Hausdorff Convergence and the Limit Shape of the Unicorns

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CONTENTS

We discuss the Hausdorff convergence of hyperbolic components in parameter space as a one-parameter family of transcendental functions is dynamically approximated by polynomials. This convergence is strongly suggested by computer experiments and is proved in a weaker form, which is illustrated with exponential, sine and cosine families. Furthermore, we consider the convergence of subhyperbolic components. Our result also applies to the antiholomorphic exponentials, which allows us to investigate the limit shape of the unicorns.

1. INTRODUCTION

Consider a family of entire functions depending on a complex parameter. By iteration it gives rise to a family of dynamical systems. The question is what can be said about the dynamical systems of the family and of their dependence on the parameter. Of particular interest are how the Fatou or stable set, where the dynamics is "well-behaved", and the Julia set, where the dynamics is "chaotic", depend on the parameter. Suppose now that an entire *transcendental* family is approximated by a sequence of families of *polynomials* (of increasing degree) locally uniformly on compact sets. The singular values, that is, values where one cannot define a local inverse, are of paramount importance for the dynamics. To make "dynamical sense" we ask all families to have the same finite number of (free) singular values, in which case we speak of *dynamical approximation.* Since it is generally easier to study polynomials rather than transcendental functions, the question is: What can be said about the transcendental family as a dynamical system

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by looking at the polynomial families as the degree tends to infinity?

This type of convergence question has first been addressed in [Devaney et al. 1986], which studied the approximation of the exponentials $E(\lambda, z) = \lambda\, e^z$ by

$$P_d(\lambda, z) = \lambda \left(1 + \frac{z}{d}\right)^d.$$

This example gives a nice connection between the connectedness locus of P_2, the well-known Mandelbrot set, and the parameter plane of E. It is shown there that certain external rays of the connectedness loci of the polynomials P_d converge to "hairs" in the parameter plane of the exponentials E, and that hyperbolic components converge pointwise. Similar results are established in [Fagella 1995] for the family $\mathcal{G}(\lambda, z) = \lambda\, z\, e^z$ approximated by

$$\mathcal{Q}_d(\lambda, z) = \lambda\, z \left(1 + \frac{z}{d}\right)^d,$$

which shows up in the study of the complex standard family. The strongest result up to now in the parameter plane is the convergence of hyperbolic components as kernels in the sense of Carathéodory in [Krauskopf and Kriete 1995b]. In the *dynamical plane* the Hausdorff convergence of the Julia sets, for suitable values of the parameter, is proved in [Krauskopf 1993] for the example from [Devaney et al. 1986]. This result has been extended to the dynamical approximation of entire transcendental functions in [Kisaka 1995; Krauskopf and Kriete 1996] and of meromorphic functions in [Krauskopf and Kriete 1995a; Kriete 1995].

In this paper we are concerned with the convergence of hyperbolic components *in the parameter plane* in the strongest sense, that is, with respect to the Hausdorff metric, which is strongly suggested by computer experiments for several examples. For families with one free singular value we consider the m-layers, defined as the set of hyperbolic components for which the free singular value converges to a cycle of minimal period m. Under a certain compactness condition we prove Hausdorff convergence of m-layers. In fact we conjecture that the compactness condition can be dropped. Furthermore we conjecture that for some of our examples the hyperbolic components themselves converge in the Hausdorff metric, which is false in general.

For illustration we use the example from [Devaney et al. 1986] and complex sine and cosine families, dynamically approximated using Chebyshev polynomials. We adapt our results to subhyperbolic components and discuss the example in [Fagella 1995]. Finally we investigate the family of antiholomorphic exponentials $\bar{E}(\lambda, z) = \lambda e^{\bar{z}}$ approximated by

$$\bar{P}_d(\lambda, z) = \lambda \Big(1 + \frac{\bar{z}}{d}\Big)^d.$$

The connectedness loci of the $\bar{P}_d$ are called *unicorns* due to their shapes and their limit shape is provided by the parameter plane of $\bar{E}$.

