Approximation of Invariant Surfaces by Periodic Orbits in High-Dimensional Maps: Some Rigorous Results

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CONTENTS

The existence of an invariant surface in high-dimensional systems greatly influences the behavior in a neighborhood of the invariant surface. We prove theorems that predict the behavior of periodic orbits in the vicinity of an invariant surface on which the motion is conjugate to a Diophantine rotation for symplectic maps and quasiperiodic perturbations of symplectic maps. Our results allow for efficient numerical algorithms that can serve as an indication for the breakdown of invariant surfaces.

1. INTRODUCTION

Periodic orbits have long served as tools to study the long-term behavior of dynamical systems (as in [Poincaré 1892], for example). In 1979, Greene proposed a numerical criterion, based on the behavior of periodic orbits, to determine the parameter values at which breakdown of certain invariant circles of twist maps of the annulus occurs. Greene's criterion (see [Greene 1979] for a precise formulation) is remarkably accurate and has provided valuable intuition that led to the formulation of a renormalization group theory for the breakdown of invariant circles for twist maps of the annulus [MacKay 1982].

The determination of the parameter values at which breakdown of invariant surfaces occurs has significant practical importance, as invariant surfaces present barriers to phase-space diffusion. One part of Greene's criterion, initially conjectured in [Greene 1979] and later proved in [MacKay 1992; Falcolini and Llave 1992], says that in twist maps of

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the annulus that admit an invariant circle with diophantine rotation number, a certain limit—taken along periodic orbits in the neighborhood of the invariant circle and based on their stability properties—is equal to zero. Moreover, if the invariant circle is analytic, the limit is reached exponentially fast. Such behavior can be, and has been, efficiently investigated numerically.

We present a similar result in higher dimensions for certain symplectic maps and quasiperiodic perturbations of symplectic maps, satisfying nondegeneracy assumptions. If an invariant surface Γ exists and is analytic, or sufficiently differentiable, and motion on Γ is conjugate to rigid rotation with a diophantine rotation vector, we show that all the eigenvalues of the derivative of the map along periodic orbits in a neighborhood of Γ tend to 1 (exponentially, if the invariant surface is analytic) as the periodic orbit approaches Γ. A precise statement is given in Section 2.

Our results are of a local nature and involve only a neighborhood of the invariant surface. Existence of an invariant surface imposes severe restrictions for the map in a neighborhood of the surface. Indeed, we show that in an appropriate neighborhood of the invariant surface the map is close to integrable, and using a perturbative argument one can control the behavior of periodic orbits. In this setting the distance from the invariant surface plays the role of a small parameter, and one can deduce that periodic orbits with rotation vectors close to the rotation vector of the invariant surface exist close to the surface. In [Perry and Wiggins 1994] similar ideas were used to deduce long-term stability for orbits that come close to an invariant surface.

2. NOTATION AND STATEMENT OF RESULTS

We will study two distinct cases: symplectic maps and quasiperiodic perturbations of such maps (that is, skew-products of symplectic maps and quasiperiodic rotation—a particular case of volume-preserving maps).

In the first case we consider maps f, either C^r or analytic, from the space $\mathbb{T}^d \times \mathbb{R}^d$ to itself, such that

(i) f preserves the natural symplectic two-form $\omega = \sum_{i=1}^{d} d\varphi_i \wedge dA_i$, and
(ii) $\partial\varphi'/\partial A$ is a nonsingular $d \times d$ matrix,

where φ' the first coordinate of $\tilde{f}(\varphi, A)$, for $\tilde{f}$ a lift of f. We will call such an f a *$2d$-dimensional nonsingular symplectic map*. C^r maps of this type for $d = 1$ are called (positive or negative) *twist maps of the annulus*.

In the second case we consider maps $f : \mathbb{T}^{d+e} \times \mathbb{R}^d \to \mathbb{T}^{d+e} \times \mathbb{R}^d$ that are periodic or quasiperiodic skew-products on $\mathbb{T}^e$ where $f|_{\mathbb{T}^d \times \mathbb{R}^d} : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ is a $2d$-dimensional nonsingular symplectic map.

Let $c = d$ in the case of the symplectic maps and $c = d+e$ in the case of quasiperiodic perturbations of symplectic maps. We say that $\boldsymbol{x}$ is a *periodic orbit of type* (P/N), for $P \in \mathbb{Z}^c$ and N a positive integer, if $f^N(\boldsymbol{x}) = \boldsymbol{x}$ and $\tilde{f}^N(\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{x}} + (P, 0)$, where $\tilde{f}, \tilde{\boldsymbol{x}}$ are (fixed) lifts of $f, \boldsymbol{x}$ to the universal cover of $\mathbb{T}^c \times \mathbb{R}^d$. We will call N the period of the orbit. Notice that only periodic skew-products can have periodic orbits. For c-vectors we will use the norm $\|\omega\|_c = \sum_{i=1}^{c} |\omega_i|$.

We define the *rotation vector* of an orbit of $\tilde{f}$ as the c-dimensional vector

$$\omega = \lim_{i\to\infty} \frac{\pi_1(\tilde{f}^i(x, y)) - x}{i}$$

if the limit exists, where π_1 the projection on the first c (angle) coordinates: $\pi_1(x, y) = x$. For a periodic orbit of type (P/N) the rotation vector is $\omega = P/N$.

We will consider sets with rotation vectors that are not well approximated by rational vectors. We define a c-dimensional vector to be (diophantine) of type (K, τ) if

$$|P\cdot\omega| \geq \frac{K}{\|P\|_c^\tau} \quad \text{for } P \in \mathbb{Z}^c,\ P \neq 0,\ K > 0. \tag{2.1}$$

It is known [Arnol'd 1988] that if $\tau > c-1$ the set of vectors of type (K,τ) has positive Lebesgue measure in the unit c-dimensional cube.

We now state our results for periodic orbits that approach invariant sets of f.

Theorem 2.1. *Let f be a $2d$-dimensional nonsingular symplectic map of class C^r, where $r > 1$. Suppose f admits a C^r invariant surface Γ, homotopic to $\mathbb{T}^d \times \{0\}$, on which the motion is C^r conjugate to rigid rotation with rotation vector ω of type (K,τ). Moreover assume that in a neighborhood of Γ there are periodic orbits $x_{(P/N)}$ of type (P/N) for $\|N\omega - P\|_d$ small enough. Then, for any nonnegative integer $k < (r-1)/\tau$, we can find $D_k > 0$ such that the eigenvalues $\lambda_1, \dots, \lambda_{2d}$ of the derivative $Df^N(x_{(P/N)})$ satisfy*

$$|\lambda_i - 1| \le D_k \|N\omega - P\|_d^{k/2} N \quad \textit{for } i = 1, \dots, 2d.$$

In the case where the map f and the invariant surface are analytic in a polystrip I_δ around the invariant surface Γ and analytically conjugate to rigid rotation, we can compute the coefficients D_k and choose the k that gives the best bound.

