Finite Subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$

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We classify maximal finite irreducible subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$, together with their natural lattices. There are 65 conjugacy classes of such groups, 41 of which consist of primitive groups. New methods for finding the maximal finite supergroups of irreducible cyclic groups are developed and applied.

INTRODUCTION

In this work we determine a set of representatives of the conjugacy classes of rational irreducible maximal finite (r.i.m.f.) groups in $\mathrm{GL}_{24}(\mathbb{Q})$. This completes the classification of the r.i.m.f. subgroups of $\mathrm{GL}_n(\mathbb{Q})$ for $n \leq 24$ started in [Plesken 1991], where the study of maximal finite subgroups of $\mathrm{GL}_n(\mathbb{Q})$ was essentially reduced to that of irreducible groups, and continued in [Plesken and Nebe 1995] and [Nebe and Plesken 1995] (compare also [Plesken 1985], where the maximal finite irreducible subgroups of $\mathrm{GL}_p(\mathbb{Z})$ for primes $p < 24$ are determined).

Finite subgroups of $\mathrm{GL}_n(\mathbb{Q})$ fix positive definite quadratic forms on the one hand and on the other hand they act on n-dimensional lattices. In particular the r.i.m.f. groups can be regarded as full automorphism groups of lattices in Euclidean spaces. The existence of the Leech lattice, the unique even unimodular lattice of dimension 24 with minimal square length 4 [Conway and Sloane 1993], makes the dimension particularly interesting. The automorphism group of this lattice is a covering group of the Conway group and an r.i.m.f. group. In close relation to this lattice are some other interesting k-modular lattices of r.i.m.f. subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$ that turn up; k-modular lattices are defined in Definition 1.4(vii). Examples are given after Remark 1.10, and detailed in my thesis [Nebe 1995,

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Chapter VI]. In fact, this paper is supposed to make one main part of the results of my thesis available to a wider audience. I have not included its second part, the discussion of the simplicial complexes $M_{24}^{\mathrm{irr}}(\mathbb{Q})$ and $M_{24}^{\mathrm{irr},F}(\mathbb{Q})$, which encode the interrelation of the r.i.m.f. groups via common irreducible subgroups.

The group $\mathrm{GL}_{24}(\mathbb{Q})$ has 65 conjugacy classes of r.i.m.f. groups, listed in Table 1. Of these, 41 consist of primitive groups (Definition 1.14).

Dimension 24 is the lowest where r.i.m.f. groups fixing a two dimensional space of invariant quadratic forms turn up (Theorem 3.1). Already in $\mathrm{GL}_{16}(\mathbb{Q})$ there exist two nonuniform r.i.m.f. groups fixing a four-dimensional space of invariant forms [Nebe and Plesken 1995]. These two examples show that for nonuniform groups it might happen that the determinant of each integral invariant positive definite quadratic form is divisible by some prime not dividing the order of the automorphism group. That this is not possible for uniform groups and under some additional assumptions also if the space of invariant quadratic forms is of dimension two has been shown in [Nebe and Plesken 1995] (see also [Feit 1974] for the absolutely irreducible case). Theorem 2.2 deals with this problem when the commuting algebra of the group is isomorphic to a number field, and gives rise to a purely arithmetic method to determine the r.i.m.f. supergroups of those groups.

The classification of the nonabelian finite simple groups and their character tables [Conway et al. 1985; Jansen et al. 1995] is used. However, the results of Section 4, where some r.i.m.f. supergroups of the irreducible finite cyclic subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$ are determined, are independent of this classification, thanks to Theorem 2.2.

Concrete number-theoretic questions, such as the computation of fundamental units and class numbers, can be dealt with using KANT [Pohst et al. 1993]. Group-theoretic problems can often be solved using GAP [Schönert et al. 1994] or CAYLEY [Cannon 1984]. The main computations are done with the help of programs developed at the Lehrstuhl B für Mathematik of the RWTH Aachen, such as the program for computing the automorphism group of a lattice implemented by B. Souvignier [Plesken and Pohst 1985; Souvignier 1994; Plesken and Souvignier 1996], the sublattice algorithm to compute all invariant lattices of a given matrix group and other C programs partly implemented by H. Brückner.

The principal strategy for the construction of the maximal finite groups is the use of normal subgroups. An important notion is that of imprimitivity (Definition 1.14), which reduces the classification of r.i.m.f. groups to the one of primitive maximal finite groups. For a primitive subgroup $G \leq \mathrm{GL}_n(\mathbb{Q})$, the restriction of the natural representation of G to a normal subgroup of G is homogenous. In particular each abelian normal subgroup of G is cyclic. Using a theorem of P. Hall, which classifies those p-groups whose abelian characteristic subgroups are cyclic, this restricts the possibilities for the maximal nilpotent normal subgroup $\mathrm{Fit}(G)$ of G.

Let $C := C_G(\mathrm{Fit}(G))$ be the centralizer in G of $\mathrm{Fit}(G)$. Then C is a normal subgroup of G and $C/Z(\mathrm{Fit}(G))$ is a subgroup of the automorphism group of a direct product of finite simple groups. Therefore the possibilities for C can be derived from the classification of finite simple groups and their character tables in the *Atlas of Finite Groups* [Conway et al. 1985]. The quotient group $G/(C\mathrm{Fit}(G))$ is isomorphic to a subgroup of the outer automorphism group $\mathrm{Out}(\mathrm{Fit}(G))$ of $\mathrm{Fit}(G)$, so in principle the group G may be constructed using only group theoretical means. But the exclusive usage of group theoretical constructions is cumbersome and not stable against errors. It is not very powerful, because it does not use the fact that G is maximal finite.

Maximal finite groups satisfy a certain closedness condition: They are full automorphism groups of all their invariant lattices with respect to all their invariant quadratic forms.

Therefore, the language of lattices and quadratic forms is introduced in Section 1.

Section 2 develops further arithmetic methods, also dealing with reducible normal subgroups (Definition 2.4). Short-cuts using the knowledge of certain irreducible but not necessarily normal subgroups of G can be obtained with the help of Theorem 2.2.

Section 3 contains the main result, the list of irreducible maximal finite subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$; see Table 1 on pages 173–174. That table also displays some information about the invariant lattices. On the one hand, these lattices have nice geometric and arithmetic properties and are of interest on their own. On the other hand, they provide powerful means for identifying the r.i.m.f. groups.

That the groups listed in Table 1 are maximal finite can easily be checked using Remark 1.3, so it remains to prove that the list of r.i.m.f. groups is complete. This is done in the last three sections.

Nearly two-thirds of the r.i.m.f. subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$ have irreducible cyclic subgroups. Therefore the r.i.m.f. supergroups of those irreducible groups are determined in Section 4, which is also interesting for the classification of cyclotomic lattices. The results of this Section are independent from the classification of finite simple groups. The latter is often used in Section 5, where we determine the r.i.m.f. groups having an irreducible subgroup that is a central product of quasisimple groups. Whereas Section 4 provides short-cuts used throughout Section 6, Section 5 is mainly intended to fix the notation for the occurring characters of the quasisimple groups.

The last section completes the proof of Theorem 3.1, classifying the primitive r.i.m.f. groups by constructing normal subgroups and determining the r.i.m.f. supergroups as automorphism groups of invariant lattices.

A table of notations may be found on page 192. An additional table, on pages 193–195, lists the invariant forms of the primitive r.i.m.f. groups of degree dividing 24 that are not tensor products of forms of smaller dimension. The invariant forms, as well as generators for the r.i.m.f. groups, are available in GAP.

1. DEFINITIONS AND FIRST PROPERTIES

This section introduces the language of lattices and quadratic forms. The main (trivial) observation is Remark 1.3, describing the maximal finite groups as full automorphism groups of all their invariant lattices. Frequently used notations from [Plesken and Nebe 1995] are briefly repeated (see also the table of notations on page 192).

Definition 1.1. Let $G \le \mathrm{GL}_n(\mathbb{Q})$ be a finite rational matrix group. The set $\mathbb{Q}^{1\times n}$ has a natural $\mathbb{Q}G$-module structure.

(i) A set $L \subseteq \mathbb{Q}^{1\times n}$ is a *full $\mathbb{Z}$-lattice* if L is a free abelian subgroup of rank n. The set of G-invariant full $\mathbb{Z}$-lattices is denoted by $\mathcal{Z}(G)$.
(ii) A quadratic form $X \in \mathbb{Q}^{n\times n}_{\mathrm{sym}}$ is *G-invariant* if $gXg^{\mathrm{tr}} = X$ for all $g \in G$. The $\mathbb{Q}$-vector space of G-invariant quadratic forms is denoted by $\mathcal{F}(G)$, and the subset of $\mathcal{F}(G)$ consisting of positive definite quadratic forms is denoted by $\mathcal{F}_{>0}(G)$.
(iii) G is called *uniform* if $\dim \mathcal{F}(G) = 1$.
(iv) The *enveloping algebra* $\bar{G}$ is the $\mathbb{Q}$-subalgebra of $\mathbb{Q}^{n\times n}$ spanned by the matrices in G.

Definition 1.2. Let L, L' be full $\mathbb{Z}$-lattices in $\mathbb{Q}^{1\times n}$, $\mathcal{F} \subseteq \mathbb{Q}^{n\times n}_{\mathrm{sym}}$ a subset of the symmetric rational $n\times n$-matrices, and $F \in \mathcal{F}$.

(i) The *automorphism group* $\mathrm{Aut}(F, L)$ of F on L is defined as the set of $g \in \mathrm{GL}_n(\mathbb{Q})$ such that $Lg = L$ and $gFg^{\mathrm{tr}} = F$.
(ii) The *Bravais group* $\mathcal{B}(\mathcal{F}, L)$ of $\mathcal{F}$ on L is defined as the intersection of all $\mathrm{Aut}(F', L)$, as F' runs over $\mathcal{F}$.
(iii) If G is a finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ and $L \in \mathcal{Z}(G)$ is a $\mathbb{Z}G$-lattice, the *Bravais group* of G on L is defined as the Bravais group of the space of G-invariant forms: $\mathcal{B}(G, L) := \mathcal{B}(\mathcal{F}(G), L)$.

Remark 1.3. Let $G \le \mathrm{GL}_n(\mathbb{Q})$ be a finite rational matrix group. Each finite supergroup $G' \le \mathrm{GL}_n(\mathbb{Q})$ of G is contained in a group $\mathrm{Aut}(F, L)$ for some $F \in \mathcal{F}_{>0}(G)$, and $L \in \mathcal{Z}(G)$. In particular, G is maximal finite if and only if $G = \mathrm{Aut}(F, L)$ for all $F \in \mathcal{F}_{>0}(G)$ and all $L \in \mathcal{Z}(G)$.

Definition 1.4. Let L, L' be full $\mathbb{Z}$-lattices in $\mathbb{Q}^{1\times n}$ and $F, F' \in \mathbb{Q}^{n\times n}_{\mathrm{sym},>0}$ positive definite symmetric matrices.

(i) The *dual lattice* $L^{\#(F)}$ of L with respect to F consists of the elements $x \in \mathbb{Q}^{1\times n}$ satisfying $xFy^{\mathrm{tr}} \in \mathbb{Z}$ for all $y \in L$.
(ii) F is called *integral* on L if $L^{\#(F)} \supseteq L$.
(iii) F is called *primitive* on L if $L^{\#(F)} \supseteq L$ and $pL^{\#(F)} \not\supseteq L$ for all primes p.
(iv) If F is integral on L, the lattice

$$L^{\mathrm{ev}(F)} := \{x \in L \mid xFx^{\mathrm{tr}} \in 2\mathbb{Z}\}$$

is called the *even sublattice* of L with respect to F. We call (L, F) *even* if $L^{\mathrm{ev}(F)} = L$.
(v) $\det(F, L)$ denotes the *determinant* of a Gram matrix of L with respect to F.
(vi) (L, F) is called *normalized* if F is integral on L and the finite abelian group $L^{\#(F)}/L$ is of square-free exponent and of rank at most $\frac{1}{2}n$ [Watson 1962].
(vii) For $k \in \mathbb{N}$, the lattice (L, F) is called *k-modular* if there is a matrix $T \in \mathrm{GL}_n(\mathbb{Q})$ with $L = L^{\#(F)}T$ and $TFT^{\mathrm{tr}} = kF$. (See [O'Meara 1973], where such a lattice is called T-modular.) A 1-modular lattice is called *unimodular*.

Remark 1.5. Let $G \leq \mathrm{GL}_n(\mathbb{Q})$ be a finite rational matrix group, $F \in \mathcal{F}_{>0}(G)$, and $c \in C_{\mathbb{Q}^{n\times n}}(G)$ with $\det(c) \neq 0$. The set $\mathcal{Z}(G)$ is closed under the operations

$$\begin{aligned} d(F)&: M \mapsto M^{\#(F)},\\ g(F)&: M \mapsto M^{\mathrm{ev}(F)},\\ m(c)&: M \mapsto Mc,\\ e&: (M_1, M_2) \mapsto \langle M_1, M_2\rangle_{\mathbb{Z}},\\ s&: (M_1, M_2) \mapsto M_1 \cap M_2, \end{aligned}$$

where $M, M_1, M_2 \in \mathcal{Z}(G)$.

A finite rational matrix group $G \leq \mathrm{GL}_n(\mathbb{Q})$ is called *lattice sparse* if any lattice in $\mathcal{Z}(G)$ can be obtained from any other by combining the five operations just defined. If p is a prime, G is called *p-lattice sparse* if any lattice $L \in \mathcal{Z}(G)$ can be obtained, by combining these operations, from any other lattice in $\mathcal{Z}(G)$ that contains L with index a p-power.

Definition 1.6. Let U be a finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ and $S \subseteq \mathcal{Z}(U)$.

(i) S is called *U-critical* if all r.i.m.f. supergroups of U are conjugate to a group $\mathrm{Aut}(F, L)$ with $F \in \mathcal{F}(U)$ and $L \in S$.
(ii) S is called *U-normal critical* if all r.i.m.f. supergroups G containing U as a normal subgroup are conjugate to a group $\mathrm{Aut}(F, L)$ with $F \in \mathcal{F}(U)$ and $L \in S$.

Remark 1.7. Let U be a finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ and let $S \subseteq \mathcal{Z}(U)$ be a set of representatives of the orbits of $N_{\mathrm{GL}_n(\mathbb{Q})}(U)$ on $\mathcal{Z}(U)$.

(i) S is a U-critical set.
(ii) If U is uniform, the subset S' of normalized elements of S is a U-critical set.
(iii) If U is uniform and lattice sparse, every lattice $L \in \mathcal{Z}(U)$ is U-critical.

Notations 1.8. Examples of r.i.m.f. groups are the automorphism groups of the following irreducible root lattices: A_n for $n \neq 7, 8$, B_n for $n \neq 4$, F_4, E_6, E_7, and E_8 [Plesken 1991]. We will use the same symbol to denote one of these root lattices, the corresponding root system, and the ($\mathrm{GL}_n(\mathbb{Q})$-conjugacy class of) its automorphism group.

For prime p, the irreducible rational representations of $\mathrm{PSL}_2(p)$ of degree $p-1$ and $p+1$ are described in [Plesken and Nebe 1995, Chapter V]. According to the notations introduced there, the lattices of dimension $p+1$ are denoted by $M_{p+1,i}$, where $i \in \{2, 3, 4, 6\}$ divides $(p-1)/2$. The corresponding representations are obtained by inducing up the representation of the Borel subgroup of $\mathrm{SL}_2(p)$ (of unimodular matrices $\left(\begin{smallmatrix} a & b\\ 0 & c\end{smallmatrix}\right)$, for $a, b, c \in \mathbb{F}_p$) onto $\langle \zeta_{p-1}^{(p-1)/i}\rangle \leq \mathbb{C}^*$.

The $\mathbb{Z}\mathrm{PSL}_2(p)$ lattices of dimension $p-1$ can be constructed as follows. The cyclic group $C_p = \langle a\rangle \leq \mathrm{GL}_{p-1}(\mathbb{Q})$ of order p acts on the root lattice A_{p-1}. The C_p-sublattices of p-power index in A_{p-1} are linearly ordered and generated by the rows of the matrices $(a - I_{p-1})^i$. Denote the unique $\mathbb{Z}C_p$-sublattice of $A_{p-1} = A_{p-1}^{(1)}$ of index p^{i-1} by $A_{p-1}^{(i)}$. The lattices $A_{p-1}^{(i)}$ are called *Craig lattices* [Conway and Sloane 1993].

If $i \in \{2,3\}$ divides $(p+1)/2$ and $p > 3$, then according to [Plesken and Nebe 1995, Theorem V.8] the automorphism group of $A_{p-1}^{((p+1)/(2i))}$ is isomorphic to $C_2 \times \mathrm{PGL}_2(p)$ and a lattice sparse r.i.m.f. group.

For the nonabelian finite simple and quasisimple groups we use the notation of [Conway et al. 1985], except that we denote the alternating group of degree n by Alt_n, to avoid confusion with the root system A_n. Split extensions are indicated by the symbol $:$, while $\boldsymbol{.}$ indicates an extension that may be either split or nonsplit. The group $\langle -I_n, G\rangle$ is denoted by $\pm G$.

For $i = 1, 2$, let $G_i \leq \mathrm{GL}_{n_i}(\mathbb{Q})$ be irreducible finite matrix groups with corresponding natural representations Δ_i and commuting algebras $A_i := C_{\mathbb{Q}^{n_i \times n_i}}(G_i)$. The A_i are $\mathbb{Q}$-division algebras. The tensor product

$$G_1 \otimes G_2 \cong G_1 \underset{C_2}{\curlyvee} G_2$$

need not be an irreducible subgroup of $\mathrm{GL}_{n_1 n_2}(\mathbb{Q})$, since the $\mathbb{Q}$-algebra $A_1 \otimes_{\mathbb{Q}} A_2$ is not necessarily a division algebra. If Q is a maximal common subalgebra of A_1 and A_2, an irreducible constituent group of $G_1 \otimes G_2$ is denoted by $G_1 \underset{Q}{\otimes} G_2$.

The following abbreviations are used: If $Q = \mathbb{Q}$, then Q is omitted in most cases. $Q \cong \mathbb{Q}[\alpha]$ is simply denoted by α. The quaternion algebra $Q \cong \mathcal{Q}_{p,q}$ with center $\mathbb{Q}$ ramified at the places p and q with Hasse invariant $\frac{1}{2}$ is abbreviated as p, q.

If G_1 or G_2 are of degree 1 over Q, then $\underset{Q}{\otimes}$ is simply denoted by $\circ$. Hence $Q_8 \circ Q_8$ denotes the absolutely irreducible subgroup of $\mathrm{GL}_4(\mathbb{Q})$ isomorphic to $Q_8 \underset{C_2}{\curlyvee} Q_8 = 2_+^{1+4}$. Alternatively, this group may be denoted by $D_8 \otimes D_8$ or 2_+^{1+4}.