This paper is organized as follows. In the next section we recall some basic notions from iteration theory. Hyperbolic components and m-layers are introduced in Section 3. The Main Theorem is stated in Section 4 and illustrated with a number of examples in Sections 5–7. The proof can be found in Section 8. The compactness condition that appears in the statement of the Main Theorem is discussed in Section 9 for our examples. Section 10 gives information on the algorithms used to compute the figures.

2. FACTS FROM ITERATION THEORY

In this section we briefly recall some notations and basic facts from iteration theory as they can be found for example in the surveys [Bergweiler 1993; Baker 1988; Erëmenko and Lyubich 1992; Milnor 1990] or the monographs [Beardon 1991; Carleson and Gamelin 1993; Steinmetz 1993]. Consider an entire function $f : \mathbb{C} \to \mathbb{C}$, that is, a function holomorphic on the whole of $\mathbb{C}$. Then f is either a polynomial or an entire transcendental function with an essential singularity at ∞. In the sequel we consider f as a map from the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself, which is perfectly justified for polynomials and a convention for transcendental functions. Let $f^{\circ n}$ denote the n-th iterate of f. For a given point $\zeta \in \widehat{\mathbb{C}}$ we call

$$O^+(\zeta) := \{f^{\circ n}(\zeta) : n \in \mathbb{N}\}$$

the (*forward*) *orbit* of ζ. The point ζ is called *m-periodic* if $f^{\circ m}(\zeta) = \zeta$, and m is called the *minimal period* if it is minimal with this property. If ζ is m-periodic its orbit $C := O^+(\zeta)$ is also called a *cycle*. Of importance is the *multiplier* of C, defined as $M(C) = (f^{\circ m})'(\zeta)$. A cycle C is called attracting (respectively repelling) if its multiplier is smaller (respectively larger) than one in absolute value. It is called indifferent if $|M(C)| = 1$. Every attracting cycle C comes with a *basin* $A(C)$, defined as the set of points $z \in \widehat{\mathbb{C}}$ whose orbits accumulate on C under iteration. In other words, $A(C) = \{z \in \mathbb{C} : \omega(z) = C\}$, where $\omega(z)$ is the set of all accumulation points of the sequence of iterates $\{f^{\circ n}(z)\}$.

The key problem is to get a global picture of the dynamics of f on $\widehat{\mathbb{C}}$, which is why one studies the following sets. The *Fatou set* $F(f)$ is defined as the set of points $\zeta \in \widehat{\mathbb{C}}$, such that on some neighborhood $V \subset \widehat{\mathbb{C}}$ of ζ the iterates $f^{\circ n}|_V$ are holomorphic and form a normal family. This means that in any sequence of these iterates one can find a subsequence that converges uniformly on compact sets to a limit, so that the Fatou set is the set of "well-behaved" points. Clearly any basin $A(C)$ is a subset of the Fatou set. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\widehat{\mathbb{C}}$, and it contains the "chaotic" points. It is a well-known theorem that the repelling periodic points of f are dense in $J(f)$, which provides an alternative way of defining these two sets. Note that we regard both $F(f)$ and $J(f)$ as subsets of $\widehat{\mathbb{C}}$ even if f is transcendental.

Of great importance are the *singular values* of f, where one cannot define a local inverse of f. The set $\operatorname{sing}(f)$ of all *finite* singular values of f consists of the *finite critical values* of f and, if f is transcendental, also of the *finite asymptotic values*. An important fact we use much in the sequel is that each basin of an attracting periodic orbit contains at least one singular value. We call f *hyperbolic* if $O^+(\operatorname{sing}(f)) \Subset F(f)$, that is, if the complete forward orbit of all singular values is relatively compact in the Fatou set. Note that in particular the singular values cannot accumulate on the Julia set if f is hyperbolic.