Theorem 2.2. *Let $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ be an analytic $2d$-dimensional nonsingular symplectic map. Suppose f admits an analytic invariant surface Γ, homotopic to $\mathbb{T}^d \times \{0\}$, on which the motion is analytically conjugate, with conjugacy γ, to rigid rotation with rotation vector ω of type (K,τ). Moreover, assume that in a neighborhood of Γ there are periodic orbits $x_{(P/N)}$ of type (P/N) for $\|N\omega - P\|_d$ small enough. If f and γ are bounded in a neighborhood of the invariant surface, the eigenvalues $\lambda_1, \dots, \lambda_{2d}$ of the derivative $Df^N(x_{(P/N)})$ satisfy*

$$|\lambda_i - 1| \le \tilde{D}_1 N \exp\bigl(-\tilde{D}_2 \|N\omega - P\|_d^{-1/(2(1+\tau))}\bigr),$$

where $\tilde{D}_1, \tilde{D}_2$ depend on the width of the domain of analyticity of f and γ, on the properties of ω (i.e., K and τ), and on the dimension d.

In the case $d = 1$, the behavior of the eigenvalues is completely determined by the trace of the derivative along the periodic orbit. In analogy with that case, we define the *residue* of a periodic orbit with period N as

$$R(\boldsymbol{x}) = \frac{1}{4d}\left(2d - \operatorname{Tr}(Df^N(\boldsymbol{x}))\right). \tag{2.2}$$

Our definition is an extension of the one used in [Greene 1979] for two-dimensional twist maps of the annulus. The factor $(4d)^{-1}$ ensures that the residue of elliptic periodic orbits (orbits for which the eigenvalues of $D\tilde{f}^N$ lie on the unit circle) is between zero and one.

Greene formulated a criterion for the breakdown of invariant curves of twist maps based on the behavior of the residue of periodic orbits [Greene 1979]. As indicated by Theorem 2.2, an analog of the criterion in higher dimensions should consider the behavior of *additional* quantities, other than the residue, such as the *eigenvalues* of Df^N along periodic orbits.

Notice that, due to invariance under cyclic permutations, the residue of a periodic orbit is the same for all the points of the orbit. Also, since the definition only involves derivatives, the residue is invariant under C^1 changes of variables. For integrable maps—that is, maps conjugate to $\tilde{g}(x,y) = (x + h(y), y)$, for $h : \mathbb{R}^d \to \mathbb{R}^d$—the residue of all periodic orbits is zero. From Theorems 2.1 and 2.2 we have the following corollary:

Corollary 2.3. *Let f be a $2d$-dimensional nonsingular symplectic map of class C^r, where $r > 1$. Suppose f admits a C^r invariant surface Γ, homotopic to $\mathbb{T}^d \times \{0\}$, on which the motion is C^r conjugate to rigid rotation with rotation vector ω of type (K,τ). Moreover assume that in a neighborhood of Γ there are periodic orbits $x_{(P/N)}$ of type (P/N) for $\|N\omega - P\|_d$ small enough. Then, for any nonnegative integer $k < (r-1)/\tau$, we can find $C_k > 0$ such that*

$$|R(x)| \le C_k \|N\omega - P\|_d^{k/2} N.$$

If f, Γ, and the conjugacy to rigid rotation are analytic, we can find $\tilde{C}_1, \tilde{C}_2 > 0$ such that

$$|R(x)| \le \tilde{C}_1 N \exp\bigl(-\tilde{C}_2 \|N\omega - P\|_d^{-1/(2(1+\tau))}\bigr).$$

Remark. In the case $d = 1$ the continued-fraction convergents to ω provide integer sequences $\{M_i\}_{i=0}^{\infty}$ and $\{N_i\}_{i=0}^{\infty}$ such that

$$|\omega - M_i/N_i| \le K N_i^{-2} \tag{2.3}$$

for all i and ω. In that case it is possible to show that if an analytic invariant curve exists then

$$\lim_{i\to\infty} \sup |R(x_i)|^{1/N_i} \le 1,$$

where the limit is taken along continued fraction convergents. Unfortunately, in higher dimensions, there is not, to our knowledge, an efficient approximation scheme that can produce convergents to an arbitrary rotation vector with d components that satisfy an inequality similar to (2.3). Such schemes do exist for certain classes of rotation vectors, such as golden vectors of the Jacobi–Perron algorithm for $d = 2$ [Kosygin 1991].

Remark. Theorems 2.1 and 2.2 are local results that apply in a neighborhood of the invariant surface. Thus, assumptions (i) and (ii) at the beginning of this section can be relaxed to hold only in a neighborhood of the invariant surface Γ.

We turn to the case of volume-preserving maps that are quasiperiodic skew-products of symplectic maps over $\mathbb{T}^e$, that is, maps of the form

$$f(\theta,\varphi,A) = \bigl(f_1(\theta,\varphi,A),\ \varphi+\omega_2,\ f_2(\theta,\varphi,A)\bigr)$$

for $f_1 : \mathbb{T}^{d+e}\times\mathbb{R}^d \to \mathbb{T}^d$, $f_2 : \mathbb{T}^{d+e}\times\mathbb{R}^d \to \mathbb{R}^d$, $\theta\in\mathbb{T}^d$, $\varphi\in\mathbb{T}^e$, and $\omega_2\in\mathbb{T}^e$ an irrational vector. We introduce the extension $f^* : \mathbb{T}^{d+e}\times\mathbb{R}^{d+e} \to \mathbb{T}^{d+e}\times\mathbb{R}^{d+e}$ by

$$\begin{aligned} f^*(\theta,\varphi,A_1,A_2) \\ &= \bigl(f_1(\theta,\varphi,A_1),\ \varphi+A_2,\ f_2(\theta,\varphi,A_1),\ A_2\bigr),\end{aligned}$$

which at $A_2=\omega_2$ reduces to f. If f admits an invariant surface Γ then f^* admits an invariant surface Γ^* at $A_2=\omega_2$. Moreover, for $\omega\in\mathbb{T}^e$, we introduce the restriction $f^*_\omega : \mathbb{T}^{d+e}\times\mathbb{R}^d \to \mathbb{T}^{d+e}\times\mathbb{R}^d$ by

$$f^*_\omega(\theta,\varphi,A) = f^*(\theta,\varphi,A,\omega).$$

If f^* admits a periodic orbit x of type $((P_1,P_2)/N)$ then $f^*_{P_2/N}$ admits a periodic orbit $\bar{x}$ of the same type.

