Does the Jones Polynomial Detect Unknottedness?

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CONTENTS

There have been many attempts to settle the question whether there exist nontrivial knots with trivial Jones polynomial. In this paper we show that such a knot must have crossing number at least 18. Furthermore we give the number of prime alternating knots and an upper bound for the number of prime knots up to 17 crossings. We also compute the number of different HOMFLY, Jones and Alexander polynomials for knots up to 15 crossings.

1. INTRODUCTION

In 1984 the Jones polynomial came into the world [Jones 1985]. Although this link invariant became an important tool for the proof of various theorems it is no magic potion for knot tabulators. There are many examples of inequivalent knots and links that have the same Jones polynomial. Even the extended versions of the Jones polynomial, such as the HOMFLY polynomial [Freyd et al. 1985] and the Kauffman polynomial [Kauffman 1987a], are only slightly better in distinguishing inequivalent knots and links.

Surprisingly, it is still unknown whether there are nontrivial knots with trivial Jones or related polynomials. For special classes of knots, such as alternating knots [Murasugi 1987], it is known that no such example can occur; see also [Lickorish and Thistlethwaite 1988; Birman 1985].

When we started our project we thought that a systematic enumeration (by crossing numbers) of nonalternating knots would lead to an example.

Now we can state:

Theorem. *Let K be a knot with trivial Jones or HOMFLY polynomial. Then K is the unknot or it has crossing number at least* 18.

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Section 2 summarizes the definition and some properties of the Jones and related polynomials. For details see, for example, [Jones 1987; Lickorish 1988; Kauffman 1987b].

An algorithm to enumerate all knots of a given crossing number is briefly described in Section 3. This algorithm was used in [Thistlethwaite 1985] to tabulate all prime knots up to 13 crossings. In Sections 4 and 6 we summarize our computational results. We did not try to classify knots with crossing number 14, 15, 16 or 17, but we can give lower and upper bounds for their numbers. The observation in Section 5 leads to a simple algorithm to decide whether a knot diagram with at most 17 crossings is a projection of the unknot.

Furthermore we can give the exact number of all (unoriented) prime alternating knots up to 17 crossings.

The reader is assumed to be familiar with the basic concepts of knot theory. For a good account see [Burde and Zieschang 1985].

2. THE JONES POLYNOMIAL

It is an open question whether all link classes are distinguishable by invariants like polynomials. One attempt was made by Jones in 1984. We choose a combinatorial way to define the Jones polynomial and the related HOMFLY polynomial. For an algebraic approach see [Jones 1987].

Let L_+, L_- and L_0 be (oriented) links with identical diagrams except near a crossing where they look like Figure 1.

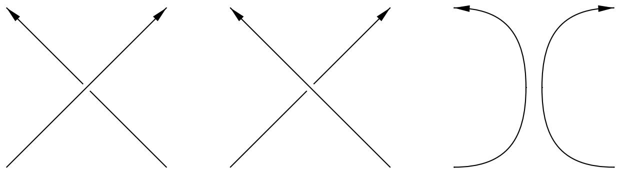

FIGURE 1. Skein relations

Let $\mathfrak{L}$ be the class of all oriented links up to equivalence. We have:

Proposition 2.1. *There is a function* (*often called the HOMFLY polynomial*)

$$P : \mathfrak{L} \to \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$$

uniquely and well-defined by $P(\text{unknot}) = 1$ *and*

$$v^{-1}\, P(L_+) - vP(L_-) - z\, P(L_0) = 0.$$

Using this polynomial, we may define the original Jones polynomial and the classical Alexander polynomial as a specialization:

Definition 2.2. *The Jones polynomial* $V(L)$ *is defined by*

$$V(L)(t) := P(L)(t,\, (t^{1/2} - t^{-1/2}))$$

and satisfies

$$t^{-1} V(L_+) - t\, V(L_-) + (t^{-1/2} - t^{1/2})\, V(L_0) = 0.$$

The Alexander polynomial $\Delta(L)$ *is defined by*

$$\Delta(L)(t) := P(L)(1,\, (t^{-1/2} - t^{1/2})).$$

We need the following property:

Let $L_1 + L_2$ be any connected sum, $L_1 \cup L_2$ the disjoint union of the oriented links L_1 and L_2, ρL the link obtained by reversing the orientation of all components of L and $\bar{L}$ the mirror image of L. $\bar{P}$ denotes the HOMFLY polynomial P with v and v^{-1} interchanged.

Then:

Proposition 2.3. (i) $P(L_1 + L_2) = P(L_1)\, P(L_2)$.
(ii) $P(L_1 \cup L_2) = (v^{-1} + v)\, z^{-1} P(L_1)\, P(L_2)$.
(iii) $P(\rho L) = P(L)$.
(iv) $\overline{P(L)} = P(\bar{L})$.