3. HYPERBOLIC COMPONENTS AND m-LAYERS

From now on we study families of entire maps depending holomorphically on a parameter λ from $\mathbb{C}^L$, or an arbitrary complex manifold of any finite dimension for that matter. That is, we consider a holomorphic function $\mathcal{F} : \mathbb{C}^L \times \mathbb{C}\,; (\lambda, z) \mapsto \mathcal{F}(\lambda, z)$, viewed as a family of entire functions $\mathcal{F}(\lambda, \cdot)$. We say that a family $\mathcal{F}$ has *one free singular value* if there is a singular value $c(\lambda)$ depending holomorphically on λ and such that $\mathcal{F}(\lambda, \cdot)$ is hyperbolic if and only if $O^+(c(\lambda)) \Subset F(\mathcal{F}(\lambda, \cdot))$. This means that all singular values other than $c(\lambda)$ are not important as far as hyperbolicity goes in the following sense. Singular values other than $c(\lambda)$ are either absorbed by attracting cycles or have the same dynamics as the free singular values, for example for symmetry reasons.

In this paper we restrict our attention to families of finite type with one free singular value. In other words, each family has only finitely many singular values, of which only one needs to be considered to check for hyperbolicity; see also the examples below. In fact a family with one free singular value is essentially a one-parameter family of entire maps, which is why we take the parameter space to be $\mathbb{C}$ in the sequel. We now define the set

$$\mathcal{H}(\mathcal{F}) := \{\lambda \in \mathbb{C} : \mathcal{F}(\lambda, \cdot) \text{ is hyperbolic}\},$$

which is open because attracting cycles persist under changes of λ; compare [Krauskopf and Kriete 1995b]. A connected component of $\mathcal{H}(\mathcal{F})$ is called a *hyperbolic component*. (We assume that $\mathcal{H}(\mathcal{F})$ is nonempty for all families considered here.)

Because there is only one free singular value $c(\lambda)$, we can define the sets

$$\mathcal{H}_m(\mathcal{F}) := \left\{ \begin{array}{r} \lambda \in \mathbb{C} : c(\lambda) \text{ is in the basin of a} \\ \text{minimal } m\text{-cycle of } \mathcal{F}(\lambda, \cdot) \end{array} \right\},$$

which we call *m-layers*. We get a natural partition of $\mathcal{H}$ into m-layers, that is, $\mathcal{H}_i(\mathcal{F}) \cap \mathcal{H}_j(\mathcal{F}) = \varnothing$ for $i \neq j$ and $\mathcal{H}(\mathcal{F}) = \dot{\bigcup}_{m\in\mathbb{N}} \mathcal{H}_m(\mathcal{F})$. The m-layers themselves consist typically of infinitely many hyperbolic components.

4. MAIN RESULT

The setting of the Main Theorem is the following. We consider a limit family $\mathcal{F}_\infty$ of entire transcendental functions of *finite type* and a family of polynomials $\mathcal{F}_d$, converging to $\mathcal{F}_\infty$ uniformly on compact sets. Clearly the $\mathcal{F}_d$ must be of increasing degree in d. To get control over the dynamics we ask that $\mathcal{F}_\infty$ and $\mathcal{F}_d$ have the same (finite) number of singular values, independently of d and λ, which is what we call *dynamical approximation*. Furthermore, we suppose that $\mathcal{F}_\infty$ and $\mathcal{F}_d$ have one free singular value $c_\infty = c_\infty(\lambda)$ and $c_d = c_d(\lambda)$, respectively.

We are interested in convergence with respect to the Hausdorff metric. Recall that for two closed sets $A, B \in \widehat{\mathbb{C}}$ the Hausdorff distance is defined as

$$h(A,B) := \inf\{\varepsilon > 0 : A \subset V_\varepsilon(B) \text{ and } B \subset V_\varepsilon(A)\},$$

where $V_\varepsilon(X)$ denotes the ε-neighborhood of a closed set $X \in \widehat{\mathbb{C}}$ with respect to the chordal metric on $\widehat{\mathbb{C}}$. When we talk of the Hausdorff convergence of hyperbolic components or m-layers, which are open sets, we always mean the convergence of the respective boundaries with respect to $\widehat{\mathbb{C}}$. For the statement of our result we need the following technical **C**ompactness property.