Theorem 2.4. *Let $f : \mathbb{T}^{d+e}\times\mathbb{R}^d \to \mathbb{T}^{d+e}\times\mathbb{R}^d$ be a quasiperiodic skew-product of a $2d$-dimensional nonsingular symplectic map over $\mathbb{T}^e$ such that $f|_{\mathbb{T}^e}$ is rigid rotation with a diophantine rotation vector. Assume that f is of class C^r, where $r>1$, and that it admits a C^r invariant surface Γ, homotopic to $\mathbb{T}^{d+e}\times\{0\}$, on which the motion is C^r conjugate to rigid rotation with rotation vector ω of type (K,τ). Moreover, assume that in the extension f^* of f there is a neighborhood of Γ^* where there are periodic orbits $x_{(P/N)}$ $(P\equiv(P_1,P_2)\in\mathbb{Z}^{d+e})$ of type (P/N) for $\|N\omega-P\|_{d+e}$ small enough. Then, for any nonnegative integer $k<(r-1)/\tau$, we can find $D_k>0$ such that $2d$ of the eigenvalues $\lambda_1,\dots,\lambda_{2d}$ of the derivative $D\bigl((f^*_{P_2/N})^N\bigr)(\bar{x}_{(P/N)})$ satisfy*

$$|\lambda_i-1| \le D_k\|N\omega-P\|_{d+e}^{k/2}N \quad \textit{for } i=1,\dots,2d,$$

the remaining e eigenvalues being identically 1.

If f, Γ, and the conjugacy to rigid rotation are analytic, we can find $\tilde{C}_1,\tilde{C}_2>0$ such that

$$|\lambda_i-1| \le \tilde{D}_1 N \exp\bigl(-\tilde{D}_2\|N\omega-P\|_{d+e}^{-1/(2(1+\tau))}\bigr).$$

Our results cover the case where f admits an invariant surface on which motion is conjugate to rotation. In [Falcolini and Llave 1992] it was shown that if f admits an invariant set on which motion is semi-conjugate to rotation, there are periodic orbits approaching the invariant set under certain conditions on the Lyapunov exponents of f on the invariant set. We quote these results for completeness:

Theorem 2.5 [Falcolini and Llave 1992, Theorem 2.3]. *Assume Γ is a hyperbolic set of rotation vector ω and that $\{x_n\}$ is a sequence of periodic points of type (M_n/N_n) such that the orbit of x_n converges to Γ. Then, for sufficiently large n, $|R(x_n)|^{1/N_n} > \lambda > 1$. Actually, if the hyperbolic set has maximum Lyapunov exponent γ, then $\lim_n R(x_n)^{1/N_n} = e^\gamma$.*

Theorem 2.6 [Falcolini and Llave 1992, Theorem 4.3]. *Let $f : M \to M$ be a C^2 diffeomorphism leaving invariant the ergodic measure μ. Assume that, with respect to this measure, f has no zero Lyapunov exponents. Then, for almost every point x_0 in the support of μ, it is possible to find a sequence $\{x_n\}_{n=0}^{\infty}$ of periodic points that converges to x_0. Moreover, the sequence of orbits can be chosen in such a way that the Lyapunov exponents of x_n converge to the Lyapunov exponents of x_0.*

3. PROOF OF THE RESULTS

The C^r Case for Symplectic Maps

We turn to the proof of Theorem 2.1, which consists of three parts. First, in Proposition 3.2, we construct a normal form in the neighborhood of the invariant surface and approximate the map in that neighborhood with an integrable mapping. The distance between our map and the integrable map can be made $O(\|H\|_d^k)$, where H are the actions in an appropriate coordinate system, for k depending on the smoothness of the invariant surface and the type of the rotation vector.

In the second part of the proof we show that in a small enough neighborhood of the invariant surface the rotation vector of periodic orbits that stay in the neighborhood cannot differ from the rotation vector of the invariant surface by more than the size of the neighborhood.

Finally, the last part is a perturbation argument that allows us to estimate the eigenvalues of the derivative along periodic orbits that stay close to the invariant surface (Lemma 3.3).

Before introducing the normal form we make a change of variables that makes it more convenient to study a neighborhood of the invariant surface.

Proposition 3.1. *Let f be a $2d$-dimensional nonsingular symplectic map of class C^r, admitting a C^r invariant surface Γ (which is a graph of a C^r function $\gamma : \mathbb{T}^d \to \mathbb{R}^d$) with $f|_\Gamma$ C^r conjugate to rigid rotation with rotation vector ω. Then we can find a symplectic C^{r-1} mapping h defined in a neighborhood of Γ and having a C^{r-1} inverse in a neighborhood of Γ, and C^{r-1} functions $v : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d$ and $u : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^d$, such that*

$$\begin{aligned} h \circ f \circ h^{-1}(\varphi, A) \\ = (\varphi + \omega + Av(\varphi, A),\ A + A^2 u(\varphi, A)), \end{aligned} \quad (3.1)$$

where A^2 implies all quadratic combinations of the various A's.

Proof. We first shift the action coordinates so that $(\varphi, 0)$ becomes the invariant surface, then we use the conjugacy to rigid rotation to deduce (3.1).

Define $h_1 : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d \times \mathbb{R}^d$ by

$$h_1(\varphi, A) = (\varphi,\ A + \gamma(\varphi)).$$

Then h_1 is of class C^r, symplectic and sends $\mathbb{T}^d \times \{0\}^d$ to the graph of γ. Thus $h_1 \circ f \circ h_1^{-1}$ leaves the surface $\mathbb{T}^d \times \{0\}^d$ invariant; in other words, there exist C^r functions $v_1 : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d$ and $u_1 : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$h_1 \circ f \circ h_1^{-1}(\varphi, A) = (v_1(\varphi, A),\ Au_1(\varphi, A)).$$

Since the motion on the surface is C^r conjugate to rigid rotation, there is a C^r function $\delta : \mathbb{T}^d \to \mathbb{T}^d$ with a C^r inverse such that $v_1(\delta(\varphi), 0) = \delta(\varphi + \omega)$. In particular, $(D\delta)^{-1}$ exists.

