and $a = (1 - \varepsilon)^2$, $\binom{an}{\theta} \ge a^\theta \cdot \binom{n}{\theta}$, which reduces the proof to showing that

$$\frac{1}{\theta}\log\frac{\theta^\theta \cdot n(n-1)\ldots(n-\theta+1)}{e^\theta \cdot \theta! \cdot n^\theta} \to 0,$$

$\nu(\mathcal{N}_\rho, \theta)$	$\theta = 2$	$\theta = 3$	$\theta = 4$	$\theta = 5$	$\theta = 6$	$\theta = 7$
$\rho = 1$	12	42	—	—	—	—
$\rho = 2$	40	578	4683	24938	94050	259308
$\rho = 3$	84	2602	46704	574718		
$\rho = 4$	144	7702	241151			
$\rho = 5$	220	18038				

TABLE 1. Values of ν for small range.

an easy consequence of Stirling's formula. The last assertion of the proposition is even easier to check. $\square$

To our annoyance, we are not able to prove either that (4–2) holds for all $\lambda < \gamma_c$, or that η_E is always finite. Let us however assume these "facts" for the sake of some speculative discussion. It seems clear that $\eta_E(\lambda)$, which is infinite at $\lambda = 0$ by Proposition 4.3, should decrease as λ increases from 0 to γ_c, since there is less and less room for distribution of sites inside $\mathcal{N}$. Is $\eta_E(\gamma_c) = 0$? We do not hazard an answer to this intriguing question. Neither do we offer a plausible scenario for the behavior of $\eta_E(\lambda)$ when $\lambda > \gamma_c$. On the one hand, the number of available sites grows in this regime since the most efficient seeds no longer need be subsets of $\mathcal{N}$. On the other hand, the severe optimization constraints inherent at γ_c should remain in effect. A likely scenario would be for η_E to have a strict minimum at γ_c, but it is conceivable, for instance, that η_E might be constant on $[\gamma_c, 2)$.

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REFERENCES

[Bohman $\geq$ 1997] T. Bohman, "Discrete threshold growth dynamics are omnivorous for box neighborhoods". To appear in *Trans. Amer. Math. Soc.*

[Durrett and Griffeath 1993] R. Durrett and D. Griffeath, "Asymptotic behavior of excitable cellular automata", *Experiment. Math.* **2**:3 (1993), 183–208.

[Fisch and Griffeath 1996] R. Fisch and D. Griffeath, "*WinCA*, a Windows-based CA experimentation environment", 1996. See http://math.wisc.edu/~griffeat/sink.html.

[Fisch et al. 1991] R. Fisch, J. Gravner, and D. Griffeath, "Threshold-range scaling for the excitable cellular automata", *Statistic and Computing* **1** (1991), 23–39.

[Fisch et al. 1993] R. Fisch, J. Gravner, and D. Griffeath, "Metastability in the Greenberg–Hastings model", *Ann. Appl. Probab.* **3**:4 (1993), 935–967.

[Gravner and Griffeath 1993] J. Gravner and D. Griffeath, "Threshold growth dynamics", *Trans. Amer. Math. Soc.* **340**:2 (1993), 837–870.

[Gravner and Griffeath 1996] J. Gravner and D. Griffeath, "First passage times for threshold growth dynamics on $\mathbb{Z}^2$", *Ann. Probab.* **24**:4 (1996), 1752–1778.

[Gravner and Griffeath 1997] J. Gravner and D. Griffeath, "Multitype Threshold Growth: convergence to Poisson–Voronoi tessellations", *Ann. Appl. Prob.* **7**:3 (1997). See http://math.wisc.edu/~griffeat/papers2.html.

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