Consider the case when $G_1 = C_5$ and $G_2 = \mathrm{SL}_2(3)$. Then the enveloping algebras are $\bar{G}_1 \cong A_1 \cong \mathbb{Q}[\zeta_5]$ and $\bar{G}_2 \cong A_2 \cong \mathcal{Q}_{\infty,2}$. Although the maximal common subalgebra of A_1 and A_2 is $\mathbb{Q}$, we have $\bar{G}_2 \leq A_1^{2\times 2}$. The irreducible subgroup of $\mathrm{GL}_8(\mathbb{Q})$ isomorphic to $C_5 \times \mathrm{SL}_2(3)$ is denoted by $C_5 \underset{\sqrt{5}'}{\otimes} \mathrm{SL}_2(3)$.

Among the commonly occurring groups are extensions of the matrix groups $G_1 \underset{Q}{\otimes} G_2$ by a cyclic group of order 2. They are denoted as follows:

Notations 1.9. For $i = 1, 2$, let $G_i \leq \mathrm{GL}_{n_i}(\mathbb{Q})$ be finite irreducible matrix groups with commuting algebras A_i in $\mathbb{Q}^{n_i \times n_i}$, and let Q be a maximal common subalgebra of dimension d of A_1 and A_2. Setting $n := n_1 n_2 / d$, we view as embedded in $\mathbb{Q}^{n\times n}$ the groups G_i, their rational algebra spans $\bar{G}_i$, as well as A_i, Q, $G_1 \underset{Q}{\otimes} G_2$, etc. Assume that $G_1 \underset{Q}{\otimes} G_2$ is an irreducible subgroup of $\mathrm{GL}_n(\mathbb{Q})$.

(i) Let $a_i \in \bar{G}_i \setminus G_i$ be units normalizing G_i such that $p^{-1}a_i^2 \in G_i$ for some square-free nonzero integer p. Then

$$G_1 \overset{2(p)}{\underset{Q}{\otimes}} G_2 := \langle G_1 \underset{Q}{\otimes} G_2,\ p^{-1}a_1 a_2\rangle$$

is an irreducible finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ containing $G_1 \underset{Q}{\otimes} G_2$ as a subgroup of index 2.

(ii) Assume there is a chain of simple $\mathbb{Q}$-algebras $\langle \bar{G}_1 \bar{G}_2\rangle \subseteq A \subset B \subseteq \mathbb{Q}^{n\times n}$ with (crossed product) $B = A \oplus Ax$ for some $x \in B$ satisfying $x^2 = \pm 1$, $xAx = A$, and $x\bar{G}_i x = \bar{G}_i$ for $i = 1, 2$. If there are units $a_i \in \bar{G}_i$ with $a_i x$ normalizing G_i and $p^{-1}(a_i x)^2 \in G_i$, for $i = 1, 2$, and some square-free integer $p \neq 0$, then

$$G_1 \overset{2(p)}{\underset{Q}{\boxtimes}} G_2 := \langle G_1 \underset{Q}{\otimes} G_2,\ p^{-1}a_1 a_2 x\rangle$$

is an irreducible finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ containing $G_1 \underset{Q}{\otimes} G_2$ with index 2.

(iii) Let $A_1 \supset \bar{G}_1$ be a simple subalgebra of $\mathbb{Q}^{n\times n}$ centralizing $\bar{G}_2$. Let $a_1 \in A_1 \setminus \bar{G}_1$ and $a_2 \in \bar{G}_2$ be units normalizing G_1 and G_2, respectively, with $p^{-1}a_i^2 \in G_i$ for some square-free nonzero integer p. Then

$$G_1 \overset{2(p)}{\underset{Q}{\boxtimes}} G_2 := \langle G_1 \underset{Q}{\otimes} G_2,\ p^{-1}a_1 a_2\rangle$$

is an irreducible finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ containing $G_1 \underset{Q}{\otimes} G_2$ with index 2.

In each case, if $p=1$, we omit (p) from the symbols. If G_1 or G_2 is of degree 1 over Q, we omit the $\times$ and the subscript from the symbols, writing instead $\overset{2(p)}{\bigcirc}$, $\overset{2(p)}{\square}$, and $\overset{2(p)}{\sqsubset\!\!\supset}$.

Examples. The symbol $[6.U_4(3).2 \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$ represents an irreducible matrix group G of degree 24 constructed from the matrix group

$$G_1 := 6.U_4(3).2_2 \le \mathrm{GL}_{12}(\mathbb{Q})$$

with commuting algebra $A_1 := C_{\mathbb{Q}^{12\times 12}}(G_1)$ and the matrix group $G_2 := \mathrm{SL}_2(3) \le \mathrm{GL}_4(\mathbb{Q})$ with commuting algebra $A_2 := C_{\mathbb{Q}^{4\times 4}}(G_2)$; both A_1 and A_2 are isomorphic to $\mathbb{Q}[\sqrt{-3}]$. The commuting algebra $Q \le \mathbb{Q}^{24\times 24}$ of the central product

$$G_1 \underset{\sqrt{-3}}{\otimes} G_2 = 6.U_4(3).2 \underset{\sqrt{-3}}{\otimes} \mathrm{SL}_2(3) \le \mathrm{GL}_{24}(\mathbb{Q})$$

is again isomorphic to $\mathbb{Q}[\sqrt{-3}]$, and we have $G = \langle G_1 \underset{\sqrt{-3}}{\otimes} G_2, x\rangle$ for a suitable matrix $x \in \mathbb{Q}^{24\times 24}$ inducing the Galois automorphism on Q.

The group G could also be denoted by the symbol $[6.U_4(3).2 \overset{2(2)}{\bigcirc} \mathrm{SL}_2(3)]_{24}$. Here $G_1 = 6.U_4(3).2_1$ is an irreducible subgroup of $\mathrm{GL}_{24}(\mathbb{Q})$ with commuting algebra $A_1 \cong \mathcal{Q}_{\infty,2}$, and G_2 is isomorphic to the unit group of the maximal $\mathbb{Z}$-order in A_1, this being unique up to conjugacy.

This notation distinguishes the two isomorphic groups $[L_2(7) \overset{2(2)}{\otimes} F_4]_{24}$ and $[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$, since it implies that the irreducible constituents of the restriction of the natural representation to $L_2(7)$ are absolutely irreducible in the first case but not in the second.

In most cases it is not necessary to construct the extensions by the cyclic group of order 2 explicitly, since these extensions occur in a natural way as subgroups of the normalizer of $G_1 \underset{Q}{\otimes} G_2$ in the automorphism group of a suitable lattice. Also, the extension cannot be read off from the symbol, as shown by the example of the two isoclinic r.i.m.f. subgroups $[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{24,i}$ of $\mathrm{GL}_{24}(\mathbb{Q})$.

The occurrence of such pairs of groups is explained as follows.

Remark 1.10. Let $U \le \mathrm{GL}_n(\mathbb{Q})$ be an irreducible Bravais group with $\dim_{\mathbb{Q}}(\mathcal{F}(U)) = 2$. Then the maximal totally real subfield K of the center of the commuting algebra of U is a real quadratic number field. Assume that there is a U-invariant lattice $L \in \mathcal{Z}(U)$ giving rise to an embedding $U \hookrightarrow \mathrm{GL}_n(\mathbb{Z})$ such that $D := N_{\mathrm{GL}_n(\mathbb{Z})}(U)/U$ is isomorphic to an infinite dihedral group. Then the two nonconjugate subgroups of D of order 2 define two uniform supergroups $G = \langle U, x\rangle$ and $H = \langle U, tx\rangle$, containing U with index two. Here tU generates the translation subgroup of D and x induces the Galois automorphism of K.

If one takes for example $U = \pm D_{10} \le \mathrm{GL}_4(\mathbb{Q})$, then G and H are conjugate in $\mathrm{GL}_4(\mathbb{Q})$. But it often occurs that the two groups G and H are not isomorphic—for instance in the last example of the two isoclinic r.i.m.f. subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$, but also in other cases, where only one of the two groups G or H is maximal finite. Two examples of such groups U are closely related to the Leech lattice: $2.J_2 \circ \mathrm{SL}_2(5)$ and $\mathrm{SL}_2(13) \circ \mathrm{SL}_2(3)$. One extension of each group—namely $[2.J_2 \overset{2}{\square} \mathrm{SL}_2(5)]_{24}$ and $[\mathrm{SL}_2(13) \overset{2(2)}{\sqsubset\!\!\supset} \mathrm{SL}_2(3)]_{24}$—is a maximal finite subgroup of $\mathrm{GL}_{24}(\mathbb{Q})$, the other one a subgroup of the r.i.m.f. group $[2.\mathrm{Co}_1]_{24}$, the automorphism group of the Leech lattice. The group $U = \mathrm{SL}_2(13) \circ \mathrm{SL}_2(3)$ is also the automorphism group of an extremal 3-modular lattice. Further examples can be found in [Nebe 1995, Chapter VI].

The following observation may be used for some of the r.i.m.f. subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$, to distinguish the two groups G and H.

Remark 1.11. Let $U \trianglelefteq G$ be a normal subgroup of G of index 2 with $Z(U) = Z(G) \cong C_2$, and assume that $G/Z(U)$ is a semidirect product $G/Z(U) = U/Z(U){:}\langle xZ(U)\rangle$ and that the conjugacy class of the complement $xZ(U)$ of $U/Z(U)$ in $G/Z(U)$ is unique. Then there are two isoclinic but nonisomorphic groups containing U with index two, G and the subcentral product $H \cong G \overset{C_2}{\curlyvee} C_4$. If $\langle xZ(U)\rangle$ is a complement of $U/Z(U)$ in $G/Z(U) \cong H/Z(U)$, the two groups G and H may be distinguished by the isomorphism type of the group $\langle x, Z(U)\rangle$, which is either $C_2 \times C_2$ or C_4. The group G is called *split* if $\langle x, Z(U)\rangle \cong C_2 \times C_2$, and *nonsplit* if $\langle x, Z(U)\rangle \cong C_4$. If the complement $xZ(U)$ of $U/Z(U)$ in $G/Z(U)$ is not unique, but for all complements $x'Z(U)$ the groups $\langle x', Z(U)\rangle \leq G$ are isomorphic to $\langle x, Z(U)\rangle$, the isomorphism type of the latter group also distinguishes the two groups G and H and the same nomenclature of split and nonsplit groups is used.

The next lemma is useful in verifying the uniqueness of the complement of $U/Z(U)$ in $G/Z(U)$.

Lemma 1.12. *Let $G = U{:}\langle x\rangle \cong U{:}C_2$ and $H = V{:}\langle y\rangle \cong V{:}C_2$ be semidirect products, where the conjugacy class of the complement $\langle x\rangle$ of U in G is unique, and likewise the conjugacy class of the complement $\langle y\rangle$ of V in H. Then there is a unique conjugacy class of complements of $U \times V$ in the subdirect product $G \overset{C_2}{\curlywedge} H = (U \times V){:}\langle xy\rangle \leq G \times H$.*

Proof. Let $\langle a\rangle \cong C_2$ be a complement of $(U \times V)$ in $(U \times V){:}\langle xy\rangle$. Then $a = (ux)(vy)$ for some $u \in U$ and $v \in V$. Since $1 = a^2 = (ux)^2(vy)^2 \in U \times V$, we get $(ux)^2 = (vy)^2 = 1$. Hence $\langle ux\rangle$ is a complement of U in G, and $\langle vy\rangle$ is a complement of V in H. Therefore there are elements $u' \in U$ and $v' \in V$ with $(ux)^{u'} = x$ and $(vy)^{v'} = y$, so $a^{(u'v')} = xy$. □

The uniqueness condition of Remark 1.11 is in particular fulfilled for these pairs $(G/Z(U), U/Z(U))$: (S_n, Alt_n) for $n \leq 5$ (where the complement is generated by a transposition); $(\mathrm{PGL}_2(q), \mathrm{PSL}_2(q))$ for q an odd prime power (where the complement is generated by an element of order two corresponding to an element in $\mathrm{GL}_2(q)$ whose determinant is not a square); and $(J_2.2, J_2)$. So it is well defined to say that the two r.i.m.f. groups $[2.J_2 \overset{2}{\square} \mathrm{SL}_2(5)]_{24}$ and $[\mathrm{SL}_2(13) \overset{2(2)}{\square} \mathrm{SL}_2(3)]_{24}$ are split in the sense of Remark 1.11. For $G := [(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)){:}2]_8$, where $U = (\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2$, the complement of $U/Z(U)$ in $G/Z(U)$ is not unique, but all groups $\langle x, Z(U)\rangle$ for $x \in G \setminus U$ such that $x^2 \in Z(U)$ are isomorphic to $C_2 \times C_2$, so G is split. The corresponding nonsplit group $G \overset{C_2}{\curlyvee} C_4$ is a proper subgroup of E_8. In this sense the group

$$[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} \mathrm{Alt}_5]_{24,1}$$

is split and

$$[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} \mathrm{Alt}_5]_{24,2}$$

is the corresponding nonsplit group.

Here is an important case where one obtains a unique matrix group:

Lemma 1.13. *Let U be a finite uniform subgroup of $\mathrm{GL}_n(\mathbb{Q})$. For $i = 1, 2$, let $G_i := \langle U, x_i\rangle$ be absolutely irreducible subgroups of $\mathrm{GL}_n(\mathbb{Q})$ containing U with index 2, where x_1 and x_2 induce the same automorphism on U. Then G_1 and G_2 are conjugate in $\mathrm{GL}_n(\mathbb{Q})$.*

Proof. By [Plesken and Nebe 1995, Lemma II.7], $\dim_{\mathbb{Q}}(C_{\mathbb{Q}^{n\times n}}(U))$ equals 1 or 2. Since U is uniform, this implies that the commuting algebra $C_{\mathbb{Q}^{n\times n}}(U)$ is either $\mathbb{Q}$ or isomorphic to an imaginary quadratic number field K. Assume first that $C_{\mathbb{Q}^{n\times n}}(U)$ has dimension 2. Then the x_i induce the Galois automorphism on K. Since both elements x_i induce the same automorphism on U one has $x_1 x_2^{-1} \in C_{\mathbb{Q}^{n\times n}}(U) = K$. Hence $x_1 = kx_2$ for some element $k \in K$ of norm ± 1, the only elements of finite order in $\mathbb{Q}^*$. But the norm form of K is positive definite, so the norm of k has to be 1, and hence $x_1^2 = x_2^2$. Therefore the map $\varphi{:}U \cup \{x_1\} \to U \cup \{x_2\}$ with $\varphi|_U = \mathrm{id}$ and $\varphi(x_1) = x_2$ extends to a group isomorphism $G_1 \to G_2$. Moreover the natural representations of G_1 and G_2 are induced from the same representation of the subgroup U of index 2, so they are equivalent. Therefore G_1 and G_2 are conjugate in $\mathrm{GL}_n(\mathbb{Q})$. If $C_{\mathbb{Q}^{n\times n}}(U) = \mathbb{Q}$ the lemma follows similarly. □

Definition 1.14 [Plesken 1991]. A finite irreducible rational matrix group $G \leq \mathrm{GL}_n(\mathbb{Q})$ is *imprimitive* if there is m dividing n and $H \leq \mathrm{GL}_m(\mathbb{Q})$ such that G is conjugate to a subgroup of the wreath product $H \wr S_k$, for $k = n/m$. (Elements of $H \wr S_k$ are $k \times k$ block matrices with entries in H and at most one nonzero entry in each row and column; formally, $H \wr S_k$ is generated by elements $\mathrm{diag}(h_1, \dots, h_k)$, for $h_i \in H$, together with elements $P \otimes I_m$, for P a $k \times k$ permutation matrix.) If G is not imprimitive, we call it *primitive*.

The imprimitive r.i.m.f. groups G have the form $G = H \wr S_k$ for some primitive r.i.m.f. group $H \leq \mathrm{GL}_m(\mathbb{Q})$ with $m = n/k$ and hence can easily be constructed if the r.i.m.f. groups of degree m are known for all proper divisors m of n. If X_m is a symbol for the primitive r.i.m.f. subgroup H of $\mathrm{GL}_m(\mathbb{Q})$, the imprimitive group $G = H \wr S_k$ is denoted by X_m^k.

Remark 1.15. Let $G \leq \mathrm{GL}_n(\mathbb{Q})$ be primitive and let $N \trianglelefteq G$ be a normal subgroup of G.

(i) If N is abelian, then N is cyclic.
(ii) If $G : N = 2$ then N is irreducible.
(iii) The natural representation of N consists of pairwise equivalent irreducible representations.
(iv) If $1 \neq N$ is a p-group for some prime p, then $p^\alpha(p-1)$ divides n for some $\alpha \geq 0$.

2. METHODS FOR DETERMINING THE R.I.M.F. GROUPS

In determining the r.i.m.f. subgroups of $\mathrm{GL}_n(\mathbb{Q})$ it is crucial to be able to determine all r.i.m.f. supergroups G of a given (irreducible) subgroup $U \leq \mathrm{GL}_n(\mathbb{Q})$. According to Remark 1.3, the group G is of the form $G = \mathrm{Aut}(F, L)$ for some $L \in \mathcal{Z}(U)$ and $F \in \mathcal{F}_{>0}(U)$. Since each lattice $L \in \mathcal{Z}(U)$ defines an embedding $U \to \mathrm{GL}_n(\mathbb{Z})$ and since $\mathrm{GL}_n(\mathbb{Z})$ has only finitely many conjugacy classes of finite subgroups (see [Buser 1985], for example), $\mathcal{Z}(U)$ decomposes into finitely many orbits under the operation of the normalizer $N_{\mathrm{GL}_n(\mathbb{Q})}(U)$ by multiplication from the right.

Thus the main problem for nonuniform groups U is to determine the relevant $F \in \mathcal{F}_{>0}(U)$.

If one only has to determine those r.i.m.f. supergroups G of U, with $\dim \mathcal{F}(G) \leq 2$, the following theorem may be applied:

Theorem 2.1 [Nebe and Plesken 1995, Theorems II.2 and II.4]. *Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ with $\dim \mathcal{F}(G) \leq 2$. If G is irreducible, assume that the class group of the maximal real subfield K of the center of $C_{\mathbb{Q}^{n\times n}}(G)$ is generated by the ideal classes represented by prime ideals containing $|G|$. Then for each $L \in \mathcal{Z}(G)$ there exists a form $F \in \mathcal{F}_{>0}(G)$ integral on L with $\det(F, L)$ having only prime divisors dividing $|G|$.*

Passing to the Bravais group $B := \mathcal{B}(U)$, one can often construct a bigger subgroup $B \leq G$ such that the commuting algebra $C_{\mathbb{Q}^{n\times n}}(B)$ is commutative. The next theorem deals with this commonly occurring situation; in order to state it, we need some additional notation for totally real fields.

Let K be a totally real number field. A finite set of rational primes is denoted by $\Pi(K)$ if

(a) each ideal class of K has an integral ideal containing $\prod_{i=1}^r p_i^{\alpha_i}$ for some $p_1, \dots, p_r \in \Pi(K)$ and $\alpha_i \in \mathbb{N}$, and
(b) for each $x \in K$ there is an integral $y \in K$ such that xy is totally positive and the prime divisors of the norm $N_{K/\mathbb{Q}}(y)$ lie in $\Pi(K)$.