We introduce (for $r > 1$) the C^{r-1} symplectic transformation

$$h_2(\varphi, A) = (\delta(\varphi), (D\delta)^{-1} A)$$

with

$$\begin{aligned} h_2^{-1} \circ h_1 \circ f \circ h_1^{-1} \circ h_2(\varphi, A) \\ = (\varphi + \omega + Av_2(\varphi, A), Au_2(\varphi, A)), \end{aligned} \quad (3.2)$$

where $v_2 : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d$ and $u_2 : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are C^{r-1} functions with

$$\begin{aligned} v_2(\varphi, A) &= A^{-1}\big(\delta^{-1}(v_1(\delta(\varphi),\ (D\delta)^{-1}A)) \\ &\qquad - \delta^{-1}(v_1(\delta(\varphi), 0))\big), \\ u_2(\varphi, A) &= u_1\big(\delta(\varphi), (D\delta)^{-1}A\big). \end{aligned}$$

For $A = 0$, we have $\partial A_i'/\partial A_j = \partial\varphi_i'/\partial\varphi_j = 0$ for $i \neq j$, $\partial\varphi_i'/\partial\varphi_i = 1$ for all i, and $\partial A_i'/\partial\varphi_j = 0$ for all i, j. Since the map is symplectic, we conclude that

$$\frac{\partial A'_i}{\partial A_i}\bigg|_{A=0} = 1$$

for all i. Moreover, from condition (ii) in the definition of a nonsingular symplectic map, we conclude that $\partial\varphi'_i/\partial A_i \neq 0$ or $v_2(\varphi, 0) \neq 0$. This concludes the proof of Proposition 3.1. □

Remark. In the case $d = 1$, Birkhoff's theorem guarantees that an invariant curve of a nonsingular symplectic map with irrational rotation number is a graph. Birkhoff's theorem fails when condition (ii) (the twist condition) is violated. Also for higher dimensions we are not aware of an analog of Birkhoff's theorem. (For $d = 2$ there is an analog of Birkhoff's theorem [Mather 1991] for the class of maps that can be expressed as a finite number of compositions of one-dimensional twist maps). In the general case, the condition that the invariant curve is a graph over $\mathbb{T}^d$ can be replaced by a more local condition (weaker in the case of the maps we have been studying and also applicable to singular symplectic maps, that is, maps with zero torsion). If Γ is homotopic to $\mathbb{T}^d$ there are coordinates in a neighborhood of Γ for which the invariant surface reduces to a graph. Then condition (ii) need only be satisfied in a neighborhood of the invariant surface in the transformed coordinates (3.2) (that is, we need only have $v_2(\varphi, 0) \neq 0$) for the conclusions of Theorem 2.1 to be valid.

We introduce some further notation. We use multi-index notation: $\{m\}$ will denote all possible combinations of indices $1_{j_1}, \dots, d_{j_d}$ such that $\sum_{l=1}^d l_{j_l} = m$. Moreover, the expression $A^{\{m\}}$ will mean all possible combinations of the different A's raised to all possible indices allowed from the condition $\sum_{l=1}^d l_{j_l} = m$. Also, a symbol $Q_{\{m\}}$ "multiplying" $A^{\{m\}}$ will denote a multitude of functions, one for each combination of the A's allowed (for example, $Q_{\{1\}}$ corresponds to d functions, $Q_{\{2\}}$ corresponds to $d(d+1)/2$ functions, etc.)

We can now construct a normal form for f in a neighborhood of the invariant surface. We first construct d independent approximate integrals in a small neighborhood of the invariant surface.

Lemma 3.2. *Let f be a C^r map as before, and let ω be a rotation vector of type (K, τ). Then, given any nonnegative integer $k < (r-1)/\tau$, we can find functions $H_{\{0\}}, H_{\{1\}}, \dots, H_{\{k\}} : \mathbb{T}^d \to \mathbb{R}^d$ and constants C_k such that the map $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^d$ defined by*

$$H = \sum_{m=0}^{k} A^{\{m\}} H_{\{m\}}(\varphi)$$

satisfies

$$\|H \circ f - H\| \leq C_{k+1} \|A\|_d^{k+1}.$$

Proof. Expanding in A we have

$$\begin{aligned}
H \circ f(\varphi, A) &= \sum_m (A + A^{\{2\}} u(\varphi, A))^{\{m\}} H_{\{m\}}(\varphi + \omega + A v(\varphi, A)) \\
&= \sum_m (A + A^{\{2\}} u(\varphi, A))^{\{m\}} \\
&\qquad \times \left(\sum_{l=0}^{k} c_{\{l\}} \frac{\partial H_{\{m\}}}{\partial A^{\{l\}}} (\varphi + \omega + A v(\varphi, A))|_{A=0} \right. \\
&\qquad\qquad \left. + O(A^{\{m+k+1\}}) \right) \\
&= \sum_{m=0}^{k} A^{\{m\}} \Big(H_{\{m\}}(\varphi + \omega) + H_{\{m-1\}}(\varphi + \omega) u(\varphi, 0) \\
&\qquad\qquad + L_{\{m\}}(\varphi) \Big) + O(A^{\{k+1\}}),
\end{aligned}$$

where $c_{\{l\}}$ are the coefficients of the Taylor expansion and $L_{\{m\}}$ depends on $H_{\{0\}}, H_{\{1\}}, \dots, H_{\{m-2\}}$ and their derivatives up to order m, as well as on the derivatives of $H_{\{m-1\}}$. Notice that changes in $H_{\{m-1\}}$ by a constant do not affect $L_{\{m\}}$.

Matching terms by order we have

$$\begin{aligned}
H_{\{0\}}(\varphi) &= H_{\{0\}}(\varphi + \omega), \\
H_{\{m\}}(\varphi) &= H_{\{m\}}(\varphi + \omega) + H_{\{m-1\}}(\varphi + \omega) u(\varphi, 0) \\
&\qquad + L_{\{m\}}(\varphi). \qquad (3.3)
\end{aligned}$$

Equations (3.3) are of the form

$$g(\varphi + \omega) - g(\varphi) = f(\varphi). \qquad (3.4)$$

It is well known [Siegel and Moser 1971; Arnol'd 1988] that for the case of ω of type (K,τ), given $f \in C^q$ with zero average over the d-torus, there exists $g \in C^{q-(\tau+\varepsilon)}$ that satisfies (3.4) for every $\varepsilon > 0$, $q > \tau$.