Definition 4.1. We say that the approximating families $\mathcal{F}_d$ have *Property C* if the following holds.

Consider a λ from a given compact set $K \subset \mathbb{C}$, such that $c_d(\lambda)$ is attracted by some m-cycle $C_d(\lambda)$ of $\mathcal{F}_d(\lambda, \cdot)$ for almost all $d \in \mathbb{N}$. Then the limit $C_\infty(\lambda)$ of the $C_d(\lambda)$ exists and lies in a disk around the origin with radius R, where R depends only on K and m.

Theorem 4.2 (Main Theorem). *Let $\mathcal{F}_\infty$ be a family of entire transcendental functions that is dynamically approximated by families of polynomials $\mathcal{F}_d$. Suppose that all families have one free singular value and that the approximating families have Property C. Then for every m-layer $\mathcal{H}_m(\mathcal{F}_\infty)$ of $\mathcal{F}_\infty$ there exists a sequence $\{\mathcal{H}_m(\mathcal{F}_d)\}$ of m-layers of $\mathcal{F}_d$ converging to $\mathcal{H}_m(\mathcal{F}_\infty)$ with respect to the Hausdorff metric as d tends to infinity.*

The purpose of Property C is to avoid the following problem. Clearly each attracting cycle $C_\infty(\lambda)$ of $\mathcal{F}_\infty(\lambda, \cdot)$ is approximated by attracting cycles $C_d(\lambda)$ of $\mathcal{F}_d(\lambda, \cdot)$. However, it is not at all clear that the limit $C_\infty(\lambda)$ of attracting cycles $C_d(\lambda)$ of $\mathcal{F}_d(\lambda, \cdot)$ is finite. It may happen that $\infty \in C_\infty(\lambda)$. This would cause great difficulties since $\mathcal{F}_\infty$ is transcendental and has ∞ as an essential singularity. However, we do not know of an example of this. In Section 9 we discuss a method of checking Property C. We conjecture that dynamical convergence is such a strong notion that Property C can be dropped from the assumptions of the Main Theorem.

We have stated the Main Theorem for entire functions of *finite type* for two reasons. First, in the proof we use the result in [Krauskopf and Kriete 1995b] on the kernel convergence of hyperbolic components, which is stated and proved for functions of finite type. Second, all the examples we present are of finite type. However, we remark that the proof in [Krauskopf and Kriete 1995b] can be generalized to the case of a limit family with infinitely many singular values that is dynamically approximated. Here this means that the limit family and the approximating families have a constant finite number of *free* singular values. As a consequence, the Main Theorem applies in this more general setting. The full proof of this is beyond the scope of this paper.

The next section illustrates the Main Theorem with a number of examples.

5. THREE EXAMPLES

Example 5.1 (Exponential Family). As mentioned in the introduction, the classic example of dynamical approximation is from [Devaney et al. 1986], the polynomials

$$P_d(\lambda, z) = \lambda \left(1 + \frac{z}{d}\right)^d$$

converging locally uniformly to

$$E(\lambda, z) = \lambda\, e^z.$$

The families E and P_d have the unique singular value 0 and the P_d have Property C; see Section 9. As a consequence of the Main Theorem, the m-layers $\mathcal{H}_m(P_d)$ converge in the Hausdorff metric to the m-layer $\mathcal{H}_m(E)$, as is illustrated in Figure 1.