3. ENUMERATION OF KNOTS

A simple (but not simple to compute!) invariant of links is given by the *crossing number*, i.e., the minimum number of crossings of all diagrams of a link.

Now it is possible to enumerate all knots with at most a prescribed crossing number in the following way: Let D be a regular knot projection of the knot K with n crossings. After choosing a starting point and a direction on K we may label all points of K which project to the n crossings by $1, \dots, 2n$. So we get an involution τ (i.e., $\tau^2 = 1$) on the set $1, \dots, 2n$ by $\tau(i) := j$ if i and j are labeling the same crossing.

This involution is completely determined by the values on odd numbers (so $\tau(i)$ is even) and we get a sequence of n even numbers, which depends on the knot projection, the starting point and the direction. Now we indicate at each element of the sequence by a sign whether the corresponding crossing is an over- or an undercrossing. If we order all sequences of a given length (for example lexicographically), we may find to each knot projection D a unique *standard sequence* $s(D)$, which is minimal and independent of the starting point and the direction. (Notice that for a given sequence it is possible to find the standard sequence without constructing the knot or the knot diagram.)

Dowker and Thistlethwaite [1983] have shown that this sequence determines the knot diagram up to homeomorphism of the extended plane. In the same work they showed that it is possible to find algorithmically all sequences arising from knot projections and not from diagrams which are connected sums of two knot diagrams. Such sequences are called *admissible.*

Thus there exists an algorithm which produces all admissible standard sequences of a given length and therefore an enumeration of all prime knots.

4. DOES THE JONES POLYNOMIAL DETECT UNKNOTTEDNESS?

It is well known that the Alexander polynomial cannot decide whether a knot is really knotted or not. But for the HOMFLY polynomial or even for the Jones polynomial no example of a nontrivial knot with trivial polynomial is known. Anstee, Przytycki and Rolfsen [Anstee et al. 1989] have unsuccessfully tried to construct such an example by applying on diagrams of the unknot transformations which do not change the Jones polynomial, but possibly the equivalence class of the knot.

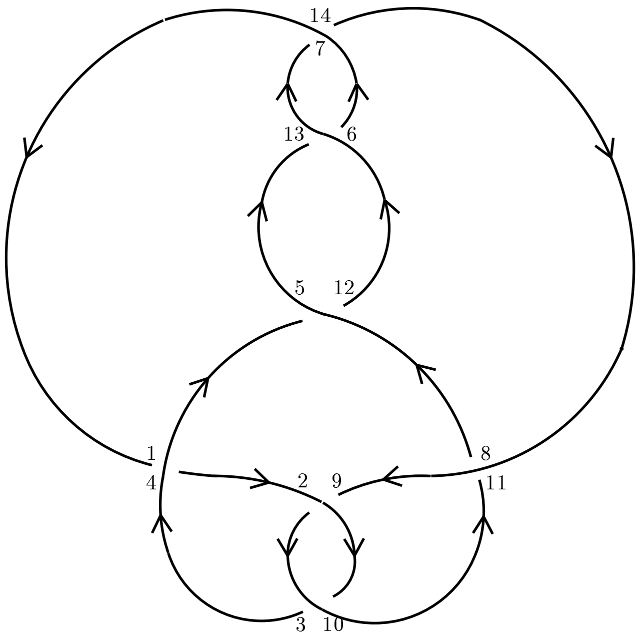

FIGURE 2. Knot with standard sequence 4 10 12 14 2 8 6.

We thought that an extensive computer search would lead to an example. Using the methods described in Section 3 we enumerated all admissible standard sequences of knot diagrams up to 17 crossings. We did not try to compile a list of all (prime) knots on 14, 15, 16 or 17 crossing in which every equivalence class is represented by just one knot. For knots on 12 and 13 crossings this work was done by Thistlethwaite [1985]. For knots up to eleven crossings see, for example, [Conway 1970].

For a given standard sequence we systematically applied all possible combinations of simple equivalence transformations called two-passes and flypes (Figure 3), which include Reidemeister moves of type II and III. If this procedure did not lead to a standard sequence that already occurred we computed the Jones polynomial by using the recursion 2.2. (For the computational complexity of the Jones polynomial see [Jaeger et al. 1990].)

Proposition 2.3 and the Jones polynomial definition ensure that we only have to regard Jones polynomials of knots with an admissible diagram to get the theorem stated in the introduction.

The flyping conjecture, proved in [Menasco and Thistlethwaite 1993], gives a method to classify all alternating knots. So as a by-product to our computations we are able to give the exact number of all (unoriented) prime alternating knots up to 17 crossings.