Note that there is a generating set $\{\bar{I}_j\}$ of the narrow ideal class group of K [Hasse 1963] represented by integral ideals I_j such that the prime divisors of the norm of the I_j ly in $\Pi(K)$.

Theorem 2.2. *Let G be a finite irreducible subgroup of $\mathrm{GL}_n(\mathbb{Q})$, and let $L \in \mathcal{Z}(G)$ be a $\mathbb{Z}G$-lattice. Assume that $C := C_{\mathbb{Q}^{n\times n}}(G)$ is commutative, and let K denote the maximal totally real subfield of C. Then there exists $F \in \mathcal{F}_{>0}(G)$ primitive on L such that the prime divisors of $\det(F, L)$ lie in the finite set $\Pi := \Pi(K) \cup \Pi(|G|)$, where $\Pi(|G|)$ denotes the set of prime divisors of $|G|$, and the set $\Pi(K)$ is as described above.*

Proof. Assume that $F \in \mathcal{F}_{>0}(G)$ is integral and primitive on L and that some prime $p \notin \Pi$ divides $\det(F, L)$. It suffices to show that F can be modified to some $F' \in \mathcal{F}_{>0}(G)$ in such a way that F' is integral on L, $p \nmid \det(F', L)$, and any prime dividing $\det(F', L)$ either divides $\det(F, L)$ or lies in Π.

Denote the completion at p of L by L_p and let $e_1, \dots, e_l$ be the primitive idempotents of $\mathbb{Q}_p \otimes C$. Since $p \nmid |G|$, the lattice L_p splits into a direct sum of irreducible $\mathbb{Z}_p G$-lattices X_i, for $i = 1, \dots, l$, each of which has only $p^\alpha X_i$ as $\mathbb{Z}_p G$-sublattices. Then $l \geq 2$, because $p \,|\, \det(F, L)$ and F is primitive on L.

Let M' be the maximal order in C. Taking the idempotents e_i modulo p one gets primitive idempotents of M'/pM'. Hence the ideal pM' splits into l different prime ideals $pM' = \pi_1' \dots \pi_l'$ that are permuted transitively by the Galois group $\mathrm{Gal}(C/\mathbb{Q})$ [Lang 1970]. For the ideal pM in the maximal order M of the maximal totally real subfield K the following two situations may occur:

1) $pM = \pi_1 \dots \pi_l$ where $\pi_i M' = \pi_i'$, or

2) $pM = \pi_1 \dots \pi_{l/2}$ where $\pi_i M' = \pi_{2i-1}' \pi_{2i}'$.

In the first case $e_i F : X_i \to \mathrm{Hom}_{\mathbb{Z}_p}(X_i, \mathbb{Z}_p)$ induces a quadratic form on the lattice X_i, for $1 \leq i \leq l$. The dual lattice $L^{\#(F)}$ is of the form $L^{\#(F)} = p^{\alpha_1} X_1 \oplus \dots \oplus p^{\alpha_l} X_l$ for some $\alpha_i \in \mathbb{Z}$, for $1 \leq i \leq l$. Since $\Pi(K)$ satisfies condition (a), there are $y_i \in M$ with $\pi_i = (y_i, p)$ such that the prime divisors of the norm of y_i lie in $\Pi \cup \{p\}$, for $1 \leq i \leq l$. Define $F_1 := y_1^{\alpha_1} \dots y_l^{\alpha_l} F$. Then p does not divide the determinant $\det(F_1, L)$ of F_1 on L and all prime divisors of $\det(F_1, L)$ either divide $\det(F, L)$ or lie in Π. Because of condition (b) on Π one may choose $y \in M$ such that yF_1 is positive definite and the prime divisors of the norm of y lie in Π. If the endomorphism ring of L is not the maximal order $M' \subseteq C$, then the form yF_1 need not be integral on L. But then there is some $m \in \mathbb{N}$ such that $F' := myF_1$ is integral on L and the prime divisors of m divide the group order $|G|$.

In the second case F induces isomorphisms

$$X_{2i-1} \cong \mathrm{Hom}_{\mathbb{Z}_p}(X_{2i}, \mathbb{Z}_p).$$

Hence the dual lattice $L^{\#(F)}$ is of the form

$$L^{\#(F)} = p^{\alpha_1}(X_1 \oplus X_2) \oplus \dots \oplus p^{\alpha_{l/2}}(X_{l-1} \oplus X_l)$$

for some $\alpha_i \in \mathbb{Z}$. As in the first case one constructs a G-invariant quadratic form $F' \in \mathcal{F}_{>0}(G)$ integral on L, such that the prime divisors of $\det(F', L)$ either lie in Π or divide $\det(F, L)$ and $p \nmid \det(F', L)$. □

As shown by the example of the two r.i.m.f. groups $[D_{120}.C_2]_{16,1}$ and $[D_{120}.C_2]_{16,2}$ of $\mathrm{GL}_{16}(\mathbb{Q})$ with 4-dimensional spaces of invariant forms, the set $\Pi(K)$ is necessary. The two r.i.m.f. groups leave invariant no positive definite lattice whose determinant is only divisible by primes < 11. The commuting algebras of the above two groups are both isomorphic to $\mathbb{Q}[\sqrt{3}, \sqrt{5}]$, a number field whose maximal order contains no totally positive prime element dividing 11.

Theorem 2.2 is applied via the following corollary, which is also referred to as the *m-parameter argument*, where $m := [K : \mathbb{Q}]$ is the dimension of $\mathcal{F}(U)$.

Corollary 2.3. *Let $U \leq \mathrm{GL}_n(\mathbb{Q})$ be a finite irreducible matrix group whose commuting algebra $C := C_{\mathbb{Q}^{n\times n}}(U)$ is commutative. Let $L \in \mathcal{Z}(U)$ be a $\mathbb{Z}U$-lattice and $G \leq \mathrm{GL}_n(\mathbb{Q})$ a finite supergroup of U acting on L. In the notation of Theorem 2.2, let K be the maximal totally real subfield of C, and denote by $\tilde{\Pi} = \tilde{\Pi}(K, |G|)$ the union of $\Pi(K')$ over all subfields K' of K. Then there exists $F \in \mathcal{F}_{>0}(G)$, primitive on L, such that the prime divisors of $\det(F, L)$ lie in $\tilde{\Pi}$.*

Proof. The commuting algebra of G is a subfield of C and its maximal totally real subfield K' is a subfield of K. Hence $\Pi(K') \subseteq \tilde{\Pi}$ and the statement follows from Theorem 2.2. □

Another important method deals with primitive r.i.m.f. supergroups G of (reducible) subgroups N of $\mathrm{GL}_n(\mathbb{Q})$ containing N as normal subgroup. The idea is to construct a G-invariant order Λ_0 in the enveloping algebra $\bar{N}$ containing the $\mathbb{Z}$-order generated by the matrices in N. Since the $\mathbb{Z}$-module generated by Λ_0 and the matrices in G is again an order, there is a Λ_0-lattice on which G acts.

We recall the radical idealizer process [Benz and Zassenhaus 1985]. Let Λ be a $\mathbb{Z}$-order in a simple $\mathbb{Q}$-algebra A. The *arithmetic (right) radical* $\mathrm{AR}_r(\Lambda)$ of Λ is defined as the intersection of all those maximal right ideals of Λ that contain the discriminant ideal of Λ. The arithmetic radical is a full $\mathbb{Z}$-module in A. Its *(right) idealizer* $\mathrm{Id}_r(\mathrm{AR}_r(\Lambda))$ is defined as the set of all elements $a \in A$ such that $\mathrm{AR}_r(\Lambda)a \subseteq \mathrm{AR}_r(\Lambda)$; this again is a $\mathbb{Z}$-order in A containing Λ. The repeated application of $\mathrm{Id}_r \circ \mathrm{AR}_r$ is called the *radical idealizer process.* It constructs a finite ascending chain of $\mathbb{Z}$-orders in A. The maximal element $(\mathrm{Id}_r \circ \mathrm{AR}_r)^\infty(\Lambda)$ of this chain is necessarily a hereditary order in A [Reiner 1975, pp. 356–358].

Definition 2.4. Let $N \leq \mathrm{GL}_n(\mathbb{Q})$ be a finite group and let $F \in \mathcal{F}_{>0}(N)$. Assume that the algebra $\bar{N} \leq \mathbb{Q}^{n\times n}$ generated by the matrices in N is simple. Then the natural $\mathbb{Q}N$-module $\mathbb{Q}^{1\times n}$ decomposes into a direct sum of l copies of an irreducible $\mathbb{Q}N$-module V. Let Λ_0 be the hereditary order in $\bar{N}$ obtained applying the radical idealizer process to the order $\langle N\rangle_{\mathbb{Z}}$. Let $L_1, \dots, L_s \subseteq V$ be representatives of the isomorphism classes of the irreducible Λ_0-lattices in V. Then the *generalized Bravais group* $\mathcal{B}^\circ(N)$ of N is defined as the set of $g \in \bar{N}$ such that $gFg^{\mathrm{tr}} = F$ and $L_i g = L_i$ for $1 \leq i \leq s$.

Proposition 2.5 [Nebe and Plesken 1995, II.10]. *Let G be a primitive r.i.m.f. group in $\mathrm{GL}_n(\mathbb{Q})$ with $N \trianglelefteq G$. Then $N \trianglelefteq \mathcal{B}^\circ(N) \trianglelefteq G$. Moreover, if X is a finite subgroup of the unit group $\bar{N}^*$ of $\bar{N}$ with $N \trianglelefteq X$, then $X \leq \mathcal{B}^\circ(N)$.*

This proposition is a good criterion for deciding which normal subgroups of primitive r.i.m.f. groups may occur. It implies that a primitive r.i.m.f. group has no normal subgroup N conjugate to $L_2(8)$, $2.\mathrm{Alt}_8$, Alt_9, $2.\mathrm{Alt}_9$ or $2.\mathrm{Sp}_6(2)$, where the natural character of N is a multiple of the absolutely irreducible rational character of degree 8 of N, since N is not normal in $\mathcal{B}^\circ(N) = 2.O_8^+(2).2$. Compare Table 4.

3. RESULTS

Theorem 3.1. *Up to conjugacy, there are 65 r.i.m.f. subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$, of which 41 are primitive groups. Four r.i.m.f. groups are not absolutely irreducible, and three of them even have a two-dimensional space of invariant forms. Twenty-eight of the r.i.m.f. groups leave modular lattices invariant.*

These 65 groups are listed in Table 1, whose information is arranged as follows.

In the first column, the number of the r.i.m.f. group $G = \mathrm{Aut}(L)$ is given, followed by a name for the group. The imprimitive groups (Definition 1.14) are the ones denoted by X_d^k, where X_d is a name for a primitive r.i.m.f. subgroup of $\mathrm{GL}_d(\mathbb{Q})$ and $dk = 24$. The additional abbreviations (s) and (ns) state whether the group G constructed according to Definition 1.9 is split or nonsplit in the sense of Definition 1.11. I thank a referee who encouraged me to make the description of some of the r.i.m.f. groups precise in this way.

The next three columns of the table give information about the G-invariant lattice L of minimal determinant. First the isomorphism type of $L^\#/L$ is given. The information in this column also allows us to recover the symbol of the invariant form in the Witt ring $W(\mathbb{Q})$, as proposed in [DeMeyer et al. 1989, p. 9]; see [Plesken and Nebe 1995] and [Nebe 1995]. The next column gives the minimum of the lattice, and the following one the number of shortest vectors of L decomposed into orbits under G. The order of G is then given. The column headed "sparse?" shows the primes p for which the group G is p-lattice sparse, or the word "yes" if the group is lattice sparse. Finally, on lines 42, 59, 60, and 65, the last column gives the isomorphism type of the commuting algebra of G; for the remaining groups this information is omitted since they are absolutely irreducible.

The groups are ordered with respect to connected components in the simplicial complexes $M_{24}^{\mathrm{irr}}(\mathbb{Q})$ and $M_{24}^{\mathrm{irr},F}(\mathbb{Q})$ [Nebe 1995] and with respect to the determinants of an invariant lattice of minimal determinant.

	lattice L	$\det L$	min	$\lvert L_{\min}\rvert$	$\lvert\mathrm{Aut}(L)\rvert$	sparse?
1	B_{24}	1	1	48	$2^{24}\cdot 24!$	$p \neq 2$
2	E_8^3	1	2	720	$2^{43}\cdot 3^{16}\cdot 5^6\cdot 7^3$	yes
3	$[2.\mathrm{Co}_1]_{24}$	1	4	196560	$2^{22}\cdot 3^9\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 23$	yes
4	$[(\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2\underset{\sqrt{5}}{\overset{2}{\boxtimes}}\mathrm{Alt}_5]_{24,2}(\mathrm{ns})$	2^8	4	3600+8640	$2^9\cdot 3^3\cdot 5^3$	$p \neq 5$
5	F_4^6	2^{12}	2	144	$2^{46}\cdot 3^{14}\cdot 5$	yes
6	$[6.U_4(3).2\underset{\sqrt{-3}}{\overset{2}{\boxtimes}}\mathrm{SL}_2(3)]_{24}$	2^{12}	4	3024	$2^{12}\cdot 3^8\cdot 5\cdot 7$	$p \neq 3$
7	E_6^4	3^4	2	288	$2^{35}\cdot 3^{17}\cdot 5^4$	yes
8	$[(\pm 3).\mathrm{PGL}_2(9)\overset{2(2)}{\circ}\mathrm{SL}_2(3)]_{24}$	3^4	4	12960+6480+2160	$2^8\cdot 3^4\cdot 5$	$p \neq 2,3$
9	$[\mathrm{Sp}_4(3)\underset{\sqrt{-3}}{\overset{2}{\boxtimes}}(3_+^{1+2}:\mathrm{SL}_2(3))]_{24}$	3^8	4	2160	$2^{11}\cdot 3^8\cdot 5$	$p \neq 3$
10	A_2^{12}	3^{12}	2	72	$12^{12}\cdot 12!$	yes
11	$[6.U_4(3).2^2]_{12}^2$	3^{12}	4	1512	$2^{21}\cdot 3^{14}\cdot 5^2\cdot 7^2$	yes
12	$F_4\otimes E_6$	$2^{12}\cdot 3^4$	4	864	$2^{14}\cdot 3^6\cdot 5$	yes
13	$[3_+^{1+2}:\mathrm{SL}_2(3)\underset{\sqrt{-3}}{\overset{2}{\boxtimes}}\mathrm{SL}_2(3)]_{12}^2$	$2^{12}\cdot 3^8$	4	432	$2^{15}\cdot 3^{10}$	$p \neq 3$
14	$[3.S_6\overset{2(2)}{\otimes}D_8]_{24}$	$2^{12}\cdot 3^8$	4	144	$2^8\cdot 3^3\cdot 5$	$p \neq 3$
15	$(A_2\otimes F_4)^3$	$2^{12}\cdot 3^{12}$	4	216	$2^{25}\cdot 3^{10}$	yes
16	$[6.L_3(4).2\overset{2(2)}{\otimes}D_8]_{24}$	$2^{12}\cdot 3^{12}$	8	3024+7560	$2^{11}\cdot 3^3\cdot 5\cdot 7$	yes
17	$[(\mathrm{SL}_2(3)\circ C_4).2\underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}}U_3(3)]_{24}$	$2^{12}\cdot 3^{12}$	8	4536+6048	$2^{11}\cdot 3^4\cdot 7$	yes
18	A_{24}	25	2	600	$2\cdot(25!)$	$p \neq 5$
19	A_4^6	5^6	2	120	$2^{28}\cdot 3^8\cdot 5^7$	yes
20	$M_{6,2}^4$	5^{12}	3	80	$2^{19}\cdot 3^5\cdot 5^4$	$p \neq 2$
21	$[(\mathrm{SL}_2(5)\overset{2}{\square}\mathrm{SL}_2(5)):2]_8^3(\mathrm{s})$	5^{12}	4	360	$2^{22}\cdot 3^7\cdot 5^6$	yes
22	$[2.J_2\overset{2}{\square}\mathrm{SL}_2(5)]_{24}(\mathrm{s})$	5^{12}	8	37800	$2^{11}\cdot 3^4\cdot 5^3\cdot 7$	yes
23	$[\pm D_{10}\underset{\sqrt{5}}{\overset{2}{\boxtimes}}\mathrm{Alt}_5]_{12}^2$	$2^8\cdot 5^6$	4	240+600	$2^{11}\cdot 3^2\cdot 5^4$	$p \neq 5$
24	$[\mathrm{SL}_2(5)\overset{2(2)}{\circ}\mathrm{SL}_2(3)]_{12}^2$	$2^4\cdot 5^8$	4	720	$2^{13}\cdot 3^4\cdot 5^2$	$p \neq 2$
25	$[\mathrm{SL}_2(5)\underset{\infty,2}{\overset{2(2)}{\otimes}}2_-^{1+4'}.\mathrm{Alt}_5]_{24}$	$2^8\cdot 5^8$	6	2400	$2^{10}\cdot 3^2\cdot 5^2$	$p \neq 2$
26	$[(\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2\underset{\sqrt{5}}{\overset{2}{\boxtimes}}\mathrm{Alt}_5]_{24,1}(\mathrm{s})$	$2^8\cdot 5^{12}$	8	1800	$2^9\cdot 3^3\cdot 5^3$	yes
27	$F_4\otimes M_{6,2}$	$2^8\cdot 5^{12}$	6	240	$2^{10}\cdot 3^3\cdot 5$	$p \neq 2$
28	$[\mathrm{SL}_2(5)\underset{\infty,3}{\overset{2}{\boxtimes}}(\pm 3_+^{1+2}).\mathrm{GL}_2(3)]_{24}$	$3^8\cdot 5^{12}$	8	1080	$2^8\cdot 3^5\cdot 5$	$p \neq 3$
29	$(A_2\otimes M_{6,2})^2$	$3^{12}\cdot 5^{12}$	6	120	$2^{11}\cdot 3^4\cdot 5^2$	yes
30	$[\pm 3.\mathrm{Alt}_6.2^2]_{12}^2$	$3^{12}\cdot 5^{12}$	8	540	$2^{13}\cdot 3^6\cdot 5^2$	yes
31	$A_2\otimes[\mathrm{SL}_2(5)\overset{2(2)}{\circ}\mathrm{SL}_2(3)]_{12}$	$2^4\cdot 3^{12}\cdot 5^8$	8	1080	$2^7\cdot 3^3\cdot 5$	$p \neq 2$
32	A_6^4	7^4	2	168	$2^{23}\cdot 3^9\cdot 5^4\cdot 7^4$	yes

TABLE 1. The r.i.m.f. groups of degree 24. The meaning of the columns is explained on the preceding page.