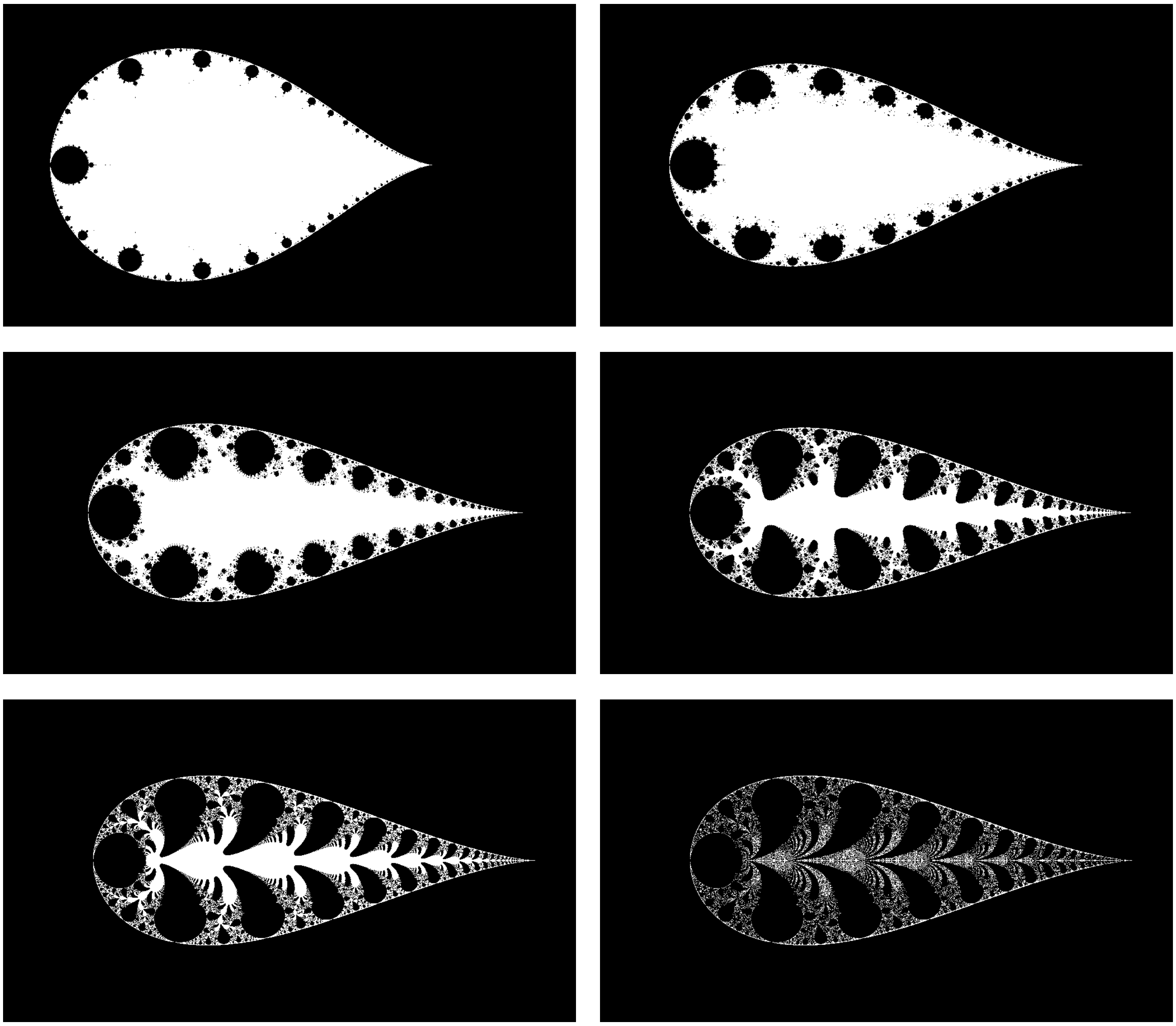

FIGURE 1. The hyperbolic components of P_2, P_4, P_{16}, P_{256}, P_{65536} and E from Example 1. All figures in this paper depict a chart in the neighborhood of the point ∞, which can be found in the middle of the pictures. For more information on how the figures were computed, see Section 10.

Consider the *Mandelbrot sets*

$$M_d := \{c \in \mathbb{C} : \{P_d^{\circ n}(0)\} \text{ stays bounded}\},$$

which are in fact the connectedness loci of the P_d, that is, the sets for which $J(P_d(\lambda, \cdot))$ is connected. The Mandelbrot sets M_d are compact and, consequently, they contain the closure of all hyperbolic components inside them. Our result implies that this closure converges to the closure of $\mathcal{H}(E)$. This is a statement about the limit of the Mandelbrot sets M_d as the degree d goes to infinity.