For $m = 0$, the only possible continuous solution is $H_{\{0\}} =$ constant: indeed, from the condition $\int_{\mathbb{T}^d} L_{\{1\}}\, d\varphi = 0$, if $\int_{\mathbb{T}^d} u(\varphi,0)\, d\varphi \neq 0$ we get $H_{\{0\}} = 0$.

Now consider the case $m > 0$. If $H_{\{0\}}, H_{\{1\}}, \dots, H_{\{m-2\}}$ are uniquely determined and $H_{\{m-1\}}$ is determined up to a constant, then $L_{\{m\}}$ is completely determined. Moreover, $H_{\{m\}}$ can be determined up to a constant if and only if

$$\int_{\mathbb{T}^d} \big(L_{\{m\}}(\varphi) + u(\varphi,0)H_{\{m-1\}}(\varphi+\omega)\big)\, d\varphi = 0, \tag{3.5}$$

which uniquely determines the average value of $H_{\{m-1\}}$ when $\int_{\mathbb{T}^d} u(\varphi,0) \neq 0$. When $\int_{\mathbb{T}^d} u(\varphi,0) = 0$ we can show that the choice $\int_{\mathbb{T}^d} H_{\{m\}}(\varphi) = 0$, $m \geq 0$ satisfies (3.5). To this end, consider the truncation $H^{[\leq m-1]} = \sum_{l=0}^{m-1} A^{\{l\}} H_{\{l\}}(\varphi)$, satisfying (3.3) up to order $m-1$. Then

$$\int_{\mathbb{T}^d} \{H^{[\leq m-1]}(\varphi,A) - H^{[\leq m-1]} \circ f(\varphi,A)\}\, d\varphi = 0,$$

since f, being symplectic, preserves volume in phase space. We have

$$\begin{aligned} H^{[\leq m-1]}&(\varphi,A) - H^{[\leq m-1]} \circ f(\varphi,A) \\ &= A^{\{m\}}\left(L_{\{m\}}(\varphi) + u(\varphi,0)H_{\{m-1\}}(\varphi+\omega)\right) \\ &\qquad\qquad + O(A^{\{m+1\}}); \end{aligned}$$

thus, condition (3.5) is satisfied.

The process can be carried out inductively as long as $L_{\{k\}}$ is smooth enough (at least of class $C^{\tau+\varepsilon}$). Since in every step of the induction the smoothness of $L_{\{k\}}$ decreases by τ, we have the bound $k\tau > r-1$, or $k < (r-1)/\tau$. If f is C^∞ or analytic the induction can be carried out for all $k \in \mathbb{N}$. $\square$

We have constructed d functions H that are approximate integrals in the vicinity of the invariant surface. Since $H_{\{0\}} = 0$, H is a small perturbation of A and the surface $H = h$, for $\|h\|_d$ small, is topologically nontrivial.

The function $\bar{H}$ defined by

$$\bar{H}(h) = \int_{H=h} A\, d\varphi$$

is conserved under f up to $O(\|A\|_d^{k+1})$ in a neighborhood of $A = 0$.

We change coordinates, in such a way that $\bar{H}$ replaces A, using a generating function S

$$S(\Phi,A) = \left(A + \int_{\mathbb{T}^d} \sum_{m=2}^{k} A^{\{m\}} H_{\{m\}}(s)\, ds\right)\Phi. \tag{3.6}$$

The function S generates the symplectic transformation

$$\begin{aligned} \bar{H} &= D_1 S(\Phi,A) = A + \int_{\mathbb{T}^d} \sum_{m=2}^{k} A^{\{m\}} H_{\{m\}}(s)\, ds, \\ \varphi &= D_2 S(\Phi,A) \\ &= \Phi\left(1 + \frac{\partial}{\partial A}\int_{\mathbb{T}^d} \sum_{m=2}^{k} A^{\{m\}} H_{\{m\}}(s)\, ds\right). \end{aligned}$$

In the new coordinates,

$$f(\Phi,\bar{H}) = (\Phi + \omega + \bar{H}\Delta(\bar{H}),\ \bar{H}) + E(\Phi,\bar{H}), \tag{3.7}$$

where the remainder satisfies $\|E\| \leq C_k \|\bar{H}\|_d^{k+1}$ (in appropriate norms) and $\Delta(0) \neq 0$.

Remark. Another way to construct the normal form would be to perform successive canonical transformations (for example using the method of Lie transforms) and reduce f to an integrable map, up to $O(A^{\{k+1\}})$, in a neighborhood of the invariant surface. This method was used in the case $d = 1$ in [MacKay 1992], whereas the method of constructing an approximate integral was used in [Falcolini and Llave 1992]. We favor the latter, since it lends itself to efficient numerical implementations.

When the map f is analytic, our estimates hold in a complex neighborhood of $\mathbb{T}^d \times \{0\}^d$ of the form $\big\{|\operatorname{Im}\Phi_i| < \xi,\ |\bar{H}_i| \leq \xi,\ \text{for } i = 1,\dots,d\big\}$ for some $\xi > 0$.

In the new $(\Phi, \bar{H})$-coordinates, we have $\|DE\| \leq C_k \|\bar{H}\|_d^k$ and

$$Df(\Phi, \bar{H}) = \begin{pmatrix} 1 & F(\bar{H}) \\ 0 & 1 \end{pmatrix} + O(\|\bar{H}\|_d^k), \tag{3.8}$$

where $F(\bar{H}) = \Delta(\bar{H}) + \bar{H}\Delta'(\bar{H})$.

In a neighborhood of the invariant surface only periodic orbits with rotation vectors close to the rotation vector of the invariant surface are allowed. Since $F(0) \neq 0$, we conclude using the implicit function theorem that the actions $\bar{H}_{\text{per}}$ of a periodic orbit of period N in the vicinity of the invariant surface are bounded by

$$C_1 \|N\omega - P\|_d \leq \|\bar{H}_{\text{per}}\|_d \leq C_2 \|N\omega - P\|_d.$$

The existence of periodic orbits for maps that are close to integrable (such as map (3.7) in a neighborhood of the invariant surface) has been studied in the case where f has a generating function [Bernstein and Katok 1987; Llave and Wayne 1989]. It was shown that some periodic orbits of the integrable system persist for small enough perturbations, and that their distance from the original periodic orbits can be bounded by the size of the perturbation. Although in [Llave and Wayne 1989] only Hamiltonian flows were considered (which correspond to maps with a generating function) the methods used could be easily extended to periodic orbits of symplectic maps that do not have a generating function.