	lattice L	$\det L$	min	$\lvert L_{\min}\rvert$	$\lvert\mathrm{Aut}(L)\rvert$	sparse?	comm. alg.
33	$[L_2(7)\overset{2(2)}{\otimes}F_4]_{24}$	7^4	4	1008+3024	$2^{11}\cdot 3^3\cdot 7$	$p\neq 2$	
34	$(A_6^{(2)})^4$	7^{12}	4	168	$2^{23}\cdot 3^5\cdot 7^4$	yes	
35	$[L_2(7)\overset{2(2)}{\boxtimes}F_4]_{24}$	7^{12}	8	1008+3024	$2^{11}\cdot 3^3\cdot 7$	$p\neq 2$	
36	$[L_2(7)\overset{2(2)}{\otimes}D_8]_{12}^2$	$2^{12}\cdot 7^4$	4	$2\cdot 336$	$2^{15}\cdot 3^2\cdot 7^2$	yes	
37	$F_4\otimes A_6$	$2^{12}\cdot 7^4$	4	504	$2^{11}\cdot 3^4\cdot 5\cdot 7$	yes	
38	$[L_2(7)\overset{2(2)}{\boxtimes}D_8]_{12}^2$	$2^{12}\cdot 7^{12}$	8	$2\cdot 336$	$2^{15}\cdot 3^2\cdot 7^2$	yes	
39	$F_4\otimes A_6^{(2)}$	$2^{12}\cdot 7^{12}$	8	504	$2^{11}\cdot 3^3\cdot 7$	yes	
40	$[\mathrm{SL}_2(13)\overset{2(2)}{\Box}\mathrm{SL}_2(3)]_{24}(\mathrm{s})$	13^{12}	12	$2\cdot 2184$+8736	$2^6\cdot 3^2\cdot 7\cdot 13$	$p\neq 2$	
41	$[\mathrm{SL}_2(7)\underset{\sqrt{-7}}{\overset{2}{\boxtimes}}L_2(7)]_{24}$	2^6	4	2352+8064+14112	$2^8\cdot 3^2\cdot 7^2$	$p\neq 7$	
42	$[6.\mathrm{Alt}_7:2]_{24}$	2^{12}	4	3024	$2^5\cdot 3^3\cdot 5\cdot 7$	yes	$\mathbb{Q}[\sqrt{-6}]$
43	$[3.M_{10}\underset{\sqrt{-3}}{\overset{2(2)}{\boxtimes}}\mathrm{SL}_2(3)]_{24}$	$2^{12}\cdot 5^{12}$	8	1080	$2^8\cdot 3^4\cdot 5$	$p\neq 3$	
44	$[\mathrm{Alt}_5\underset{\sqrt{5}}{\overset{2}{\boxtimes}}(C_3\overset{2(2)}{\boxtimes}D_8)]_{24}$	$2^8\cdot 3^{12}\cdot 5^{12}$	10	144	$2^7\cdot 3^2\cdot 5$	$p\neq 2$	
45	$[3.M_{10}\overset{2(2)}{\boxtimes}D_8]_{24}$	$2^{12}\cdot 3^{12}\cdot 5^{12}$	16	1080+1080	$2^8\cdot 3^3\cdot 5$	yes	
46	$(A_2\otimes A_6)^2$	$3^{12}\cdot 7^4$	4	252	$2^{13}\cdot 3^6\cdot 5^2\cdot 7^2$	yes	
47	$(A_2\otimes A_6^{(2)})^2$	$3^{12}\cdot 7^{12}$	8	252	$2^{13}\cdot 3^4\cdot 7^2$	yes	
48	$A_2\otimes[L_2(7)\overset{2(2)}{\otimes}D_8]_{12}$	$2^{12}\cdot 3^{12}\cdot 7^4$	8	$2\cdot 504$	$2^8\cdot 3^2\cdot 7$	yes	
49	$A_2\otimes[L_2(7)\overset{2(2)}{\boxtimes}D_8]_{12}$	$2^{12}\cdot 3^{12}\cdot 7^{12}$	16	$2\cdot 504$	$2^8\cdot 3^2\cdot 7$	yes	
50	A_{12}^2	13^2	2	312	$(2\cdot 13!)^2\cdot 2$	yes	
51	$[(\pm L_3(3)).2\overset{2}{\Box}C_3]_{24}$	13^2	4	936+5616+8424	$2^7\cdot 3^4\cdot 13$	$p\neq 3$	
52	$A_2\otimes A_{12}$	$3^{12}\cdot 13^2$	4	468	$12\cdot 13!$	yes	
53	$[(\pm D_{78}).C_{12}]_{24}$	$3^{12}\cdot 13^2$	6	624+936	$2^4\cdot 3^2\cdot 13$	$p\neq 13$	
54	$A_4\otimes E_6$	$3_+^4\cdot 5_+^6$	4	720	$2^{11}\cdot 3^5\cdot 5^2$	yes	
55	$(A_2\otimes A_4)^3$	$3_+^{12}\cdot 5_+^6$	4	180	$2^{16}\cdot 3^7\cdot 5^3$	yes	
56	$[\pm 3.\mathrm{PGL}_2(9)\underset{\sqrt{5}}{\overset{2}{\boxtimes}}D_{10}]_{24}$	$3_+^{12}\cdot 5_+^6$	8	2700+2160+1080	$2^7\cdot 3^3\cdot 5^2$	$p\neq 5$	
57	$A_2\otimes[\pm D_{10}\underset{\sqrt{5}}{\overset{2}{\boxtimes}}\mathrm{Alt}_5]_{12}$	$2^8\cdot 3_+^{12}\cdot 5_+^6$	8	360+900	$2^6\cdot 3^2\cdot 5^2$	$p\neq 5$	
58	$[\pm U_4(2).2]_{24}$	$2^8\cdot 3_+^{10}\cdot 5_+$	6	240+1440	$2^8\cdot 3^4\cdot 5$	$p\neq 2$	
59	$[\mathrm{SL}_2(7)\circ\tilde{S}_4]_{24}$	7^4	4	$2\cdot 1008$+2016	$2^7\cdot 3^2\cdot 7$	$p\neq 2$	$\mathbb{Q}[\sqrt{2}]$
60	$[\mathrm{SL}_2(7)\overset{2}{\circ}Q_{16}]_{24}$	$2^6\cdot 7^4$	4	336	$2^8\cdot 3\cdot 7$	yes	$\mathbb{Q}[\sqrt{2}]$
61	$M_{8,3}^3$	$3_+^3\cdot 7^9$	4	252	$2^{16}\cdot 3^4\cdot 7^3$	yes	
62	$A_4\otimes A_6$	$5_+^6\cdot 7_+^4$	4	420	$2^8\cdot 3^3\cdot 5^2\cdot 7$	yes	
63	$A_4\otimes A_6^{(2)}$	$5_+^6\cdot 7_+^{12}$	8	420	$2^8\cdot 3^2\cdot 5\cdot 7$	yes	
64	$[\mathrm{SL}_2(11)\underset{\sqrt{-11}}{\overset{2(2)}{\boxtimes}}\mathrm{SL}_2(3)]_{24}$	$2^{12}\cdot 11^{12}$	12	1320	$2^6\cdot 3^2\cdot 5\cdot 11$	$p\neq 2$	
65	$[\pm L_2(11):2]_{24}$	$5\cdot 11_+^8$	6	$2\cdot 220$+660	$2^4\cdot 3\cdot 5\cdot 11$	yes	$\mathbb{Q}[\sqrt{5}]$

4. THE R.I.M.F. SUPERGROUPS OF IRREDUCIBLE CYCLIC GROUPS

We now determine the r.i.m.f. supergroups of the irreducible cyclic groups in $\mathrm{GL}_{24}(\mathbb{Q})$. Since 40 of the r.i.m.f. subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$ contain irreducible cyclic subgroups, Table 2 provides many shortcuts, used in the main proof of the classification theorem given in Section 6. The table is of independent interest in view of the study of cyclotomic lattices. Note that the unimodular lattices fixed by the irreducible cyclic subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$ have already been determined in [Bayer-Fluckiger 1984].

Let $\pm 1 \leq U \cong C_m$ be an irreducible subgroup of $\mathrm{GL}_{24}(\mathbb{Q})$. Then $m = 70, 78, 90, 52, 56, 72$, or 84. The isomorphism classes of $\mathbb{Z}U$-lattices in $\mathbb{Q}^{1\times 24}$ correspond to the ideal classes of the respective cyclotomic fields $\mathbb{Q}[\zeta_m]$. The respective class numbers are 1, 2, 1, 3, 2, 3, 1 [Washington 1982].

For brevity's sake, we restrict ourselves to a brief discussion of the case $U \cong C_{78}$, which illustrates all

U	Π	r.i.m.f. supergroups
C_{70}	2, 3, 5, 7	$[2.\mathrm{Co}_1]_{24}$, $[2.J_2 \overset{2}{\square} \mathrm{SL}_2(5)]_{24}$, $A_4 \otimes A_6$, $A_4 \otimes A_6^{(2)}$
C_{78}	2, 3, 7, 11, 13	$[2.\mathrm{Co}_1]_{24}$, $[\mathrm{SL}_2(13) \overset{2(2)}{\sqsupset} \mathrm{SL}_2(3)]_{24}$, $[\pm L_3(3).2 \overset{2}{\square} C_3]_{24}$, $A_2 \otimes A_{12}$, $[(\pm D_{78}).C_{12}]_{24}$
C_{90}	2, 3, 5, 7, 13	E_8^3, $[\mathrm{Sp}_4(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} (3_+^{1+2}:\mathrm{SL}_2(3))]_{24}$, $[(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8^3$, $[\mathrm{SL}_2(5) \underset{\infty,3}{\overset{2}{\boxtimes}} (\pm 3_+^{1+2}).\mathrm{GL}_2(3)]_{24}$, $A_4 \otimes E_6$, $(A_2 \otimes A_4)^3$
C_{52}	2, 3, 7, 11, 13	$[2.\mathrm{Co}_1]_{24}$, $[\mathrm{SL}_2(13) \overset{2(2)}{\sqsupset} \mathrm{SL}_2(3)]_{24}$, A_{12}^2
C_{56}	2, 3, 7	$[2.\mathrm{Co}_1]_{24}$, $[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, $[6.U_4(3).2^2]_{12}^2$, $[(\mathrm{SL}_2(3) \circ C_4).2 \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} U_3(3)]_{24}$, A_6^4, $[L_2(7) \overset{2(2)}{\otimes} F_4]_{24}$, $(A_6^{(2)})^4$, $[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$, $[L_2(7) \overset{2(2)}{\otimes} D_8]_{12}^2$, $F_4 \otimes A_6$, $[L_2(7) \overset{2(2)}{\boxtimes} D_8]_{12}^2$, $F_4 \otimes A_6^{(2)}$, $[\mathrm{SL}_2(7) \underset{\sqrt{-7}}{\overset{2}{\boxtimes}} L_2(7)]_{24}$, $[\mathrm{SL}_2(7) \circ \tilde{S}_4]_{24}$, $[\mathrm{SL}_2(7) \overset{2}{\circ} Q_{16}]_{24}$
C_{72}	2, 3	E_8^3, F_4^6, $[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, E_6^4, $[\mathrm{Sp}_4(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} (3_+^{1+2}:\mathrm{SL}_2(3))]_{24}$, A_2^{12}, $F_4 \otimes E_6$, $[3_+^{1+2}:\mathrm{SL}_2(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{12}^2$, $(A_2 \otimes F_4)^3$
C_{84}	2, 3, 5, 7	$[2.\mathrm{Co}_1]_{24}$, $[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, $[6.U_4(3).2^2]_{12}^2$, $[6.L_3(4).2 \overset{2(2)}{\otimes} D_8]_{24}$, $[(\mathrm{SL}_2(3) \circ C_4).2 \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} U_3(3)]_{24}$, $[L_2(7) \overset{2(2)}{\otimes} F_4]_{24}$, $[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$, $F_4 \otimes A_6$, $F_4 \otimes A_6^{(2)}$, $(A_2 \otimes A_6)^2$, $(A_2 \otimes A_6^{(2)})^2$, $A_2 \otimes [L_2(7) \overset{2(2)}{\otimes} D_8]_{12}$, $A_2 \otimes [L_2(7) \overset{2(2)}{\boxtimes} D_8]_{12}$

TABLE 2. Supergroups of the irreducible cyclic subgroups of $\mathrm{GL}_{24}(\mathbb{Q})$. The first column gives the irreducible cyclic subgroup $U \leq \mathrm{GL}_{24}(\mathbb{Q})$, the second column contains a set Π of primes, and the last column gives the r.i.m.f. supergroups G of U such that either the prime divisors of $|G|$ or those of the determinant of an integral lattice $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ lie in Π. It turns out that this list of r.i.m.f. supergroups in the third column of the table is complete except for $U \cong C_{56}$, which is also a subgroup of the r.i.m.f. group $[2.J_2 \overset{2}{\square} \mathrm{SL}_2(5)]_{24}$.

the difficulties. The other six subgroups U can be dealt with similarly. For a detailed discussion of all seven cases see [Nebe 1995].

Let G be an r.i.m.f. supergroup of $U := C_{78}$ such that either the prime divisors of $|G|$ or those of the determinant of an integral lattice $(L, F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ lie in $\Pi := \{2, 3, 7, 11, 13\}$. Since the class number of $\mathbb{Q}[\zeta_{78}]$ is 2 [Washington 1982], there are two isomorphism classes of $\mathbb{Z}U$-lattices in $\mathbb{Q}^{1\times 24}$. Representatives L and L' can be chosen such that L contains L' of index 13. A convenient method to see that the two lattices L and L' are nonisomorphic is to compute the respective Bravais groups, which are $\mathcal{B}(U, L) \cong \pm D_{78}$ and $\mathcal{B}(U, L') = U$.

One finds a form $F_0 \in \mathcal{F}_{>0}(U)$ with $\det(F_0, L) = 13^2$. The space $\mathcal{F}(U)$ may be identified with the maximal totally real subfield $K := \mathbb{Q}[\zeta_{78} + \zeta_{78}^{-1}]$ of $\mathbb{Q}[\zeta_{78}]$ via the $\mathbb{Q}$-isomorphism $K \to \mathcal{F}(U)$ mapping c to cF_0. The positive definite forms in $\mathcal{F}(U)$ correspond to the totally positive elements in K.

By Corollary 2.3, G is of the form $G = \mathrm{Aut}(F, L)$ or $G = \mathrm{Aut}(F, L')$ for some $F \in \mathcal{F}_{>0}(U)$ such that the prime divisors of $\det(F, L)$ or $\det(F, L')$ lie in the finite set $\tilde{\Pi} := \tilde{\Pi}(K, |G|)$. With KANT [Pohst et al. 1993] one decides that $\tilde{\Pi}$ may be chosen as $\tilde{\Pi} := \{2, 3, 7, 11, 13, 17\}$ if $|G|$ only involves the prime divisors 2, 3, 7, 11 and 13. A more detailed analysis of the decomposition of 17 in the subfields of K shows that the prime 17 can be omitted. The primes 2, 7, and 11 are inert in K; 13 is totally ramified since $13 = p_{13}^{12}e$ for some prime element p_{13} and unit $e \in K$, and 3 decomposes as $3 = p_3^2(p_3')^2$ for some prime elements $p_3, p_3' \in K$. For both lattices L and L' there is an element $n \in N_{\mathrm{GL}_{24}(\mathbb{Z})}(U)$ conjugating the ideal generated by p_3 into the one generated by p_3'. There is no totally positive prime dividing 3 or 13, but p_3, p_3' and p_{13} can be chosen such that p_3p_3', p_3p_{13} and $p_3'p_{13}$ are totally positive. The set $I := \{uF_0u^{\mathrm{tr}}F_0^{-1} \mid u \in \mathbb{Z}[\zeta_{78}]^*\}$ forms a subgroup of index two in the group of all totally positive units of the maximal order of K. Note that if $^-$ denotes the "complex conjugation" in $\mathbb{Q}[\zeta_{78}]$, that is, the automorphism defined by $\zeta_{78} \mapsto \zeta_{78}^{-1}$, then $F_0u^{\mathrm{tr}}F_0^{-1} = \bar{u}$ for all $u \in \mathbb{Q}[\zeta_{78}]$. Since u is an isometry between the lattices $(L, u\bar{u}cF_0)$ and

F	$\mathrm{Aut}(F, L)$			$\mathrm{Aut}(v_0F, L)$			$\mathrm{Aut}(F, L')$			$\mathrm{Aut}(v_0F, L')$		
F_0	G_{51}	4	14976	G_{51}	4	14976	B_1	4	936	B_1	4	936
$p_{13}^2F_0$	B_2	6	8424	B_2	6	8424	B_1	6	1248	B_1	6	1248
$p_{13}^4F_0$	B_2	6	156	B_2	6	156	G_{40}	12	13104	G_{40}	12	13104
$p_{13}^{-2}F_0$							G_3	4	196560	G_3	4	196560
$p_3p_{13}F_0$	B_3	6	3588	B_3	6	3588	B_3	8	8190	B_3	8	8190
$p_3p_{13}^3F_0$	B_3	8	468	B_3	8	468	B_3	10	468	B_3	10	468
$p_3p_{13}^5F_0$	B_3	12	468	B_3	12	468						
$p_3p_{13}^{-1}F_0$	B_3	4	7020	B_3	4	7020	B_3	4	468	B_3	4	468
$p_3^2F_0$	B_2	6	2028	B_2	6	2964	B_4	6	312	B_4	6	312
$p_3^2p_{13}^2F_0$	B_2	8	468	B_2	10	3276	B_4	12	2808	B_4	12	2808
$p_3^2p_{13}^4F_0$	B_2	12	156	B_2	12	312	B_5	22	13104	B_5	22	13104
$p_3^2p_{13}^{-2}F_0$							B_5	6	26208	B_5	6	26208
$p_3p_3'F_0$	G_{52}	4	468	G_{53}	6	1560	B_6	6	312	B_6	6	312
$p_3p_3'p_{13}^2F_0$	B_7	8	1170	G_{53}	8	468	B_6	12	2028	B_6	12	2028
$p_3p_3'p_{13}^4F_0$	B_7	12	936	G_{53}	12	156	B_6	16	234	B_6	16	234
$p_3p_3'p_{13}^{-2}F_0$							B_6	4	468	B_6	4	468

TABLE 3. For each relevant $\mathbb{Z}C_{78}$ lattice, the table shows the corresponding automorphism group, the minimum, and the number of shortest vectors.

(L, cF_0) for all totally positive $c \in K$, one only has to consider representatives of the I-orbits on the set of totally positive elements of K.

Let v_0 be any totally positive unit of K not contained in I. Taking only normalized lattices, one concludes that G is one of the groups in Table 3.