It is generally believed that each Mandelbrot set M_d is the closure of the hyperbolic components it contains; this is known as the Generic Hyperbolicity Conjecture. For the limit, the conjecture is that $\mathcal{H}(E)$ is dense in

$$M_\infty := \{c \in \mathbb{C} : \{E^{\circ n}(0)\} \text{ stays bounded}\},$$

which in turn is conjecturally dense in $\widehat{\mathbb{C}}$. If this is indeed true, then the closure of the limit of the Mandelbrot sets M_d, as d goes to infinity, is $\widehat{\mathbb{C}}$.

Example 5.2 (Approximation by Chebyshev polynomials). It is well-known that the Chebyshev polynomials provide an excellent approximation of the cosine and sine functions; see for example [Rivlin 1974]. Recall that these polynomials can be defined recursively by

$$\begin{aligned} T_0(z) &= 1, \\ T_1(z) &= z, \\ T_d(z) &= 2\,z\,T_{d-1}(z) - T_{d-2}(z). \end{aligned}$$

For our purposes it is important that $T_{4d}(z/(4d))$ and $T_{4d+1}(z/(4d+1))$ converge locally uniformly and dynamically to $\cos(z)$ and $\sin(z)$, respectively. Consider now the families

$$\mathcal{S}_d(\lambda, z) = \lambda T_{4d+1}\left(\frac{z}{4d+1}\right)$$

that dynamically approximate the family

$$\mathcal{S}_\infty(\lambda, z) = \lambda \sin(z).$$

These families have the singular values $\pm\lambda$. Since the functions are odd the orbit of $-\lambda$ is the negative of the orbit of λ, so that both singular values behave in the same way. Consequently, to check for hyperbolicity it is sufficient to consider the orbit of, say, the singular value λ, which we call the free singular value. The convergence of the $\mathcal{H}_m(\mathcal{S}_d)$ to $\mathcal{H}_m(\mathcal{S}_\infty)$ appears to be very rapid and is illustrated in Figure 2.

In the same way $\lambda\cos(z)$ can be approximated, but here we consider a slightly different example, namely the family

$$\mathcal{C}_\infty(\lambda, z) = \lambda(\cos(z) - 1),$$

which is dynamically approximated by

$$\mathcal{C}_d(\lambda, z) = \lambda\left(T_{4d}\left(\frac{z}{4d}\right) - 1\right).$$

For these families the critical value 0 is an attracting fixed point for any λ, so that the second critical value -2λ is the free singular value. Again, the convergence of $\mathcal{H}(\mathcal{C}_d)$ to $\mathcal{H}(\mathcal{C}_\infty)$ seems to be very rapid, as shown in Figure 3.

We have not been able to show that our examples involving the Chebyshev polynomials have Property C; compare Section 9. Nonetheless, Figures 2 and 3 suggest that even the hyperbolic components themselves converge.

Figures 1–3 give the impression that the hyperbolic components themselves converge in the Hausdorff metric. We conjecture that this is true for the above families and for many other examples, but in general this is false, as we show now.

Counterexample 5.3. The polynomial counterexample in [Krauskopf and Kriete 1995b, Example 2] can be adapted to this setting. Here we present an (even worse) example, where a single hyperbolic component is pinched into infinitely many hyperbolic components in the limit, with ∞ as their common boundary point.

Consider the families

$$\mathcal{F}_d(\lambda, z) = \left(1 - \frac{1}{d} + \frac{1}{2}\left(1 + \frac{\lambda}{d}\right)^d\right)\left(\left(1 + \frac{z}{d}\right)^d - 1\right)$$

and

$$\mathcal{F}_\infty(\lambda, z) = \left(1 + \tfrac{1}{2}e^\lambda\right)\left(e^z - 1\right).$$