The last part of the proof of Theorem 2.1 consists of a simple perturbative argument. Since we are interested in the eigenvalues of the derivative along periodic orbits, we will estimate the norm of products of matrices close to the ones appearing in (3.8).

Lemma 3.3. *Let $\{A_i\}_{i=1}^N$ be a set of $2d \times 2d$ matrices of the form*

$$A_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix},$$

with

$$\max\bigl(1, \sup_{1 \leq i \leq N} (\sup_{1 \leq l,k \leq d} |(a_i)_{lk}|)\bigr) \leq A,$$

and let $\{B_i\}_{i=1}^N$ satisfy

$$\sup_{1 \leq i \leq N,\, 1 \leq j,k \leq 2d} |(B_i)_{jk} - (A_i)_{jk}| \leq \varepsilon$$

with $\varepsilon < A$. Then all the eigenvalues $\lambda_1, \dots, \lambda_{2d}$ of $B \equiv B_1 \dots B_N$ satisfy

$$|1 - \lambda_i| \leq 2\bigl((1 + 3d\sqrt{A}\sqrt{\varepsilon})^N - 1\bigr).$$

Proof. We introduce norms for vectors and matrices. For a vector in $\mathbb{R}^{2d}$ we define

$$\|v\|_\delta = \sum_{i=1}^d (|v_i|\delta + |v_{i+d}|).$$

For any $2d \times 2d$ matrix C, we define

$$\|C\|_\delta = \sup_{v \in \mathbb{R}^{2d}} \|Cv\|_\delta / \|v\|_\delta.$$

Then, if λ is an eigenvalue of C, we have $|\lambda| \leq \|C\|_\delta$.

For the matrices A_i, B_i and for $\delta < 1$, we have

$$\begin{aligned} \|A_i\|_\delta &\leq 1 + d\max(1, |(a_i)_{jk}|)\delta \leq 1 + dA\delta \\ \|A_i - B_i\|_\delta &\leq \varepsilon \max(d + d\delta, d + d\delta^{-1}) \\ &= \varepsilon d(1 + \delta^{-1}). \end{aligned} \tag{3.9}$$

To prove the claim about the eigenvalues of B, notice that the eigenvalues $\mu_1, \dots, \mu_{2d}$ of $B - I$ satisfy

$$\begin{aligned} |\mu_i| &\leq \|B - A_1 \cdots A_N + A_1 \cdots A_N - I\|_\delta \\ &\leq \|B - A_1 \cdots A_N\|_\delta + \|A_1 \cdots A_N - I\|_\delta. \end{aligned}$$

We write $B = B_1 \dots B_N = (A_1 + (B_1 - A_1)) \times (A_2 + (B_2 - A_2)) \dots (A_N + (B_N - A_N))$. Expanding and grouping terms, we get

$$\begin{aligned} B = {} & A_1 \cdots A_N \\ & + \sum_i A_1 \cdots A_{i-1}(B_i - A_i)A_{i+1} \cdots A_N \\ & + \sum_{i,j} \bigl(A_1 \cdots A_{i-1}(B_i - A_i)A_{i+1} \\ & \qquad \cdots A_{j+1}(B_j - A_j)A_{j+1} \cdots A_N\bigr) \\ & + \cdots \\ & + (B_1 - A_1) \cdots (B_N - A_N) \end{aligned}$$

or

$$\|B - A_1 \cdots A_N\|_\delta \le \binom{N}{1} \max_i \|A_i\|_\delta^{N-1} \|B_i - A_i\|_\delta + \binom{N}{2} \max_i \|A_i\|_\delta^{N-2} \|B_i - A_i\|_\delta^2 + \cdots + \binom{N}{N} \max_i \|B_i - A_i\|_\delta^N.$$

Using the estimates (3.9) we conclude that

$$\|B - A_1 \dots A_N\|_\delta \le \big(1 + dA\delta + d(1+\delta^{-1})\varepsilon\big)^N - (1 + d\delta A)^N.$$

Choosing $\delta = (\varepsilon/A)^{1/2} < 1$ we obtain

$$\|B - A_1 \dots A_N\|_\delta \le \big(1 + 3d\sqrt{A}\sqrt{\varepsilon}\big)^N - 1.$$

Following the same steps, $\|A_1 \cdots A_N - I\|_\delta$ can be bounded by

$$\|A_1 \cdots A_N - I\|_\delta \le (1 + d\sqrt{A}\sqrt{\varepsilon})^N - 1.$$

Since $\mu_i = \lambda_i - 1$ we have

$$|\lambda_i - 1| \le (1 + 3d\sqrt{A}\sqrt{\varepsilon})^N + (1 + d\sqrt{A}\sqrt{\varepsilon})^N - 2 \le 2\big((1 + 3d\sqrt{A}\sqrt{\varepsilon})^N - 1\big). \qquad \square$$

Putting all these estimates together, for N large enough, we can bound all the eigenvalues of $Df^N(\boldsymbol{x})$ for a (P/N) periodic orbit by

$$|\lambda_i - 1| \le D_k \|N\omega - P\|_d^{k/2} N.$$

This concludes the proof of Theorem 2.1.

The Analytic Case

To prove Theorem 2.2 we only need to compute the values of the constants C_k and D_k, and choose the best value for k. The optimal bound depends on the diophantine properties of the rotation vector ω. Let

$$\mathcal{T}_\delta = \{(\varphi, A) : \operatorname{Re}\varphi_i \in [0,1],\ |\operatorname{Im}\varphi_i| \le \delta,\ |A_i| \le \delta\}$$

be a complex product neighborhood of the invariant surface. For an analytic function F on $\mathcal{T}_\delta$, set

$$\|F\|_\delta \equiv \sup_{\mathcal{T}_\delta} |F|;$$

or, if F denotes several functions, set

$$\|F\|_\delta \equiv \max_i \|F_i\|_\delta.$$

We first state a lemma that provides quantitative bounds for the solution to equations similar to (3.4).

Lemma 3.4. *Let L be a bounded analytic function on $\mathcal{T}_\delta$ and assume that L has zero average over $\mathbb{T}^d$. For ω diophantine of type (K, τ) we can find a solution of the equation*

$$H(\varphi) - H(\varphi + \omega) = L(\varphi)$$

unique, up to an additive constant, on $\mathcal{T}_\delta$. Moreover, the solution is bounded on any smaller domain $\mathcal{T}_{\delta-\eta}$ by

$$\|H\|_{\delta-\eta} \le C_{K,\tau,d}\, \eta^{-\tau} \|L\|_\delta$$

for any $0 < \eta < \delta$.