The subscripts i of the groups G_i refer to the number of the r.i.m.f. group G_i in Table 1. The groups B_i are not maximal finite; we have

$$
\begin{array}{ll}
B_1 = C_{26}\,.\,C_6 \overset{2(2)}{\Box} \mathrm{SL}_2(3), & \\
B_2 = (\pm C_{13} : C_3 \otimes S_3)\,.\,2 & \text{with algebra } \mathbb{Q}[\sqrt{13}], \\
B_3 = \pm C_3 \overset{2}{\boxtimes} (C_{13} : C_3) & \text{with algebra } K', \\
B_4 = C_{26}\,.\,C_6 \bigcirc \mathrm{SL}_2(3) & \text{with algebra } \mathbb{Q}[\sqrt{13}], \\
B_5 = \mathrm{SL}_2(13) \bigcirc \mathrm{SL}_2(3) & \text{with algebra } \mathbb{Q}[\sqrt{13}], \\
B_6 = \pm C_{13} : C_{12} \otimes C_3 & \text{with algebra } \mathbb{Q}[\sqrt{-3}], \\
B_7 = \pm C_{13} : C_{12} \otimes S_3\,. &
\end{array}
$$

(Here we give, for the not absolutely irreducible groups B_i, the isomorphism type of the commuting algebra; K' means the subfield of $\mathbb{Q}[\zeta_{78} + \zeta_{78}^{-1}]$ with $[K' : \mathbb{Q}] = 4$.)

Note that the isometric lattices $(L', p_3^2 p_{13}^{-2} F_0)$ and $(L',\, v_0 p_3^2 p_{13}^{-2} F_0)$ are extremal 3-modular lattices.

5. ESSENTIALLY SEMISIMPLE GROUPS

In this section we determine the primitive r.i.m.f. groups G such that $G^{(\infty)}$ is an irreducible central product of quasisimple groups.

Table 4 summarizes the information that we need from the classification of finite simple groups. It displays all quasisimple groups having an irreducible rational representation of degree d dividing 24. A list of candidates for quasisimple normal subgroups N of the r.i.m.f. groups $G \leq \mathrm{GL}_{24}(\mathbb{Q})$ can be obtained from this table by taking those groups N that are normal in their generalized Bravais group $\mathcal{B}^{\circ}(N)$ (Definition 2.4); this group is given in the second column. The fourth column fixes the notation for the natural character of $N \leq \mathrm{GL}_d(\mathbb{Q})$, used in Section 6. The isomorphism type of the commuting algebra $C_{\mathbb{Q}^{d\times d}}(N)$ is also given. The last column refers to the page in [Holt and Plesken 1989] where a detailed description of the $\mathbb{Z}N$-lattice in $\mathbb{Q}^{1\times d}$ may be found.

That the table follows from the classification of finite simple groups and from [Conway et al. 1985] can be seen as follows. In [Landazuri and Seitz 1974; Seitz and Zalesskii 1993] it is shown that most of the finite Chevalley groups having a projective representation of degree at most 24 are contained in [Conway et al. 1985]. The exceptions are:

1. some linear groups that have no representation of degree $d \mid 24$ because of their group order and the formula in [Schur 1905];
2. the group $L_2(49)$, whose 24-dimensional representation has real Schur index 2; and
3. the group $\mathrm{Sp}_4(7)$, for which the character field of the two conjugate 24-dimensional representations is $\mathbb{Q}[\sqrt{-7}]$ [Srinivasan 1968].

A case-by-case discussion using Theorem 2.1, Tables 2 and 4, and the inclusions of the finite simple groups from the lists of maximal subgroups in [Conway et al. 1985] yields the following two propositions:

Proposition 5.1. *Let G be a primitive r.i.m.f. subgroup of $\mathrm{GL}_{24}(\mathbb{Q})$ such that $G^{(\infty)}$ is irreducible and quasisimple. Then G is conjugate to one of these eight r.i.m.f. groups*: $[2\,.\,\mathrm{Co}_1]_{24}$, $[6\,.\,\mathrm{Alt}_7 : 2]_{24}$, A_{24}, $[\pm U_4(2)\,.\,2]_{24}$, $[\mathrm{SL}_2(7) \bigcirc \tilde{S}_4]_{24}$, $[\mathrm{SL}_2(7) \overset{2}{\bigcirc} Q_{16}]_{24}$, $[\mathrm{SL}_2(13) \overset{2(2)}{\Box} \mathrm{SL}_2(3)]_{24}$, *or* $[\pm L_2(11) : 2]_{24}$.

Proposition 5.2. *Let G be a primitive r.i.m.f. subgroup of $\mathrm{GL}_{24}(\mathbb{Q})$ such that $G^{(\infty)}$ is irreducible and a central product of at least two quasisimple groups. Then G is conjugate to one of these r.i.m.f. groups*: $[\mathrm{SL}_2(7) \underset{\sqrt{-7}}{\overset{2}{\boxtimes}} L_2(7)]_{24}$, $[(\mathrm{SL}_2(5) \bigcirc \mathrm{SL}_2(5)) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} \mathrm{Alt}_5]_{24,2}$, $[2\,.\,J_2 \overset{2}{\Box} \mathrm{SL}_2(5)]_{24}$, $[(\mathrm{SL}_2(5) \bigcirc \mathrm{SL}_2(5)) : 2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} \mathrm{Alt}_5]_{24,1}$, $A_4 \otimes E_6$, $A_4 \otimes A_6$, *or* $A_4 \otimes A_6^{(2)}$.

As a referee pointed out, one might think that other elements of the Suzuki chain in the Conway group might give rise to r.i.m.f. groups G isoclinic to a maximal subgroup of $[2\,.\,\mathrm{Co}_1]_{24}$. That only the

N	$\mathcal{B}^\circ(N)$	d	character	comm. alg.	page
Alt_5	$\pm S_5$	4	χ_4	$\mathbb{Q}$	272
Alt_5	$\pm\mathrm{Alt}_5$	6	$\chi_{3a}+\chi_{3b}$	$\mathbb{Q}[\sqrt{5}]$	273
$\mathrm{SL}_2(5)$	$\mathrm{SL}_2(5)$	8	$2(\chi_{2a}+\chi_{2b})$	$\mathcal{Q}_{\sqrt{5},\infty,\infty}$	275
$\mathrm{SL}_2(5)$	$\mathrm{SL}_2(9)$	8	$2\chi_4$	$\mathcal{Q}_{\infty,3}$	274
$\mathrm{SL}_2(5)$	$\mathrm{SL}_2(5)$	12	$2\chi_6$	$\mathcal{Q}_{\infty,2}$	
$L_2(7)$	$\pm L_2(7)$	6	$\chi_{3a}+\chi_{3b}$	$\mathbb{Q}[\sqrt{-7}]$	290
$L_2(7)$	$\pm L_2(7)$	6	χ_6	$\mathbb{Q}$	291
$L_2(7)$	$\pm L_2(7){:}2$	8	χ_8	$\mathbb{Q}$	293
$\mathrm{SL}_2(7)$	$\mathrm{SL}_2(7)$	8	$\chi_{4a}+\chi_{4b}$	$\mathbb{Q}[\sqrt{-7}]$	295
$\mathrm{SL}_2(7)$	$\mathrm{SL}_2(7)$	24	$2(\chi_{6a}+\chi_{6b})$	$\mathcal{Q}_{\sqrt{2},\infty,\infty}$	
$\mathrm{SL}_2(9)$	$\mathrm{SL}_2(9)$	8	$2\chi_{4a}$ or $2\chi_{4b}$	$\mathcal{Q}_{\infty,3}$	311
$3.\mathrm{Alt}_6$	$\pm 3.\mathrm{Alt}_6$	12	$\chi_{3a}+\chi'_{3a}+\chi_{3b}+\chi'_{3b}$	$\mathbb{Q}[\sqrt{5},\sqrt{-3}]$	
$3.\mathrm{Alt}_6$	$\pm 3.\mathrm{Alt}_6$	12	$\chi_6+\chi'_6$	$\mathbb{Q}[\sqrt{-3}]$	
$6.\mathrm{Alt}_6$	$6.\mathrm{Alt}_6$	24	$\chi_{6a}+\chi'_{6a}+\chi_{6b}+\chi'_{6b}$	$\mathbb{Q}[\sqrt{2},\sqrt{-3}]$	
$L_2(8)$	$2.O_8^+(2).2$	8	χ_8	$\mathbb{Q}$	328
$L_2(11)$	$\pm L_2(11){:}2$	24	$\chi_{12a}+\chi_{12b}$	$\mathbb{Q}[\sqrt{5}]$	
$\mathrm{SL}_2(11)$	$\mathrm{SL}_2(11)$	12	$\chi_{6a}+\chi_{6b}$	$\mathbb{Q}[\sqrt{-11}]$	
$\mathrm{SL}_2(13)$	$\mathrm{SL}_2(13)$	24	$2(\chi_{6a}+\chi_{6b})$	$\mathcal{Q}_{\sqrt{13},\infty,\infty}$	
Alt_7	$\pm S_7$	6	χ_6	$\mathbb{Q}$	316
$2.\mathrm{Alt}_7$	$2.\mathrm{Alt}_7$	8	$\chi_{4a}+\chi_{4b}$	$\mathbb{Q}[\sqrt{7}]$	317
$3.\mathrm{Alt}_7$	$6.U_4(3).2$	12	$\chi_6+\chi'_6$	$\mathbb{Q}[\sqrt{-3}]$	
$6.\mathrm{Alt}_7$	$6.\mathrm{Alt}_7$	24	$\chi_{6a}+\chi'_{6a}+\chi_{6b}+\chi'_{6b}$	$\mathbb{Q}[\sqrt{2},\sqrt{-3}]$	
$L_3(3)$	$\pm L_3(3)$	12	χ_{12}	$\mathbb{Q}$	
$U_3(3)$	$\pm U_3(3)$	12	$2\chi_6$	$\mathcal{Q}_{\infty,3}$	
$\mathrm{SL}_2(23)$	$\mathrm{SL}_2(23)$	24	$\chi_{12a}+\chi_{12b}$	$\mathbb{Q}[\sqrt{-23}]$	
$\mathrm{SL}_2(25)$	$\mathrm{SL}_2(25)$	24	$2\chi_{12}$	$\mathcal{Q}_{\infty,5}$	
$2.\mathrm{Alt}_8$	$2.O_8^+(2).2$	8	χ_8	$\mathbb{Q}$	320
$6.L_3(4)$	$6.L_3(4)$	12	$\chi_6+\chi'_6$	$\mathbb{Q}[\sqrt{-3}]$	
$U_4(2)$	$\pm U_4(2){:}2$	6	χ_6	$\mathbb{Q}$	336
$U_4(2)$	$\pm U_4(2){:}2$	24	χ_{24}	$\mathbb{Q}$	
$\mathrm{Sp}_4(3)$	$\mathrm{Sp}_4(3)\circ C_3$	8	$\chi_{4a}+\chi_{4b}$	$\mathbb{Q}[\sqrt{-3}]$	338
$U_3(4)$	$2.G_2(4)$	24	$2\chi_{12}$	$\mathcal{Q}_{\infty,2}$	
$2.M_{12}$	$2.M_{12}$	12	χ_{12}	$\mathbb{Q}$	
Alt_9	$2.O_8^+(2).2$	8	χ_8	$\mathbb{Q}$	323
$2.\mathrm{Alt}_9$	$2.O_8^+(2).2$	8	χ_8	$\mathbb{Q}$	324
$2.J_2$	$2.J_2$	24	$2(\chi_{6a}+\chi_{6b})$	$\mathcal{Q}_{\sqrt{5},\infty,\infty}$	
$2.\mathrm{Sp}_6(2)$	$2.O_8^+(2).2$	8	χ_8	$\mathbb{Q}$	340
$6.U_4(3)$	$6.U_4(3).2$	12	$\chi_6+\chi'_6$	$\mathbb{Q}[\sqrt{-3}]$	
$2.O_8^+(2)$	$2.O_8^+(2).2$	8	χ_8	$\mathbb{Q}$	341
$2.G_2(4)$	$2.G_2(4)$	24	$2\chi_{12}$	$\mathcal{Q}_{\infty,2}$	
Alt_{13}	$\pm S_{13}$	12	χ_{12}	$\mathbb{Q}$	
$6.\mathrm{Suz}$	$6.\mathrm{Suz}$	24	$\chi_{12}+\chi'_{12}$	$\mathbb{Q}[\sqrt{-3}]$	
$2.\mathrm{Co}_1$	$2.\mathrm{Co}_1$	24	χ_{24}	$\mathbb{Q}$	
Alt_{25}	$\pm S_{25}$	24	χ_{24}	$\mathbb{Q}$	

TABLE 4. The quasisimple irreducible rational matrix groups of degree $d \mid 24$. The meaning of the columns is explained in the second paragraph of Section 5.

group $[2.J_2 \overset{2}{\square} \mathrm{SL}_2(5)]_{24}$ turns up is explained by Lemma 1.12, since the unique subgroup of index two in the other absolutely irreducible maximal subgroups of $[2.\mathrm{Co}_1]_{24}$ corresponding to elements of the Suzuki chain is uniform.

6. PROOF OF THEOREM 3.1

We now complete the proof of the classification in Theorem 3.1. The primitive r.i.m.f. groups are built up using normal subgroups. Therefore the proof is organized according to the possibilities for normal p-subgroups of the r.i.m.f. groups. The different primes p are dealt with in decreasing order, always assuming that p is the biggest prime with $O_p(G) \neq 1$. The ordering of the case-by-case discussion is adapted to the one in Table 5. In particular, the first eight possibilities for $O_2(G)$ are excluded in Lemma 6.14 by a uniform argument.

Theorem 6.1. *Let G be a primitive r.i.m.f. group in $\mathrm{GL}_{24}(\mathbb{Q})$ and let p is a prime such that $N := O_p(G)$ is nontrivial. Then N is one of the groups listed in Table 5.*

N	$\mathcal{B}^\circ(N)$	$\deg_{\mathbb{Q}}$	comm. alg.	$\mathrm{Out}(N)$
C_{13}	$\pm N$	12	$\mathbb{Q}[\zeta_{13}]$	C_{12}
C_7	$\pm N$	6	$\mathbb{Q}[\zeta_7]$	C_6
C_5	$\pm N$	4	$\mathbb{Q}[\zeta_5]$	C_4
C_9	$\pm N$	6	$\mathbb{Q}[\zeta_9]$	C_6
3_+^{1+2}	$\pm N : \mathrm{SL}_2(3)$	6	$\mathbb{Q}[\sqrt{-3}]$	$\mathrm{GL}_2(3)$
C_3	$\pm N$	2	$\mathbb{Q}[\sqrt{-3}]$	C_2
$Q_8 \circ Q_8 \otimes D_8$	$N.\mathrm{Alt}_8$	8	$\mathbb{Q}$	S_8
$Q_8 \circ Q_8 \otimes C_4$	$N.S_6$	8	$\mathbb{Q}[\sqrt{-1}]$	$S_6 \times C_2$
$Q_{16} \circ Q_8$	$N.S_3$	8	$\mathbb{Q}[\sqrt{2}]$	$S_3 \times C_2 \times C_2$
$QD_{16} \underset{\sqrt{-2}}{\otimes} Q_8$	$N.S_3$	8	$\mathbb{Q}[\sqrt{-2}]$	$S_3 \times C_2$
$Q_8 \underset{\sqrt{-1}}{\otimes} C_8$	$N.S_3$	8	$\mathbb{Q}[\zeta_8]$	$S_3 \times C_2 \times C_2$
D_{32}	N	8	$\mathbb{Q}[\zeta_{16} + \zeta_{16}^{-1}]$	$C_4 \times C_2$
QD_{32}	N	8	$\mathbb{Q}[\zeta_{16} - \zeta_{16}^{-1}]$	C_4
C_{16}	N	8	$\mathbb{Q}[\zeta_{16}]$	$C_4 \times C_2$
$D_8 \otimes Q_8$	$N.\mathrm{Alt}_5$	8	$\mathcal{Q}_{\infty,2}$	S_5
$Q_8 \circ Q_8$	$N : (S_3 \times S_3) = \mathrm{Aut}(F_4)$	4	$\mathbb{Q}$	$S_3 \wr C_2$
$Q_8 \circ C_4$	$N.S_3$	4	$\mathbb{Q}[\sqrt{-1}]$	$S_3 \times C_2$
Q_{16}	N	8	$\mathcal{Q}_{\sqrt{2},\infty,\infty}$	$C_2 \times C_2$
D_{16}	N	4	$\mathbb{Q}[\sqrt{2}]$	$C_2 \times C_2$
QD_{16}	N	4	$\mathbb{Q}[\sqrt{-2}]$	C_2
C_8	N	4	$\mathbb{Q}[\zeta_8]$	$C_2 \times C_2$
Q_8	$N : C_3$	4	$\mathcal{Q}_{\infty,2}$	S_3
D_8	N	2	$\mathbb{Q}$	C_2
C_4	N	2	$\mathbb{Q}[\sqrt{-1}]$	C_2
C_2	N	1	$\mathbb{Q}$	1

TABLE 5. The first column shows the possibilities for the normal p-subgroups N of G. The second gives the group $\mathcal{B}^\circ(N)$ (Definition 2.4), which is, according to Proposition 2.5, also a normal subgroup of G. Columns 3 and 4 allow us to restrict the possibilities for the centralizer $C_G(N) = C_G(\mathcal{B}^\circ(N))$. Since $G/NC_G(N)$ embeds in the outer automorphism group $\mathrm{Out}(N)$ of N, this group is given in the last column.

Proof. Let $G \leq \mathrm{GL}_{24}(\mathbb{Q})$ be primitive and let p be a prime such that $N := O_p(G) \neq 1$. It follows from Remark 1.15 that all abelian characteristic subgroups of N are cyclic. The possible p-groups with this property were classified by P. Hall [Huppert 1967, p. 357], and are either cyclic, extraspecial of exponent p, or central products of a cyclic and an extraspecial group. For $p = 2$ there also occur dihedral, quasidihedral, and quaternion groups, and central products of these groups with extraspecial groups. Remark 1.15 also implies that $p-1$ divides 24, so that $p \in \{2, 3, 5, 7, 13\}$. Since 24 is not divisible by 5, 7, or 13, one has $N = C_p$, if $p = 5$, 7, or 13. In the case $p = 3$, either the degree of the irreducible constituents of the natural representation of N is 2 and $N \cong C_3$, or the degree is 6 and N is isomorphic to C_9 or 3_+^{1+2}. In the case $p = 2$ the degree of the irreducible constituents of the natural representation of N divides 8 and the Theorem of P. Hall implies that N is one of the 2-groups in Table 5. The groups $\mathcal{B}^\circ(N)$ can be obtained by considering the lattice of invariant lattices in the irreducible constituent module of the natural $\mathbb{Q}N$-module in the respective cases. □

We now consider the various possibilities for primitive r.i.m.f. groups G, according to the highest p such that $O_p(G) \neq 1$.