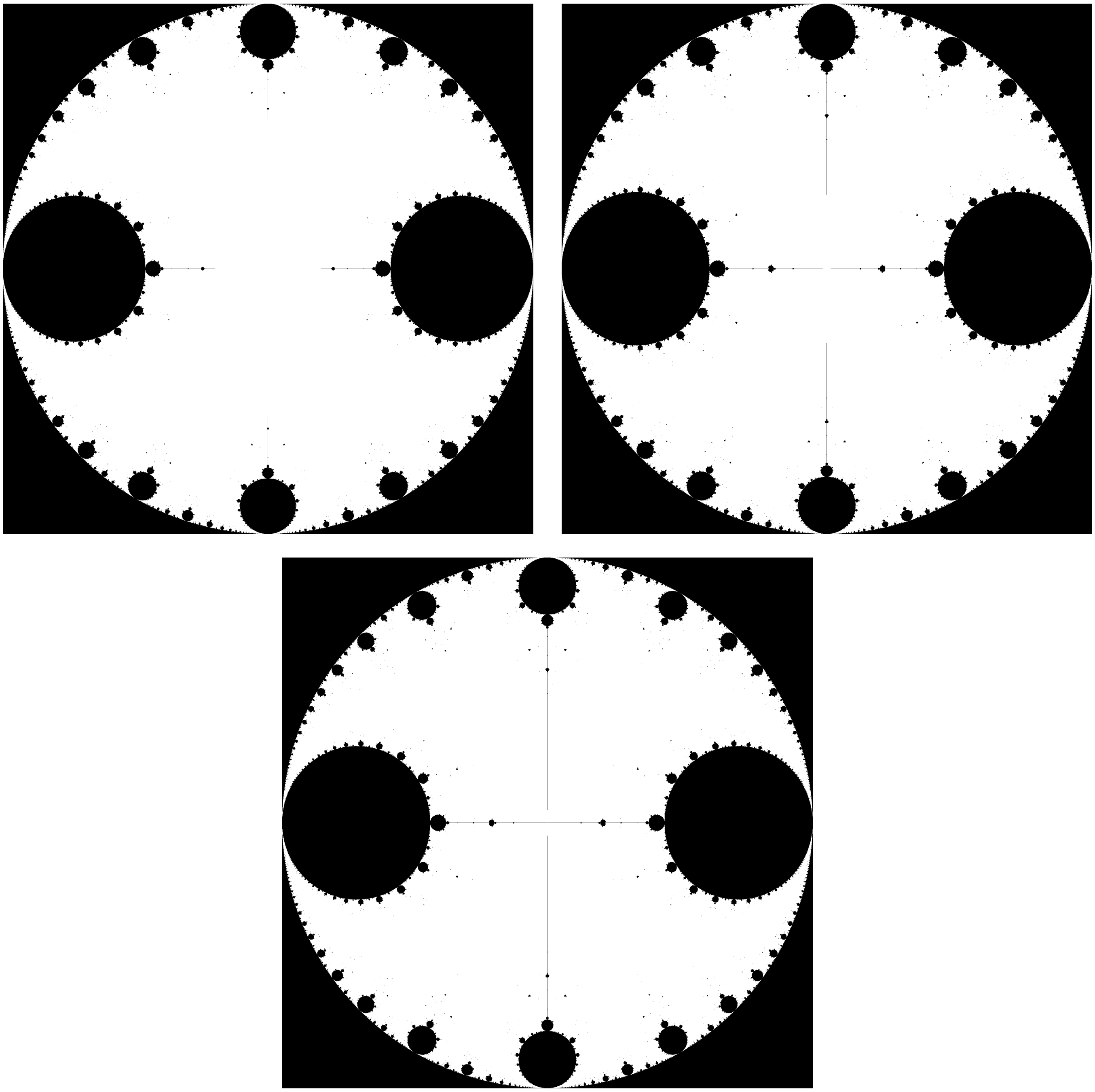

FIGURE 2. The hyperbolic components of $\mathcal{S}_1$, $\mathcal{S}_{16}$ and $\mathcal{S}_\infty$ from Example 5.2.