Proofs of this lemma can be found in [Rüssmann 1975; Rüssmann 1976; Arnol'd 1988; Fassò and Benettin 1989].

In the process of constructing d approximate integrals in the neighborhood of the invariant surface we need to solve the equations

$$H_{\{m\}}(\varphi) - H_{\{m\}}(\varphi + \omega) = H_{\{m-1\}}(\varphi + \omega) u(\varphi, 0) + L_{\{m\}}(\varphi),$$

where $L_{\{m\}}(\varphi) = L^1_{\{m\}}(\varphi) - L^2_{\{m\}}(\varphi)$ with $L^1_{\{m\}}(\varphi)$ equal to

$$\sum_{j=1}^{m} \frac{1}{\{j\}!} \Big(\frac{\partial}{\partial A}\Big)^{\{j\}} H_{\{m-j\}}(\varphi + \omega + Av(\varphi, A))|_{A=0}$$

and $L^2_{\{m\}}(\varphi)$ equal to

$$\sum_{j=2}^{m} H_{\{m-j\}}(\varphi) \frac{1}{\{j\}!} \Big(\frac{\partial}{\partial A}\Big)^{\{j\}} (A + A^{\{2\}} u(\varphi, A))^{\{j\}}|_{A=0}$$

under the condition (3.5).

We will use induction to estimate bounds on the H's.

Theorem 3.5. *If the invariant surface is analytic in $\mathcal{T}_\delta$ and ω is diophantine of type (K, τ), there are* *numbers $\tilde{K}$ and E (depending on the system, the invariant surface, the dimension and ω) such that $\|\tilde{H}_{\{m\}}\|_{\delta-m\eta} \le ED^m$ and $\max|\bar{H}_{\{m\}}| \le ED^m$, for $\bar{H} = \int_{\mathbb{T}^d} H\,d\varphi$, $\tilde{H} = H - \bar{H}$, $\delta - k\eta > 0$, and $D = \tilde{K}\eta^{-1-\tau}$.*

Proof. Using induction, the hypothesis holds for $m = 1$. Assuming that all $H_{\{m\}}$'s are determined completely up to order $m-2$ and up to an additive constant for $H_{\{m-1\}}$ and satisfy the bounds in the assumption, we have

$$\sup_{\substack{\|A\|_d \le \eta/2V \\ \mathcal{T}_{\delta-(m-1/2)\eta}}} |H_{\{m-j\}}(\varphi+\omega+Av(\varphi,A))| \le \|H_{\{m-j\}}\|_{\delta-(m-1)\eta} \le \|H_{\{m-j\}}\|_{\delta-j\eta},$$

where $V = \sup_{\mathcal{T}_\delta} |v(\varphi,A)|$.

We can use Cauchy estimates to bound derivatives with respect to A; this is justified for the case of max norms in $\mathbb{C}^d$ by the arguments in [Perry and Wiggins 1994]. We derive the inequalities

$$\sup_{\substack{\|A\|_d \le \eta/2V \\ \mathcal{T}_{\delta-(m-1/2)\eta}}} \left| \frac{1}{\{j\}!}\Big(\frac{\partial}{\partial A}\Big)^{\{j\}} H_{\{m-j\}}(\varphi+\omega+Av(\varphi,A))|_{A=0}\right| \le \|H_{\{m-j\}}\|_{\delta-j\eta}\frac{(2V)^j}{\eta^j}$$

and

$$\sup_{\substack{\|A\|_d \le \eta/2V \\ \mathcal{T}_{\delta-(m-1/2)\eta}}} \left| \frac{1}{\{j\}!}\Big(\frac{\partial}{\partial A}\Big)^{\{j\}} (A+A^{\{2\}}u(\varphi,A))^{\{j\}}|_{A=0}\right| \le \frac{1}{\eta^j}.$$

From them we deduce

$$\|L^1_{\{m\}}\|_{\delta-(m-1/2)\eta} \le D^{m-1}E\frac{4V}{\eta},$$
$$\|L^2_{\{m\}}\|_{\delta-(m-1/2)\eta} \le D^{m-1}E\frac{2}{\eta}.$$

From condition (3.5) it follows that

$$\|\bar{H}_{\{m-1\}}\| \le ED^{m-1}$$

for η fixed and E large enough.

Using 3.4 and fixing $\eta \le \delta/2k$ we have

$$\|\tilde{H}_m\|_{\delta-m\eta} \le ED^{m-1}\tilde{K}\eta^{-1-\tau} \le ED^m,$$

which concludes the induction. □

To conclude the proof of Theorem 2.2 we fix $\eta = \delta/2k$ and have $C_k \le \tilde{K}(k/\delta)^{k(1+\tau)}$. Using a simple maximization argument over k, we get

$$\max_{k\in\mathbb{N}}\Big(\frac{k}{\delta}\Big)^{k(1+\tau)}B^k \le \exp\big(-(1+\tau)B^{-1/(1+\tau)}\delta e^{-1}\big).$$

Letting $B = \|N\omega - P\|_d^{1/2}$ concludes the proof.

Remark. Theorem 2.2 is also valid for the case of complex maps with complex invariant surfaces, as long as the nondegeneracy condition (ii) is satisfied in a neighborhood of the invariant surface.

The Quasiperiodic Skew-Product Case

The proof for the case of a quasiperiodic perturbation of a symplectic map is similar to the proofs of Theorems 2.1 and 2.2. We sketch the proof, referring to the preceding ones and emphasize the differences.

We study invariant sets of maps $f : \mathbb{T}^{d+e}\times\mathbb{R}^d \to \mathbb{T}^{d+e}\times\mathbb{R}^d$ on which motion is conjugate to rigid rotation with rotation vector $\omega = (\omega_1,\omega_2)$, for $\omega_1 \in \mathbb{T}^d$ and $\omega_2 \in \mathbb{T}^e$, and satisfying

$$f(\varphi_1,\varphi_2,A) = (f_1(\varphi_1,\varphi_2,A),\ \varphi_2+\omega_2),$$

where $f_1 : \mathbb{T}^{d+e}\times\mathbb{R}^d \to \mathbb{T}^d\times\mathbb{R}^d$ and $f_1(\,\cdot\,,\varphi_2,\,\cdot\,)$ is symplectic.