Case $\mathrm{O}_{13}(\mathrm{G}) \neq 1$

Proposition 6.2. *If G is a primitive r.i.m.f. group of degree 24 with $O_{13}(G) \neq 1$, then G is conjugate to $[(\pm D_{78}).C_{12}]_{24}$ and has a presentation*

$$G = \langle a, b, c \mid a^{78}, b^2, a^b = a^{-1}, c^{12}, a^c = a^{67}, [b, c] = a^{23} \rangle.$$

Proof. Let G be a primitive r.i.m.f. group of degree 24 with $O_{13}(G) \neq 1$. Theorem 6.1 implies that $O_{13}(G) \cong C_{13}$. Since the centralizer $C := C_G(O_{13}(G))$ embeds in $\mathrm{GL}_2(\mathbb{Q}[\zeta_{13}])$, it is soluble. Moreover $O_p(G) = 1$ for $p \neq 2, 3, 13$ and $O_2(G) \cdot O_3(G) = O_2(C) \cdot O_3(C)$ is one of these groups: C_2, C_4, D_8, Q_8, or C_6. Hence the prime divisors of $|G|$ are contained in $\{2, 3, 13\}$. If $O_2(G) \cdot O_3(G) \neq \pm 1$ then G contains an irreducible subgroup $\cong C_{52}$ or C_{78}. Table 2 implies $G = [(\pm D_{78}).C_{12}]_{24}$ in this case.

Otherwise $O_2(G) \cdot O_3(G) = \pm 1$ and $C \cong C_{26}$ is a reducible normal subgroup of G. The factor group G/C is isomorphic to a subgroup of $C_{12} = \mathrm{Aut}(C_{13})$. The split extension $C_{26} : C_{12}$ is reducible and the unique maximal nonsplit extension $C_{26}.C_{12}$ is a proper subgroup of $[\mathrm{SL}_2(13) \overset{2(2)}{\square} \mathrm{SL}_2(3)]_{24}$. □

Case $\mathrm{O}_7(\mathrm{G}) \neq 1$, $\mathrm{O}_{13}(\mathrm{G}) = 1$

Proposition 6.3. *All primitive r.i.m.f. groups $G \leq \mathrm{GL}_{24}(\mathbb{Q})$ satisfy $O_7(G) = 1$.*

Proof. Let G be a primitive r.i.m.f. group of $\mathrm{GL}_{24}(\mathbb{Q})$ with $O_7(G) \neq 1$. Theorem 6.1 implies $O_7(G) \cong C_7$. The centralizer $C := C_G(O_7(G))$ embeds in $\mathrm{GL}_4(\mathbb{Q}[\zeta_7])$ and hence $O_p(G) = O_p(C) = 1$ for all primes $p > 7$.

If C is insoluble, C contains one of these five groups as its normal subgroup $C^{(\infty)}$: Alt_5, $\mathrm{SL}_2(5)$, $\mathrm{SL}_2(7)$, $\mathrm{SL}_2(9)$, or $2.\mathrm{Alt}_7$ (see Table 4). In the first, second, and fourth cases G contains an irreducible subgroup $\cong C_{70}$ and the prime divisors of $|G|$ are 2, 3, 5, and 7. Table 2 then shows that $O_7(G) = 1$. In the third case G contains an irreducible subgroup $\cong C_{56}$ and the prime divisors of $|G|$ are 2, 3, and 7. Table 2 then shows that $O_7(G) = 1$ again.

Also in the last case G contains an irreducible subgroup $\cong C_{56}$ but 5 divides the order of G. Since $O_7(G)C^{(\infty)}$ is irreducible modulo 5, the determinants $\det(F, L)$ are not divisible by 5 for all

$$(L, F) \in \mathcal{Z}(O_7(G)C^{(\infty)}) \times \mathcal{F}(O_7(G)C^{(\infty)})$$

with F primitive on L. Hence Table 2 shows that $O_7(G) = 1$ also in this case.

Now assume that $O_5(C) = O_5(G) > 1$. Then $O_5(G) \cong C_5$ and G contains an irreducible self-centralizing normal subgroup $\cong C_{70}$. Hence the only primes dividing $|G|$ are 2, 3, 5, and 7, and again Table 2 shows that $O_7(G) = 1$.

Now let $O_3(C) = O_3(G) > 1$. Then $O_3(G) \cong C_3$ and G contains a normal subgroup $N \cong C_{21}$. The centralizer $C_G(N)$ embeds in $\mathrm{GL}_2(\mathbb{Q}[\zeta_{21}])$, hence is soluble, and $O_2(G) = O_2(C_G(N))$ is one of C_2, C_4, D_8, or Q_8.

In the last three cases, G contains an irreducible subgroup $\cong C_{84}$. Since $|G|$ is only divisible by the primes 2, 3, and 7, Table 2 shows that $O_7(G) = 1$.

If $O_2(G) = C_2$, G contains a self-centralizing normal subgroup $Z \cong C_{42}$. The factor group G/Z is isomorphic to a subgroup of $C_6 \times C_2$. Since the split extension $Z{:}(C_6 \times C_2)$ is reducible, the primitivity of G implies that 2^2 divides $G{:}Z$ and every subgroup of index 2 of G is nonsplit over Z. But the unique maximal extension with these properties is imprimitive too, since it contains the reducible normal subgroup $C_7{:}C_3 \otimes \tilde{S}_3$ of index 2.

Next assume that C is soluble and $O_5(C) = O_3(C) = 1$. Then $O_2(G) = O_2(C)$ is one of C_2, C_4, D_8, Q_8, C_8, D_{16}, Q_{16}, QD_{16}, $Q_8 \circ C_4$, or $Q_8 \circ Q_8$.

In the first case C_{14} is a reducible self-centralizing normal subgroup of G and G is reducible. In cases 5–8, G contains a self-centralizing irreducible normal subgroup $\cong C_{56}$ and is not maximal finite by Table 2. In the second and third cases, $O_7(G) \times O_2(G) =: N \trianglelefteq G$ is a reducible normal subgroup of index ≤ 6 in G. Since the unique extension $N{:}C_3$ is reducible this contradicts the primitivity of G.

Now assume that $N := O_2(G) \times O_7(G)$ is isomorphic to $Q_8 \otimes C_7$. Since $\mathbb{Q}[\zeta_7]$ does not split the quaternion algebra $\mathcal{Q}_{\infty,2}$, N is irreducible with commuting algebra $C_{\mathbb{Q}^{24\times 24}}(N) \cong \mathcal{Q}_{\zeta_7,2,2}$ and has a 12-dimensional space of invariant forms. The Bravais group B of a normal critical lattice is $B := C_7 \otimes \mathrm{SL}_2(3)$ and has the same commuting algebra as N. Consider two cases:

(a) 3^2 divides $|G|$: Then $O_{2,7,3}(G)$ is conjugate to $C_7{:}C_3 \otimes \mathrm{SL}_2(3)$ and has a 4-dimensional space of invariant forms. The Bravais group of a normal critical lattice is $L_2(7) \otimes \mathrm{SL}_2(3)$, contradicting the assumption $O_7(G) = C_7$.

(b) 3^2 does not divide $|G|$: Then B is a normal subgroup of G with $G/B \le C_2 \times C_2$. If $G/B \cong C_2$, G is not maximal, since we have, for example, $G < G{:}C_3 \le \mathrm{GL}_{24}(\mathbb{Q})$. Hence $G/B \cong C_2 \times C_2$ and G contains an irreducible subgroup $\cong C_{56}$. Since the prime divisors of $|G|$ are 2, 3, and 7, Table 2 shows that $O_7(G) = 1$.

Next assume that $O_2(G) \times O_7(G) = D_8 \otimes C_{28} =: N \trianglelefteq G$. Then $\mathcal{B}^\circ(N) = C_7 \otimes (\mathrm{SL}_2(3) \circ C_4).2 \trianglelefteq G$ and hence G contains an irreducible subgroup $\cong C_{84}$. Since the prime divisors of $|G|$ are 2, 3, and 7, Table 2 shows that $O_7(G) = 1$.

If $O_2(G) \times O_7(G) = Q_8 \circ Q_8 \otimes C_7 =: N \trianglelefteq G$, then $\mathcal{B}^\circ(N) = F_4 \otimes C_7 \trianglelefteq G$ and G contains an irreducible subgroup $\cong C_{84}$. Since the prime divisors of $|G|$ are 2, 3, and 7, Table 2 shows that $O_7(G) = 1$. $\square$

Case $O_5(G) \neq 1$, $O_{13}(G) = O_7(G) = 1$

Lemma 6.4. *Let G be an r.i.m.f. group of degree 24 containing a subgroup U conjugate to $C_{15} \underset{\sqrt{5}}{\otimes} \mathrm{Alt}_5$. Assume that the prime divisors of $|G|$ lie in the set $\{2,3,5,7\}$, or that there is a lattice $(L,F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ with F integral on L, such that the prime divisors of $\det(F,L)$ lie in $\{2,3,5,7\}$. Then G is one of the following six r.i.m.f. groups:* $[2.\mathrm{Co}_1]_{24}$, $[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{24,2}$, $[2.J_2 \overset{2}{\square} \mathrm{SL}_2(5)]_{24}$, $[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{24,1}$, $A_2 \otimes [\pm D_{10} \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{12}$, *or* $[\pm 3.\mathrm{PGL}_2(9) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{24}$.

Proof. The commuting algebra of U is isomorphic to $\mathbb{Q}[\zeta_{15}]$ and U fixes up to isomorphism four lattices $L_1, \ldots, L_4$. The Bravais group $\mathcal{B}(U, L_1) = \mathcal{B}(U, L_4)$ is conjugate to $\pm D_{30} \overset{2}{\underset{\sqrt{5}}{\otimes}} \mathrm{Alt}_5$, and $\mathcal{B}(U, L_2)$ and $\mathcal{B}(U, L_3)$ are both conjugate to $\pm 3.\mathrm{Alt}_6 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} C_5$. Using the 4-parameter argument (see Corollary 2.3 and the paragraph before it), one finds that the r.i.m.f. supergroups of U are conjugate to $A_2 \otimes [\pm D_{10} \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{12}$, $[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{24,1}$, or $[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{24,2}$ (on L_1 and L_4), or $[2.\mathrm{Co}_1]_{24}$, $[2.J_2 \overset{2}{\square} \mathrm{SL}_2(5)]_{24}$, or $[\pm 3.\mathrm{PGL}_2(9) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{24}$ (on L_2 and L_3). $\square$

Lemma 6.5. *Let G be an r.i.m.f. group of degree 24 containing a subgroup U conjugate to $C_{20} \underset{\sqrt{5}}{\otimes} \mathrm{Alt}_5$. Assume that the prime divisors of $|G|$ lie in the set $\{2,3,5,7\}$, or that there is a lattice $(L,F) \in \mathcal{Z}(G) \times \mathcal{F}_{>0}(G)$ with F integral on L, such that the prime divisors of* $\det(F,L)$ *lie in $\{2,3,5,7\}$. Then G is one of these five r.i.m.f. groups*: $[2.\mathrm{Co}_1]_{24}$, $[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{24,2}$, $[2.J_2 \overset{2}{\Box} \mathrm{SL}_2(5)]_{24}$, $[\pm D_{10} \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{12}^2$, *or* $[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)) : 2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{24,1}$.

Proof. Similar to the proof of Lemma 6.4. □

Proposition 6.6. *Let $G \le \mathrm{GL}_{24}(\mathbb{Q})$ be a primitive r.i.m.f. group with $O_5(G) > 1$. Then $O_5(G) = C_5$ and G is conjugate to either $A_2 \otimes [\pm D_{10} \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{12}$ or $[\pm 3.\mathrm{PGL}_2(9) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{24}$.*

Proof. Let $G \le \mathrm{GL}_{24}(\mathbb{Q})$ be a primitive r.i.m.f. group with $O_5(G) > 1$. Theorem 6.1 implies that $O_5(G) \cong C_5$. Moreover the centralizer $C_G(O_5(G))$ embeds in $\mathrm{GL}_6(\mathbb{Q}[\zeta_5])$.

Because of Proposition 6.3 one has $O_7(G) = 1$. Hence $O_p(G) = 1$ for all primes $p > 5$.

Assume first that $O_3(G) > 1$. Then Theorem 6.1 implies that $O_3(G) \cong C_3$, C_9, or 3_+^{1+2}.

In the second case G contains an irreducible self-centralizing normal subgroup $\cong C_{90}$, and Table 2 then shows that $O_5(G) = 1$.

In the third case G has a normal subgroup

$$\mathcal{B}^\circ(O_5(G) \otimes O_3(G)) \cong \pm C_5 \otimes 3_+^{1+2} : \mathrm{SL}_2(3).$$

Hence G contains an irreducible subgroup $\cong C_{90}$. Since the prime divisors of $|G|$ are 2, 3, and 5, Table 2 then shows that $O_5(G) = 1$.

Now assume $O_3(G) \cong C_3$. Since

$$C := C_G(O_3(G) \times O_5(G))$$

embeds in $\mathrm{GL}_3(\mathbb{Q}[\zeta_{15}])$, the last term of the derived series $C^{(\infty)}$ of C is one of 1, Alt_5, or $3.\mathrm{Alt}_6$ (see Table 4). Furthermore G/C is isomorphic to a subgroups of $\mathrm{Aut}(C_{15}) = C_4 \times C_2$.

In the first case C is soluble, hence equal to $\cong C_{30}$, and its irreducible constituents are of degree 8. Since G/C is a 2-group, G is reducible.

In the other two cases, G contains a subgroup U conjugate to $C_{15} \underset{\sqrt{5}}{\otimes} \mathrm{Alt}_5$. Hence Lemma 6.4 implies that G is conjugate to $A_2 \otimes [\pm D_{10} \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{12}$ or $[\pm 3.\mathrm{PGL}_2(9) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{24}$.

Now let $O_3(G) = 1$. Then, by Theorem 6.1, $O_2(G)$ is isomorphic to C_2, C_4, D_8, or Q_8. In all cases the centralizer $C := C_G(O_2(G) \times O_5(G))$ is not soluble, because otherwise G contains a self-centralizing normal subgroup $B := \mathcal{B}^\circ(O_2(G)) \times O_5(G)$ conjugate to C_{10}, C_{20}, $C_5 \otimes D_8$, or $C_5 \underset{\sqrt{5}}{\otimes} \mathrm{SL}_2(3)$. The irreducible constituents of B are of degree 4 or 8 over $\mathbb{Q}$. Since B is of 2-power index in G this implies that G is reducible.

If $O_2(G) \ne C_2$, then C embeds in $\mathrm{GL}_3(\mathbb{Q}[\zeta_{20}])$ or $\mathrm{GL}_3(\mathbb{Q}[\zeta_5])$, and G contains a subgroup $C_{20} \underset{\sqrt{5}}{\otimes} \mathrm{Alt}_5$, contradicting Lemma 6.5.

Hence $O_2(G) = \pm 1 \cong C_2$ and

$$C := C_G(O_5(G)) \hookrightarrow \mathrm{GL}_6(\mathbb{Q}[\zeta_5])$$

is a normal subgroup of index 1, 2, or 4 in G. Table 4 implies that $C^{(\infty)}$ is one of the matrix groups Alt_5, $\mathrm{SL}_2(5)$ (2 groups), $\mathrm{Alt}_5 \underset{\sqrt{5}}{\otimes} \mathrm{SL}_2(5)$, $L_2(7)$ (2 groups), Alt_7, $U_3(3)$, $U_4(2)$, or $2.J_2$. We take the cases separately.

Assume first $C^{(\infty)} \cong \mathrm{Alt}_5$. Then $N := \pm O_5(G) \times C^{(\infty)}$ is reducible. The outer automorphism group of N is isomorphic to $C_4 \times C_2$. Since G is primitive, N is of index 4 or 8 in G. In particular, there is an element $x \in G$ (of order 2 or 4) centralizing $C^{(\infty)}$ and inducing the automorphism of order 2 on $O_5(G)$.

If x is of order 2, the group $\langle N, x\rangle \sim \pm\mathrm{Alt}_5 \otimes D_{10}$ is still reducible. Hence $G{:}N = 8$ and $U := \pm\mathrm{Alt}_5 \otimes (C_5 : C_4)$ is an irreducible normal subgroup of index 2 in G. The Bravais group of a normal critical $\mathbb{Z}U$-lattice is conjugate to $(\pm\mathrm{Alt}_5 \underset{\sqrt{5}}{\otimes} D_{10}) \wr C_2$ contradicting the primitivity of G.

Hence x is of order 4 and G contains an irreducible subgroup $U := \langle N, x\rangle \sim \mathrm{Alt}_5 \underset{\sqrt{5}}{\otimes} Q_{20}$. Using Theorem 2.1 one gets a contradiction to $C^{(\infty)} \cong \mathrm{Alt}_5$.

Now assume that $C^{(\infty)}$ is conjugate to $\mathrm{SL}_2(5)$, where the restriction of the natural character of G to $C^{(\infty)}$ is $6(\chi_{2a} + \chi_{2b})$. Then G contains the reducible normal subgroup $N := O_5(G) \times C^{(\infty)}$ with index a 2-power. Since the irreducible constituents of N are of degree 8, G is reducible.

If instead $C^{(\infty)}$ is conjugate to $\mathrm{SL}_2(5)$, where the restriction of the natural character of G to $C^{(\infty)}$ is $4\chi_6$, then $N := O_5(G) \times C^{(\infty)}$ is an irreducible normal subgroup of G. The Bravais group of N of a normal critical lattice is conjugate to $\mathrm{Alt}_5 \underset{\sqrt{5}}{\otimes} (\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2$, contradicting the fact that $C^{(\infty)} \cong \mathrm{SL}_2(5)$.

If $C^{(\infty)}$ is conjugate to $\mathrm{Alt}_5 \underset{\sqrt{5}}{\otimes} \mathrm{SL}_2(5)$ or $2.J_2$, then $G^{(\infty)}$ is irreducible and Proposition 5.2 or 5.1 yield a contradiction to $O_5(G) = C_5$.

If $C^{(\infty)} = U_4(2)$, then $N := O_5(G) \times C^{(\infty)}$ is already irreducible with commuting algebra

$$C_{\mathbb{Q}^{24\times 24}}(N) \cong \mathbb{Q}[\zeta_5].$$

Using the 2-parameter argument one finds that G is a proper subgroup of $A_4 \otimes E_6$.

In the remaining cases G contains an irreducible subgroup $\cong C_{70}$. Since $2, 3, 5$, and 7 are the only prime divisors of $|G|$, one gets a contradiction with Table 2. $\square$

Case $\mathrm{O}_3(\mathrm{G}) \neq 1$, $\mathrm{O}_{13}(\mathrm{G}) = \mathrm{O}_7(\mathrm{G}) = \mathrm{O}_5(\mathrm{G}) = 1$

Proposition 6.7. *All primitive r.i.m.f. groups* $G \leq \mathrm{GL}_{24}(\mathbb{Q})$ *satisfy* $O_3(G) \not\cong C_9$.

Proof. Let G be an r.i.m.f. group of degree 24 with $O_3(G) = C_9$. Because of Proposition 6.6 one has $O_p(G) = 1$ for all primes $p > 3$. Since the centralizer $C := C_G(O_3(G))$ embeds in $\mathrm{GL}_4(\mathbb{Q}[\zeta_9])$, the possibilities for $C^{(\infty)}$ are 1, Alt_5, $\mathrm{SL}_2(5)$, $\mathrm{SL}_2(9)$, or $\mathrm{Sp}_4(3)$ (see Table 4).