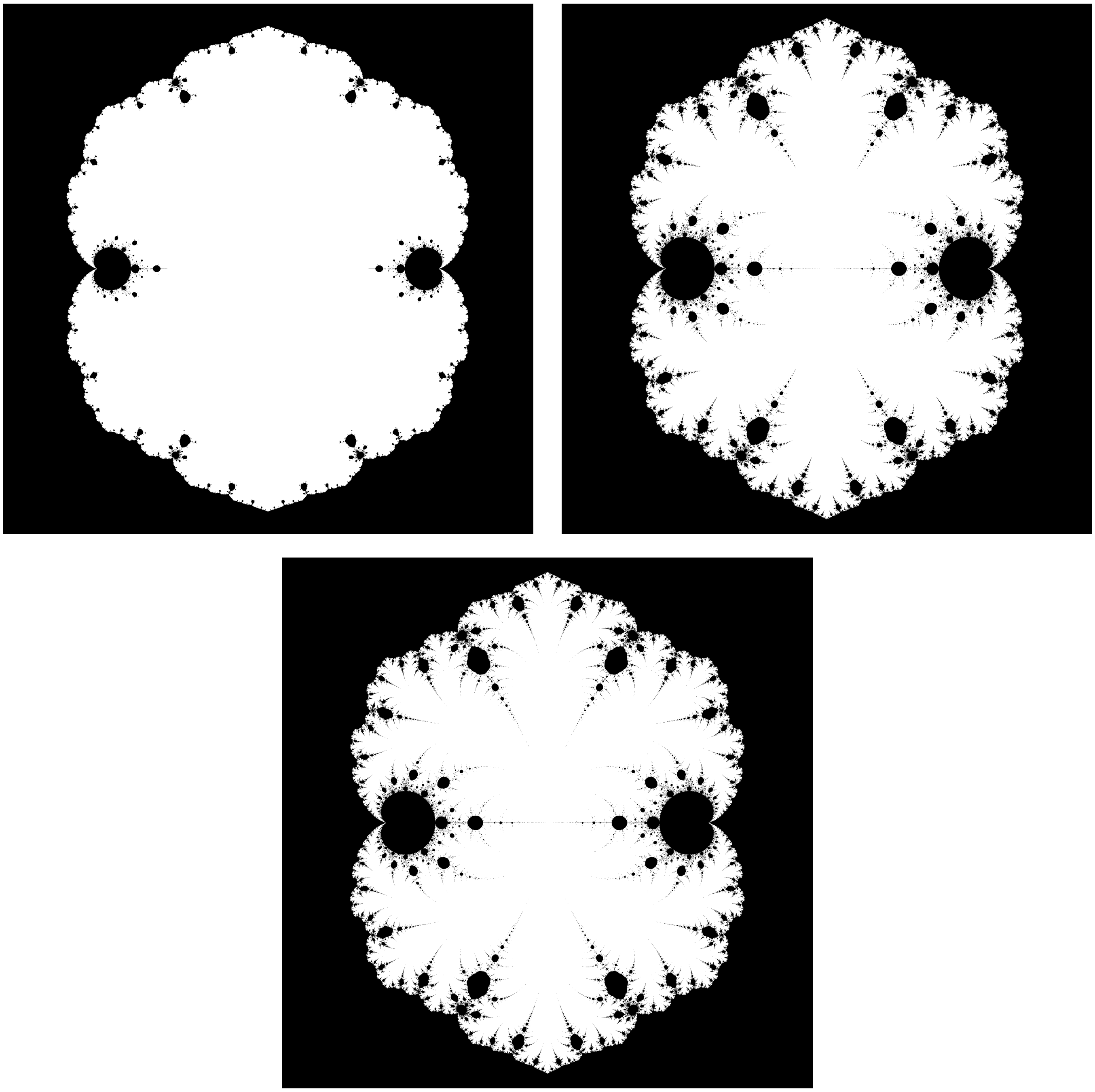

FIGURE 3. The hyperbolic components of $\mathcal{C}_1$, $\mathcal{C}_{16}$ and $\mathcal{C}_\infty$ from Example 5.2.