A word is in order on possible faults in the source code of our program. The Jones polynomial is an invariant of the equivalence class of a knot and by applying the transformations outlined above a knot stays in its class. So we can use the computation of the Jones polynomial for a verification of the transformations and vice versa.

After we did the computations for all knots up to 16 crossings a paper of another group was published [Arnold et al. 1994] in which they enumerate all (unoriented) prime alternating knots up to 14 crossings. They obtained the same numbers as we did. This gives further evidence for the correctness of our program.

To show the complexity of the problem: It took about a week on a modern workstation to compute the results for 16 crossings.

5. PROJECTIONS OF THE UNKNOT

Given a knot diagram it is natural to ask how to decide in an easy way whether it is a projection of the unknot. Ochiai [1990] has shown that for every n there is a diagram of the unknot with no n-waves. An *n-wave* is given if in the diagram there is an overpass (underpass) τ_1 with more than n crossings that may be replaced by another overpass (underpass) τ_2 connecting the two ends of τ_1 and having n crossings without changing the knot type. In this sense there are "nontrivial projections of the trivial knot" [Ochiai 1990].

As outlined above we have chosen another approach to find out whether a knot is knotted or not. By our computational results we have

Observation 5.1. Let D be a diagram of the unknot having at most 17 crossings. Then D may be transformed into the canonical diagram of the unknot by some flypes and two-passes (including Reidemeister moves of type II and III) and Reidemeister moves of type I without increasing the number of crossings.

6. ESTIMATING THE NUMBER OF KNOTS

Let $k(n)$ be the number of unoriented prime knot classes with crossing number n, and let $l(n)$ be the number of unoriented link classes with crossing number n. In these and following definitions, cheiral pairs (a knot and its mirror image) count as one knot.

Ernst and Sumners [1987] have shown that $k(n)$ grows exponentially with n. They give the lower bound

$$\liminf_{n\to\infty} k(n)^{1/n} \geq 2.68.$$

Welsh [1992] obtained an upper and lower bound for the growth of $l(n)$:

$$4 \leq \liminf_{n\to\infty} l(n)^{1/n} \leq \limsup_{n\to\infty} l(n)^{1/n} \leq \tfrac{27}{2}.$$

For low-crossing knots and links the lower bounds are far from optimal.

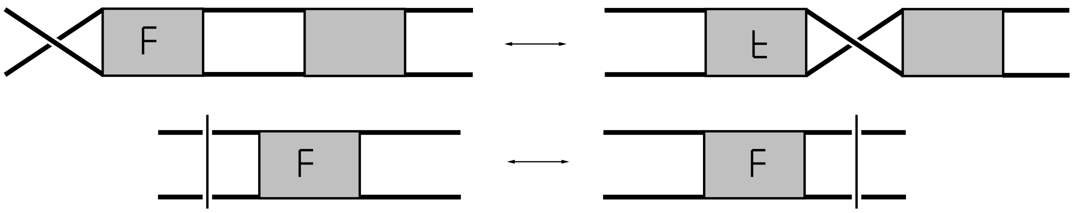

FIGURE 3. Top: a flype. Bottom: a two-pass.

n	$P(n)$	$V(n)$	$\Delta(n)$	$k(n)$	$a(n)$
3	1	1	1	1	1
4	1	1	1	1	1
5	2	2	2	2	2
6	3	3	3	3	3
7	7	7	7	7	7
8	21	21	21	21	18
9	49	49	44	49	41
10	160	151	132	165	123
11	509	452	339	552	367
12	1907	1596	1222	2176	1288
13	7935	6180	3866	9988	4878
14	35395	25074	14557	$\leq$ 50345	19536
15	178866	114409	56708	$\leq$ 279556	85263
16				$\leq$ 1608280	379799
17				$\leq$ 9821800	1769979

TABLE 1. For each crossing number n, $k(n)$ is the number of unoriented prime knots with n crossings, $a(n)$ is the number of unoriented prime alternating knots with n crossings, and $P(n)$ is the number of different HOMFLY polynomials that occur for knots with crossing number n but not for knots with smaller crossing number. $V(n)$ and $\Delta(n)$ are the analogous numbers for the Jones and Alexander polynomials. Cheiral pairs count as one knot.

Table 1 lists the value of $k(n)$ for $n \leq 13$, taken from [Thistlethwaite 1985], and an unambitious upper bound for $k(n)$ in the range $14 \leq n \leq 17$, which we got for free while proving our theorem. The table also gives the exact number $a(n)$ of (unoriented) prime alternating knots up to 17 crossings, and the number of new polynomials found for each value of n. A similar table for alternating knots may be found in [Dasbach and Hougardy 1993].

The numbers given shed light on the efficiency of the different polynomials.

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