In all cases where $C^{(\infty)} \neq 1$, the group G contains an irreducible subgroup $\cong C_{90}$. Since 2, 3, and 5 are the only prime divisors of $|G|$, the assumption $O_3(G) \cong C_9$ contradicts Table 2.

If $C^{(\infty)} = 1$ then G is soluble and the possibilities for $O_2(G)$ are C_2, C_4, D_8, Q_8, C_8, D_{16}, Q_{16}, QD_{16}, $Q_8 \circ C_4$, $Q_8 \circ Q_8$, and $D_8 \otimes Q_8$.

In the last seven cases the normal subgroup

$$\mathcal{B}^\circ(O_2(G)) \times O_3(G) \trianglelefteq G$$

contains an irreducible subgroup $\cong C_{72}$ (or $\cong C_{90}$ in the last case). Since the prime divisors of $|G|$ are 2 and 3 (also 5 in the last case) one gets a contradiction to Table 2.

If $O_2(G)$ is isomorphic to C_2, C_4, or D_8, then $\mathrm{Aut}(O_2(G))$ is a 2-group. Therefore G contains the reducible normal subgroup $O_2(G) \times O_3(G)$ of index 2 and hence is imprimitive.

Finally assume that $O_2(G) \cong Q_8$. Then $B := \mathcal{B}^\circ(O_2(G)) \underset{\sqrt{-3}}{\otimes} C_9 \trianglelefteq G$ is a reducible normal subgroup of G. The factor group G/B is isomorphic to a subgroup of $\mathrm{Out}(B) \cong C_2 \times S_3$. Since $O_3(G) \cong C_9$ and $O_3(G/B)$ centralizes the normal subgroup $\mathcal{B}^\circ(O_2(G))$ it follows that $3 \nmid [G{:}B]$. Hence the primitivity of G implies that $G/B \cong C_2 \times C_2$. Therefore $C_G(O_3(G))$ is isomorphic to one of $\mathrm{GL}_2(3) \times C_9$ or $\tilde{S}_4 \times C_9$. In particular G contains an irreducible subgroup $\cong C_{72}$. Since 2 and 3 are the only prime divisors of $|G|$, one gets a contradiction to Table 2. $\square$

Proposition 6.8. *Let* $G \leq \mathrm{GL}_{24}(\mathbb{Q})$ *be a primitive r.i.m.f. group with* $O_3(G) \cong 3_+^{1+2}$. *Then* G *is either* $[\mathrm{Sp}_4(3) \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} (3_+^{1+2}{:}\mathrm{SL}_2(3))]_{24}$ *or* $[\mathrm{SL}_2(5) \overset{2}{\underset{\infty,3}{\boxtimes}} (\pm 3_+^{1+2}).\mathrm{GL}_2(3)]_{24}$.

Proof. Let G be an r.i.m.f. group of degree 24 with $O_3(G) = 3_+^{1+2}$. Since $\mathcal{B}^\circ(O_3(G)) = 3_+^{1+2}{:}\mathrm{SL}_2(3) \trianglelefteq G$ contains a subgroup $\cong C_9$, the same arguments as in the proof of Proposition 6.7 show that G is conjugate to one of the two desired groups. $\square$

Lemma 6.9. *Let* G *be a primitive r.i.m.f. group of degree* 24 *with* $O_3(G) = C_3$. *Then* $O_2(G)$ *is one of* C_2, C_4, D_8, *or* Q_8.

Proof. Let G be a primitive r.i.m.f. group of degree 24 with $O_3(G) = C_3$. The centralizer $C := C_G(O_3(G))$ embeds in $\mathrm{GL}_{12}(\mathbb{Q}[\zeta_3])$ and is a normal subgroup of index ≤ 2 in G. The primitivity of G implies that C is irreducible. According to Theorem 6.1, the possibilities for $O_2(G) = O_2(C)$ are C_2, C_4, D_8, Q_8, C_8, D_{16}, QD_{16}, $Q_8 \circ C_4$, $Q_8 \circ Q_8$, Q_{16}, or $D_8 \otimes Q_8$.

Let $B := \mathcal{B}^\circ(O_2(G)) \times O_3(G)$. If $O_2(G)$ is not conjugate to one of the four groups of the lemma, $N := C_G(B)$ embeds in $\mathrm{GL}_3(\mathbb{Q}[\sqrt{-3}, \zeta_8])$. In particular, N is soluble and $O_p(N) = 1$ for all primes $p > 3$. Hence $N \leq B$ and B is a normal subgroup of 2-power index in G. Since 3 does not divide the degrees of the irreducible constituents of the natural representation of B, this implies that G is reducible. $\square$

Using the 2-parameter argument one gets the following two lemmas:

Lemma 6.10. *Let $G \leq \mathrm{GL}_{24}(\mathbb{Q})$ be an r.i.m.f. group. If G contains a subgroup conjugate to $\mathrm{SL}_2(5){:}2 \otimes C_3$, then G is one of these three groups*: $[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} \mathrm{Alt}_5]_{24,1}$, $[\mathrm{SL}_2(5) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{12}^2$, *or* $A_2 \otimes [\mathrm{SL}_2(5) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{12}$.

Lemma 6.11. *Let $G \leq \mathrm{GL}_{24}(\mathbb{Q})$ be an r.i.m.f. group. If G contains a subgroup conjugate to $\mathrm{SL}_2(5).2 \otimes C_3$ (nonsplit extension), then G is one of these five groups*: $[2.\mathrm{Co}_1]_{24}$, $[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)){:}2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} \mathrm{Alt}_5]_{24,2}$, $[6.\mathrm{Alt}_7{:}2]_{24}$, $[\mathrm{SL}_2(5) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{12}^2$, *or* $[\mathrm{SL}_2(5) \underset{\infty,2}{\overset{2(2)}{\otimes}} 2_-^{1+4'}.\mathrm{Alt}_5]_{24}$.

Proposition 6.12. *Let G be a primitive r.i.m.f. group of degree 24 with $O_3(G) \cong C_3$ and $O_2(G) > \pm 1$. The possibilities for G are* $[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, $[(\pm 3).\mathrm{PGL}_2(9) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{24}$, $[3.S_6 \overset{2(2)}{\otimes} D_8]_{24}$, $[6.L_3(4).2 \overset{2(2)}{\otimes} D_8]_{24}$, $[3.M_{10} \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, $A_2 \otimes [\mathrm{SL}_2(5) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{12}$, $[\mathrm{Alt}_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{24}$, $[3.M_{10} \overset{2(2)}{\boxtimes} D_8]_{24}$, $A_2 \otimes [L_2(7) \overset{2(2)}{\otimes} D_8]_{12}$, *or* $A_2 \otimes [L_2(7) \overset{2(2)}{\boxtimes} D_8]_{12}$.

Proof. Because of Proposition 6.2 and 6.3 we have $O_p(G) = 1$ for all primes $p > 3$. Lemma 6.9 implies that $O_2(G)$ is one of the three groups C_4, D_8, or Q_8. Let $N := O_2(G) \times O_3(G)$. As in the proof of Lemma 6.9 one gets that $C := C_G(N)$ is not soluble. Hence $C^{(\infty)}$ is one of Alt_5, $\mathrm{SL}_2(5)$, $L_2(7)$ (2 matrix groups), $3.\mathrm{Alt}_6$ (2 matrix groups), Alt_7, $3.\mathrm{Alt}_7$, $U_3(3)$, $6.L_3(4)$, $U_4(2)$, or $6.U_4(3)$ (see Table 4).

If $C^{(\infty)}$ is isomorphic to one of $L_2(7)$, Alt_7, $3.\mathrm{Alt}_7$, $U_3(3)$, $6.L_3(4)$, or $6.U_4(3)$, the group G contains an irreducible subgroup $\cong C_{84}$. Since all primes dividing $|G|$ are ≤ 7, Table 2 implies that G is conjugate to one of $[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, $[6.L_3(4).2 \overset{2(2)}{\otimes} D_8]_{24}$, $A_2 \otimes [L_2(7) \overset{2(2)}{\otimes} D_8]_{12}$, or $A_2 \otimes [L_2(7) \overset{2(2)}{\boxtimes} D_8]_{12}$.

Now assume that $C^{(\infty)} \cong \mathrm{Alt}_5$. If $O_2(G) \cong C_4$ or D_8, then G contains an irreducible normal subgroup conjugate to $\mathrm{Alt}_5 \otimes C_{12}$. Using the 4-parameter argument one finds that G is conjugate to $[\mathrm{Alt}_5 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{24}$. If $O_2(G) \cong Q_8$, then G contains an irreducible normal subgroup conjugate to $\mathrm{Alt}_5 \otimes \mathrm{SL}_2(3) \circ C_3$. The Bravais group of a normal critical lattice is conjugate to $F_4 \otimes \mathrm{Alt}_5$ contradicting $O_3(G) > 1$.

Now let $C^{(\infty)} \cong \mathrm{SL}_2(5)$. If $O_2(G) \cong C_4$, or D_8, then G contains an irreducible normal subgroup conjugate to $\mathrm{SL}_2(5) \underset{\sqrt{-1}}{\otimes} C_{12}$. The Bravais group of a normal critical lattice is conjugate to $2^{10}.\mathrm{Alt}_6.2^2 \overset{2}{\otimes} C_{12}$, contradicting the primitivity of G. If, on the other hand, $O_2(G) \cong Q_8$, then G contains a uniform normal subgroup conjugate to $\mathrm{SL}_2(5) \underset{\sqrt{-3}}{\otimes} \mathrm{SL}_2(3) \circ C_3$. In this case one finds that G is conjugate to $A_2 \otimes [\mathrm{SL}_2(5) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{12}$

Next assume that $C^{(\infty)} \cong 3.\mathrm{Alt}_6$, where the natural character of $C^{(\infty)}$ is $2(\chi_{3a} + \chi'_{3a} + \chi_{3b} + \chi'_{3b})$. Then G contains a subgroup $3.\mathrm{Alt}_6 \otimes C_4$. An application of the 4-parameter argument yields the conclusion that G is one of $[3.M_{10} \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$ or $[3.M_{10} \overset{2(2)}{\boxtimes} D_8]_{24}$.

Assume instead that $C^{(\infty)} \cong 3.\mathrm{Alt}_6$, where the natural character of $C^{(\infty)}$ is $2(\chi_6 + \chi_6')$. If $O_2(G) \cong C_4$ or D_8, then G has an irreducible normal subgroup $3.\mathrm{Alt}_6 \otimes C_4$. With the 2-parameter argument one finds that G is conjugate to $[3.S_6 \overset{2(2)}{\otimes} D_8]_{24}$. If $O_2(G) \cong Q_8$, then G has a uniform normal subgroup $3.\mathrm{Alt}_6 \underset{\sqrt{-3}}{\otimes} \mathrm{SL}_2(3)$. One finds that G is conjugate to $[(\pm 3).\mathrm{PGL}_2(9) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{24}$.

Finally assume that $C^{(\infty)} \cong U_4(2)$. If $O_2(G) \cong C_4$ or D_8, then G has an irreducible normal subgroup $U_4(2) \otimes C_{12}$. The Bravais group of a normal critical lattice is $6.U_4(3).2 \overset{2}{\square} C_4$ contradicting $C^{(\infty)} \cong U_4(2)$. If $O_2(G) \cong Q_8$, then G has a uniform normal subgroup $U_4(2) \otimes \mathrm{SL}_2(3) \circ C_3$, whose r.i.m.f. supergroups are $E_6 \otimes F_4$ and $[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, a contradiction with either $O_3(G) \cong C_3$ or $C^{(\infty)} \cong U_4(2)$. ☐

Proposition 6.13. *If $G \leq \mathrm{GL}_{24}(\mathbb{Q})$ is a primitive r.i.m.f. group with $O_3(G) \cong C_3$ and $O_2(G) = \pm 1$, then G is conjugate to one of these six groups*: $[6.\mathrm{Alt}_7 : 2]_{24}$, $[\pm 3.\mathrm{PGL}_2(9) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]$, $A_2 \otimes [\pm D_{10} \underset{\sqrt{5}}{\overset{2}{\boxtimes}} \mathrm{Alt}_5]_{12}$, $[(\pm L_3(3)).2 \overset{2}{\square} C_3]_{24}$, $A_2 \otimes A_{12}$, *or* $[(\pm D_{78}).C_{12}]_{24}$.

Proof. If $O_{13}(G) > 1$, Proposition 6.2 implies that G is conjugate to $[(\pm D_{78}).C_{12}]_{24}$. Because of Proposition 6.3 one has $O_7(G) = 1$. If $O_5(G) > 1$, Proposition 6.6 implies that G is conjugate to either $[\pm 3.\mathrm{PGL}_2(9) \underset{\sqrt{5}}{\overset{2}{\boxtimes}} D_{10}]$ or $A_2 \otimes [\pm D_{10} \underset{\sqrt{5}}{\overset{2}{\boxtimes}} \mathrm{Alt}_5]_{12}$. Assume for the rest of the proof that $O_p(G) = 1$ for all primes $p > 3$. The centralizer $C := C_G(O_3(G))$ embeds in $\mathrm{GL}_{12}(\mathbb{Q}[\zeta_3])$ and, being a normal subgroup of index ≤ 2 in G, it is an irreducible subgroup of $\mathrm{GL}_{24}(\mathbb{Q})$. The last term of the derived series $C^{(\infty)}$ is a central product of quasisimple groups with center $\leq C_6$. Let Δ denote the natural representation of G. The primitivity of G implies that $\Delta|_{C^{(\infty)}} = k \cdot \Gamma$ for some rational irreducible representation $\Gamma : C^{(\infty)} \to \mathrm{GL}_d(\mathbb{Q})$ with $d = 24/k$.

Since all subgroups of $\mathrm{GL}_3(\mathbb{Q})$ are soluble, $d > 3$.

If $d = 4$, then $C^{(\infty)}$ is conjugate to Alt_5 and C is reducible.

If $d = 6$, the possibilities for $C^{(\infty)}$ are Alt_5, $L_2(7)$ (2 matrix groups), Alt_7, or $U_4(2)$.

In all cases the index of $N := \mathcal{B}^{\circ}(C^{(\infty)}) \otimes O_3(G)$ divides the order of $\mathrm{Out}(C^{(\infty)} O_3(G))$, which divides 4. Hence $G : N = 4$, and all proper supergroups of N (in particular C) are irreducible. Since $C^{(\infty)} \otimes S_3$ is reducible, G contains the irreducible normal subgroup

$$M := C^{(\infty)} C_G(C^{(\infty)}) \cong C^{(\infty)} \times \tilde{S}_3$$

of index 2.

If $C^{(\infty)} \cong \mathrm{Alt}_5$, then $C \cong (\pm \mathrm{Alt}_5).2 \times C_3$ is reducible and hence G is imprimitive.

Now assume $C^{(\infty)} \cong L_2(7)$, where the natural character of $C^{(\infty)}$ is $4(\chi_{3a} + \chi_{3b})$. Since $\mathbb{Q}[\sqrt{-7}]$ splits the quaternion algebra $\mathcal{Q}_{\infty,3}$ the group M is reducible and hence G is imprimitive.

If $C^{(\infty)} \cong L_2(7)$, where the natural character of $C^{(\infty)}$ is $4\chi_6$, then M is conjugate to $L_2(7) \otimes \tilde{S}_3$ and already uniform. One finds that the r.i.m.f. supergroups are $[6.U_4(3).2^2]_{12}^2$, $(A_2 \otimes A_6)^2$, and $A_2 \otimes [L_2(7) \overset{2(2)}{\otimes} D_8]_{12}$, contradicting $O_2(G) O_3(G) = C_6$.

In the last two cases, $C^{(\infty)} \cong \mathrm{Alt}_7$ or $U_4(2)$, we get $N \cong \pm S_7 \times C_3$ or $N \cong \pm U_4(2) : 2 \times C_3$, respectively, so N is reducible and G is imprimitive.

Now consider the case $d = 8$. Table 4 implies that $C^{(\infty)}$ is conjugate to one of $\mathrm{SL}_2(5)$, $\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)$, $\mathrm{SL}_2(9)$, or $\mathrm{Sp}_4(3)$. In all cases $\mathrm{Out}(O_3(G) \times C^{(\infty)})$ is a 2-group, and therefore G is reducible.

If $d = 12$, Table 4 implies that the possibilities for $C^{(\infty)}$ are $\mathrm{SL}_2(5)$, $3.\mathrm{Alt}_6$ (2 matrix groups), $\mathrm{SL}_2(11)$, $L_3(3)$, $U_3(3)$, $6.L_3(4)$, $2.M_{12}$, $6.U_4(3)$, or Alt_{13}.

If $C^{(\infty)} \cong \mathrm{SL}_2(5)$, then C is conjugate to either $\mathrm{SL}_2(5) : 2 \otimes C_3$ or $\mathrm{SL}_2(5).2 \otimes C_3$. Using Lemma 6.10 or 6.11, respectively, we obtain a contradiction with $C^{(\infty)} \cong \mathrm{SL}_2(5)$.

If $C^{(\infty)} \cong 3.\mathrm{Alt}_6$ then $C^{(\infty)} = O_3(G) C^{(\infty)}$ and $G / \pm C^{(\infty)}$ is isomorphic to a subgroup of the group

$\mathrm{Out}(C^{(\infty)}) \cong C_2 \times C_2$. If the natural character of $C^{(\infty)}$ is $2(\chi_{3a} + \chi'_{3a} + \chi_{3b} + \chi'_{3b})$, then $C^{(\infty)}$ is reducible and the primitivity of G implies that $G/(\pm C^{(\infty)})$ is isomorphic to $C_2 \times C_2$. In particular G contains one of the isoclinic groups $\pm(3.S_6)$ or $\pm 3.S_6$ of index 2. Since both groups are reducible this contradicts the primitivity of G.

If the natural character of $C^{(\infty)}$ is $2(\chi_6 + \chi'_6)$, then again $C^{(\infty)}$ is reducible and as above G contains one of the isoclinic groups $\pm(3.S_6)$ or $\pm 3.S_6$ of index 2. Since the split extension $\pm(3.S_6)$ is reducible, G contains the uniform group $U := \pm 3.S_6$. The only primitive r.i.m.f. supergroup of U is $[3.S_6 \overset{2(2)}{\otimes} D_8]_{24}$, contradicting $O_2(G) = \pm 1$.

Now assume that $C^{(\infty)} \cong \mathrm{SL}_2(11)$. Then $N := C^{(\infty)}O_3(G) = \mathrm{SL}_2(11) \otimes C_3$ is an irreducible normal subgroup of G. Using the 2-parameter argument one gets that the r.i.m.f. supergroups of N are conjugate to $[2.\mathrm{Co}_1]_{24}$, A_2^{12}, or $[\mathrm{SL}_2(11) \underset{\sqrt{-11}}{\overset{2(2)}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, contradicting $O_3(G) \cong C_3$.