The first part of the proof consists of constructing a normal form for f in a neighborhood of the invariant surface with rotation vector ω. As in Proposition 3.1 we can find a map h, defined in a neighborhood of the invariant surface, such that $h\circ f\circ h^{-1}(\varphi_1,\varphi_2,A)$ equals

$$(\varphi_1+\omega_1+A_1v(\varphi_1,\varphi_2,A_1),\ \varphi_2+\omega_2,\ A_1+A_1^2u(\varphi_1,\varphi_2))$$

with $v(\varphi_1,\varphi_2,0) \ne 0$.

We can now construct d approximate integrals for f in a neighborhood of the invariant surface, by expanding and matching by orders, just as in Lemma 3.2. The difference at this point is that not only the properties of ω_1 (the rotation vector for the symplectic coordinates) but also the combined properties of ω_1 and ω_2 are important.

After constructing the approximate integrals, we perform a transformation (using a generating function in the "symplectic" coordinates, identity in the remaining coordinates) to substitute the approximate integrals for the original "actions".

The normal form for f in a neighborhood of the invariant surface is

$$f(\Phi_1, \varphi_2, \tilde{A}_1) = (\Phi_1+\omega_1+\tilde{A}_1\Delta(\tilde{A}_1), \varphi_2+\omega_2, \tilde{A}_1) + (E_1(\Phi_1, \varphi_2, \tilde{A}_1), 0_e, E_2(\Phi_1, \varphi_2, \tilde{A}_1)),$$

where $\Delta(0, \omega_2) \neq 0$ and $\|E_{1,2}\| \leq C_k \|\tilde{A}_1\|_d^{k+1}$ in appropriate norms.

Instead of studying the normal form for f itself, we will study the extension $f^* : \mathbb{T}^{d+e} \times \mathbb{R}^{d+e} \to \mathbb{T}^{d+e} \times \mathbb{R}^{d+e}$ with

$$\begin{aligned} f^*(\Phi_1, \varphi_2, \tilde{A}_1, A_2) &= (\Phi_1 + \omega_1 + \tilde{A}_1\Delta(\tilde{A}_1), \varphi_2 + A_2, \tilde{A}_1, A_2) \\ &\quad + (E_1(\Phi_1, \varphi_2, \tilde{A}_1), 0_e, E_2(\Phi_1, \varphi_2, \tilde{A}_1), 0_e). \end{aligned}$$

The map f^* is also area-preserving and, for $A_2 \equiv \omega_2$, motion in the $\Phi_1, \varphi_2, \tilde{A}_1$ coordinates under f^* is identical to motion in the $\Phi_1, \varphi_2, \tilde{A}_1$ coordinates under f. The map f^* has the advantage that in a neighborhood of an invariant surface with rotation vector of type (K, τ) one can find periodic orbits (by simply changing A_2 to nearby rational numbers).

The bounds on the eigenvalues of the derivative follow from Lemma 3.3. The $2e$ eigenvalues corresponding to rotation in the φ_2, A_2 coordinates are identically 1.

Using arguments similar to those in the preceding subsection, we can also reproduce the proof for the analytic case. This proves Theorem 2.4.

Remark. In the case of a general volume-preserving map $f : \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^d \times \mathbb{R}$ under conditions similar to the ones in Theorem 2.4 it is possible to construct one approximate integral in the neighborhood of the invariant surface. However, no result similar to Theorem 2.4 is possible, since we have no control for the motion along the angle coordinates as we did in the symplectic skew-product case.

4. CONCLUSIONS

Theorems 2.1, 2.2, and 2.4 suggest that the eigenvalues of the derivative of a symplectic map along a periodic orbit are a higher-dimensional analog of the residue (as used in Greene's criterion for two-dimensional twist maps—for a justification and an application of Greene's criterion in the case of a particular dissipative map see [Llave and Tompaidis 1994]). Based on this analogy the following efficient numerical algorithm can be implemented to indicate existence of a close-by invariant surface.

- Compute periodic orbits with rotation vectors close to the rotation vector of an invariant set of interest.
- Check whether the periodic orbits thus computed stay within a small neighborhood in phase space.
- Compute the eigenvalues of the derivative of the map along the periodic orbits.
- If all the eigenvalues approach 1 as the rotation vector of the periodic orbit approaches the rotation vector of the sought-for invariant set, existence is indicated. If, on the other hand, the distance from the eigenvalues to 1 increases, we have a numerical indication for the nonexistence of the invariant surface.

Since convergence to the limit behavior (either 1 for the case of an invariant surface or ∞ for the case of a uniformly hyperbolic invariant set) is exponentially fast, relatively low-period orbits can be used. In [Tompaidis 1996] (the next article in this issue) we implement such an algorithm for the case of a quasiperiodic excitation of a two-dimensional symplectic map.

Periodic orbits can also be used [Greene 1979; MacKay 1982] to investigate behavior at breakdown. If transition can be described in terms of a fixed point of a renormalization group operator with a stable manifold of codimension one, the eigenvalues of the periodic orbits scale with the period of the orbit and the distance from breakdown. Kosygin [1991] constructed a renormalization group operator and showed that if, under repeated action of the operator, the map converges to a trivial fixed point, then the original map admits an invariant surface. No such description is known for the behavior at breakdown. Numerical studies and analytical arguments suggest that if such a renormalization operator exists, there are regions in parameter space where behavior at breakdown is governed by dynamics more complex than a simple fixed point [MacKay et al. 1994; Artuso et al. 1991; Tompaidis 1996].

Another interesting problem is to determine the existence of lower-dimensional hyperbolic tori on which motion is conjugate to rigid rotation with a resonant rotation vector. One can separate phase space in the neighborhood of the low-dimensional torus to the center manifold of the torus and the hyperbolic directions. Arguments similar to the ones we used in this paper can be used to show that along the center manifold the map is close to an integrable normal form. Along the hyperbolic directions behavior can be described using arguments similar to those of [Falcolini and Llave 1992]. The natural result appears to be that $2d^*$ eigenvalues (where d^* the dimension of the low-dimensional torus) of the derivative of the map along periodic orbits will approach 1, while the rest will approach $e^{\lambda_i T}$, where the λ_i are the nonzero Lyapunov exponents of the orbits on the low-dimensional torus and T is the period. Unfortunately a numerical algorithm to estimate domains of existence of lower dimensional hyperbolic tori would be difficult to implement, since we cannot numerically isolate the eigenvalues that tend to 1, from eigenvalues that become exponentially large.

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