If $C^{(\infty)} \cong L_3(3)$, then $C^{(\infty)} \otimes C_3$ is already uniform and its r.i.m.f. supergroups are conjugate to $[2.\mathrm{Co}_1]_{24}$, $A_2 \otimes A_{12}$, or $[(\pm L_3(3)).2 \overset{2}{\square} C_3]_{24}$. Hence G is conjugate to $[(\pm L_3(3)).2 \overset{2}{\square} C_3]_{24}$ in this case.

In the case $C^{(\infty)} \cong U_3(3)$ the group C has to be isomorphic to the nonsplit extension $(\pm U_3(3)).2 \times C_3$, because the split extension $U_3(3){:}2 \times C_6$ is reducible. Using the 2-parameter argument one finds that the r.i.m.f. supergroups of C are conjugate to $[2.\mathrm{Co}_1]_{24}$, $[6.U_4(3).2^2]_{12}^2$, $[(\mathrm{SL}_2(3) \circ C_4).2 \underset{\sqrt{-1}}{\overset{2(3)}{\boxtimes}} U_3(3)]_{24}$, or $[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, which is a contradiction.

Next assume that $C^{(\infty)} \cong 6.L_3(4)$. Then $C^{(\infty)} = C^{(\infty)}O_3(G)$ is reducible and $G/C^{(\infty)} \cong C_2 \times C_2$. Hence G contains one of the isoclinic groups $U_1 := 6.L_3(4){:}2_2$ or $U_2 := 6.L_3(4).2_2$ (nonsplit extension). Since the commuting algebra of U_1 is isomorphic to $\mathcal{Q}_{\infty,3}$ the group U_1 is uniform. The $\mathbb{Z}U_1$-lattices having a maximal order as endomorphism ring, they are imprimitive, whereas the automorphism groups of the other $\mathbb{Z}U_1$-lattices are conjugate to $[6.L_3(4).2 \overset{2(2)}{\otimes} D_8]_{24}$—a contradiction. The group U_2 is reducible and hence G is imprimitive in this case.

If $C^{(\infty)} \cong 2.M_{12}$, then G contains a uniform normal subgroup isomorphic to $2.M_{12} \times C_3$ whose r.i.m.f. supergroups are $[2.\mathrm{Co}_1]_{24}$ and A_2^{12}.

If $C^{(\infty)} \cong 6.U_4(3)$, then $O_3(G)C^{(\infty)} = C^{(\infty)}$ is reducible and therefore $G/C^{(\infty)} \cong C_2 \times C_2$. In particular G contains one of the isoclinic groups $6.U_4(3){:}2_1$ or $6.U_4(3).2_1$ (nonsplit extension). Since the first group contains the group U_1 of the case $C^{(\infty)} \cong 6.L_3(4)$ and the second group is reducible, one gets a contradiction to the primitivity of G.

If $C^{(\infty)} \cong \mathrm{Alt}_{13}$, then $\mathrm{Alt}_{13} \otimes C_3$ is a uniform normal subgroup of G and G is conjugate to $A_2 \otimes A_{12}$.

In the case $d = 24$, the group $G^{(\infty)}$ is already irreducible. Using Propositions 5.1 and 5.2 one gets the statement of Proposition 6.13. $\square$

Case $\mathrm{O}_p(\mathrm{G}) = 1$ for All Odd Primes p

Lemma 6.14. *If $G \le \mathrm{GL}_{24}(\mathbb{Q})$ is a primitive r.i.m.f. group then $O_2(G)$ is one of $D_8 \otimes Q_8$, $Q_8 \circ Q_8$, $Q_8 \circ C_4$, C_8, D_{16}, QD_{16}, Q_{16}, Q_8, C_4, D_8, or C_2. Moreover, $C_G(O_2(G)) \not\le O_2(G)$.*

Proof. Set $B := \mathcal{B}^\circ(O_2(G))$. If $O_2(G)$ is conjugate to one of the other 2-groups of Table 5, then $N := C_G(B)$ embeds in $\mathrm{GL}_3(Q[\zeta_{16}])$. In particular N is soluble and $O_p(N) = 1$ for all odd primes p. Hence $N \le B$. Since $\mathrm{Out}(B)$ is a 2-group, B is of 2-power index in G and therefore G is reducible. Also in the other cases, $\mathrm{Out}(B)$ is a 2-group and therefore $C_G(O_2(G)) \not\le O_2(G)$. $\square$

To finish the proof of Theorem 3.1 it remains to determine those primitive r.i.m.f. groups G such that $O_2(G)$ is one of the 11 groups listed in Lemma 6.14. The same lemma also implies that the centralizer $C_G(O_2(G))$ contains a normal subgroup that is a central product of some of the quasisimple groups listed in Table 4. For the rest of this section, assume that G is a primitive r.i.m.f. group with $O_p(G) = 1$ for all odd primes p.

Proposition 6.15. *If* $O_2(G) \cong D_8 \otimes Q_8$, *then* G *is conjugate to* $[\mathrm{SL}_2(5) \overset{2(2)}{\underset{\infty,2}{\otimes}} 2_-^{1+4'}.\mathrm{Alt}_5]_{24}$.

Proof. The normal subgroup $B := \mathcal{B}^{\circ}(O_2(G))$ is conjugate to $2_-^{1+4'}.\mathrm{Alt}_5$. The centralizer

$$\pm 1 \neq C_G(B)$$

embeds in $\mathrm{GL}_3(\mathcal{Q}_{\infty,2})$. Table 4 implies that

$$C_G(B)^{(\infty)} \cong \mathrm{SL}_2(5).$$

Hence G contains the uniform normal subgroup $\mathrm{SL}_2(5) \underset{\infty,2}{\otimes} 2_-^{1+4'}.\mathrm{Alt}_5$ and is conjugate to $[\mathrm{SL}_2(5) \overset{2(2)}{\underset{\infty,2}{\otimes}} 2_-^{1+4'}.\mathrm{Alt}_5]_{24}$. □

Lemma 6.16. *The r.i.m.f. supergroups of*

$$U := \mathrm{Alt}_7 \otimes C_8$$

are $A_6 \otimes F_4$ *and* A_6^4.

Proof. U fixes four lattices up to isomorphism. The Bravais groups are conjugate to $S_7 \otimes D_{16}$. The Lemma follows with the 2-parameter argument. □

Proposition 6.17. *If* $O_2(G) \cong Q_8 \circ Q_8$, *then* G *is conjugate to one of* $F_4 \otimes E_6$, $F_4 \otimes M_{6,2}$, $[L_2(7) \overset{2(2)}{\otimes} F_4]_{24}$, $[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$, $F_4 \otimes A_6$, *or* $F_4 \otimes A_6^{(2)}$.

Proof. The normal subgroup $B := \mathcal{B}^{\circ}(O_2(G))$ of G is conjugate to F_4. The centralizer $\pm 1 \neq C_G(B) =: C$ embeds in $\mathrm{GL}_6(\mathbb{Q})$. With Table 4 one finds that $C^{(\infty)}$ is one of Alt_5, $L_2(7)$ (2 matrix groups), Alt_7, or $U_4(2)$.

If $C^{(\infty)} \cong L_2(7)$, then $C^{(\infty)}B$ contains an irreducible subgroup $\cong C_{56}$. Since 2, 3 and 7 are the only primes dividing $|G|$, Table 2 implies that G is conjugate to $[L_2(7) \overset{2(2)}{\otimes} F_4]_{24}$, $[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$, or $F_4 \otimes A_6^{(2)}$.

If $C^{(\infty)} \cong \mathrm{Alt}_7$, then $C^{(\infty)}B$ contains an irreducible subgroup $\mathrm{Alt}_7 \otimes C_8$. From Lemma 6.16 one concludes that G is conjugate to $F_4 \otimes A_6$.

If $C^{(\infty)} \cong \mathrm{Alt}_5$, then $C^{(\infty)}B$ is irreducible with commuting algebra $\cong \mathbb{Q}[\sqrt{5}]$. With the 2-parameter argument one gets that G is conjugate to $F_4 \otimes M_{6,2}$.

If $C^{(\infty)} \cong U_4(2)$, then $C^{(\infty)}B$ is uniform and lattice sparse and hence has a unique r.i.m.f. supergroup. This is $F_4 \otimes E_6$. □

Proposition 6.18. *If* $O_2(G) \cong Q_8 \circ C_4$, *then* G *is conjugate to* $[(\mathrm{SL}_2(3) \circ C_4).2 \overset{2(3)}{\underset{\sqrt{-1}}{\boxtimes}} U_3(3)]_{24}$.

Proof. The normal subgroup $B := \mathcal{B}^{\circ}(O_2(G))$ is conjugate to $(\mathrm{SL}_2(3) \circ C_4).2$. The centralizer $\pm 1 \neq C_G(B) =: C$ embeds in $\mathrm{GL}_6(\mathbb{Q}[\sqrt{-1}])$. Hence $C^{(\infty)}$ is one of Alt_5, $\mathrm{SL}_2(5)$, $L_2(7)$ (2 matrix groups), Alt_7, $U_3(3)$, or $U_4(2)$.

If $C^{(\infty)} \cong L_2(7)$ or $U_3(3)$, the group $C^{(\infty)}B$ contains an irreducible subgroup $\cong C_{56}$. Since 2, 3, and 7 are the only prime divisors of $|G|$, Table 2 implies that G is conjugate to $[(\mathrm{SL}_2(3) \circ C_4).2 \overset{2(3)}{\underset{\sqrt{-1}}{\boxtimes}} U_3(3)]_{24}$.

If $C^{(\infty)} \cong \mathrm{Alt}_7$, then $C^{(\infty)}B$ contains the subgroup $\mathrm{Alt}_7 \otimes C_8$ contradicting Lemma 6.16.

If $C^{(\infty)} \cong \mathrm{Alt}_5$, then $C^{(\infty)}B$ is already irreducible. The Bravais group $\mathcal{B}(C^{(\infty)}B, L)$ of a normal critical lattice L is conjugate to $F_4 \otimes \mathrm{Alt}_5$, contradicting $O_2(G) \cong Q_8 \circ C_4$.

If $C^{(\infty)} \cong \mathrm{SL}_2(5)$ or $\cong U_4(2)$, the group $C^{(\infty)}B$ is uniform and one arrives at a contradiction. □

Proposition 6.19. *If* $O_2(G) \cong C_8$, D_{16}, QD_{16}, *or* Q_{16}, *then* G *is conjugate to* $[\mathrm{SL}_2(7) \overset{2}{\circ} Q_{16}]_{24}$.

Proof. In all cases, G contains a normal subgroup $N \cong C_8$. The centralizer $\pm 1 \neq C_G(N) =: C$ embeds in $\mathrm{GL}_6(\mathbb{Q}[\zeta_8])$. Table 4 implies that $C^{(\infty)}$ is one of Alt_5, $\mathrm{SL}_2(5)$, $L_2(7)$ (2 matrix groups), Alt_7, $U_3(3)$, or $U_4(2)$.

If $C^{(\infty)} \cong L_2(7)$ or $U_3(3)$, then $C^{(\infty)}N$ contains an irreducible subgroup $\cong C_{56}$. Since the prime divisors of $|G|$ are 2, 3, and 7, one concludes from Table 2 that G is conjugate to $[\mathrm{SL}_2(7) \overset{2}{\circ} Q_{16}]_{24}$.

If $C^{(\infty)} \cong \mathrm{Alt}_7$, then $C^{(\infty)}N$ is conjugate to $\mathrm{Alt}_7 \otimes C_8$ contradicting Lemma 6.16.

If $C^{(\infty)} \cong \mathrm{Alt}_5$, then $C^{(\infty)}N$ is irreducible. Applying the 4-parameter argument one gets a contradiction.

If $C^{(\infty)} \cong \mathrm{SL}_2(5)$, the Bravais group of $C^{(\infty)}N$ of a normal critical lattice is conjugate to $\mathrm{SL}_2(5). C_2 \circ \tilde{S}_4$, contradicting the fact that $N \trianglelefteq G$.

If $C^{(\infty)} \cong U_4(2)$, then $C^{(\infty)}N$ is irreducible. An application of the 2-parameter argument yields a contradiction. $\square$

Lemma 6.20. *Let* $U := L_2(7) \otimes \mathrm{SL}_2(3)$, *and assume that the commuting algebra of* U *is isomorphic to* $\mathcal{Q}_{\infty,2}$. *Then the r.i.m.f. supergroups of* U *are conjugate to* $[2.\mathrm{Co}_1]_{24}$, $[6.U_4(3).2 \overset{2}{\underset{\sqrt{-3}}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, $[L_2(7) \overset{2(2)}{\otimes} F_4]_{24}$, *or* $F_4 \otimes A_6$.

Proof. Since U is already uniform, the lemma follows by an easy inspection of the $\mathbb{Z}U$-lattices. $\square$

Proposition 6.21. *If* $O_2(G) \cong Q_8$, *then* G *is conjugate to* $[\mathrm{SL}_2(7) \circ \tilde{S}_4]_{24}$, $[\mathrm{SL}_2(11) \overset{2(2)}{\underset{\sqrt{-11}}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$, *or* $[\mathrm{SL}_2(13) \overset{2(2)}{\mathsf{D}} \mathrm{SL}_2(3)]_{24}$.

Proof. G has a normal subgroup $B := \mathcal{B}^\circ(O_2(G))$ conjugate to $\mathrm{SL}_2(3)$. Since the centralizer $\pm 1 \neq C_G(B) =: C$ embeds in $\mathrm{GL}_6(\mathcal{Q}_{\infty,2})$ the group $C^{(\infty)}$ is one of Alt_5, $\mathrm{Alt}_5 \underset{\sqrt{5}}{\otimes} \mathrm{SL}_2(5)$, $\mathrm{SL}_2(5)$, $L_2(7)$ (2 matrix groups), $\mathrm{SL}_2(7)$, $\mathrm{SL}_2(11)$, $\mathrm{SL}_2(13)$, Alt_7, $U_3(3)$, $U_4(2)$, $U_3(4)$, $2.J_2$, or $2.G_2(4)$ (see Table 4).

If $C^{(\infty)} \cong \mathrm{Alt}_5 \underset{\sqrt{5}}{\otimes} \mathrm{SL}_2(5)$, $\mathrm{SL}_2(7)$, $\mathrm{SL}_2(13)$, $U_3(4)$, $2.J_2$, or $2.G_2(4)$, then $G^{(\infty)} = C^{(\infty)}$ is already irreducible. Propositions 5.2 and 5.1 imply that G is conjugate to either $[\mathrm{SL}_2(7) \circ \tilde{S}_4]_{24}$ or $[\mathrm{SL}_2(13) \overset{2(2)}{\mathsf{D}} \mathrm{SL}_2(3)]_{24}$.

If $C^{(\infty)} \cong \mathrm{Alt}_5$, then $C^{(\infty)}B$ is irreducible with commuting algebra isomorphic to $\mathcal{Q}_{\sqrt{5},\infty,\infty}$. The Bravais group on a normal critical lattice is conjugate to $2.J_2 \circ \mathrm{SL}_2(5)$, contradicting $O_2(G) \cong Q_8$.

If $C^{(\infty)} \cong \mathrm{SL}_2(5)$, then G is imprimitive, because it contains the reducible normal subgroup $\mathcal{B}^\circ(C^{(\infty)}B) = \mathrm{SL}_2(5) \overset{2(2)}{\circ} \mathrm{SL}_2(3)$ of index two.

Now assume that $C^{(\infty)} \cong L_2(7)$, where the natural character of $C^{(\infty)}$ is $4(\chi_{3a} + \chi_{3b})$. Since $\mathbb{Q}[\sqrt{-7}]$ does not split the quaternion algebra $\mathcal{Q}_{\infty,2}$, the group $C^{(\infty)}B$ is an irreducible subgroup of $\mathrm{GL}_{24}(\mathbb{Q})$ with commuting algebra $\mathcal{Q}_{\sqrt{-7},2,2}$. Moreover, G contains $C^{(\infty)}B$ of index ≤ 4 and one of the following possibilities occurs:

(i) G is conjugate to one of the three groups $L_2(7){:}2 \otimes \mathrm{SL}_2(3)$, $(\pm L_2(7).2) \underset{\sqrt{-1}}{\otimes} \mathrm{SL}_2(3)$, or the split extension $L_2(7) \overset{2}{\boxtimes} \mathrm{SL}_2(3)$.

(ii) G is conjugate to the nonsplit extension $L_2(7) \overset{2}{\boxtimes} \mathrm{SL}_2(3)$.

(iii) G contains a subgroup conjugate to $L_2(7) \otimes \mathrm{GL}_2(3)$ or $L_2(7) \underset{\sqrt{-7}}{\otimes} \tilde{S}_4$.

In the first case G is a subgroup of $F_4 \otimes A_6^{(2)}$, and in the second case G is a subgroup of $[L_2(7) \overset{2(2)}{\otimes} F_4]_{24}$. In the last case G contains an irreducible subgroup $\cong C_{56}$, contradicting Table 2.

If $C^{(\infty)} \cong L_2(7)$, where the natural character of $C^{(\infty)}$ is $4\chi_6$, then G contains the subgroup U of Lemma 6.20 and one gets a contradiction to $O_2(G) \times O_3(G) \cong Q_8$.

If $C^{(\infty)} \cong \mathrm{SL}_2(11)$, then $C^{(\infty)}B$ is uniform fixing the same lattices as its unique r.i.m.f. supergroup $[\mathrm{SL}_2(11) \overset{2(2)}{\underset{\sqrt{-11}}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$. Hence G is conjugate to this latter group.

If $C^{(\infty)} \cong \mathrm{Alt}_7$, then G contains the subgroup U of Lemma 6.20 and one gets a contradiction to $O_2(G) \times O_3(G) \cong Q_8$.

Now assume $C^{(\infty)} \cong U_3(3)$. Then $C^{(\infty)}B$ is an irreducible normal subgroup of G with commuting algebra $\cong \mathcal{Q}_{2,3}$. Moreover $|G/C^{(\infty)}B| \leq 4$, and one of the following possibilities occurs:

(i) G is conjugate to $U_3(3){:}2 \underset{\sqrt{-3}}{\otimes} \mathrm{SL}_2(3)$.

(ii) G is conjugate to $(\pm U_3(3).2) \circ \mathrm{SL}_2(3)$ or to the split extension $U_3(3) \overset{2}{\underset{\sqrt{-1}}{\otimes}} \mathrm{SL}_2(3)$.

(iii) G is conjugate to the nonsplit extension

$$U_3(3) \overset{2}{\underset{\sqrt{-1}}{\otimes}} \mathrm{SL}_2(3).$$

(iv) G has a subgroup conjugate to $U_3(3) \underset{\sqrt{-2}}{\otimes} \mathrm{GL}_2(3)$ or $U_3(3) \underset{\infty,3}{\otimes} \tilde{S}_4$.

In the last case G contains an irreducible subgroup $\cong C_{56}$. Since 2, 3, and 7 are the only prime divisors of $|G|$, this contradicts Table 2. In the first case G is a proper subgroup of $[6.U_4(3).2\overset{2}{\underset{\sqrt{-3}}{\boxtimes}}\mathrm{SL}_2(3)]_{24}$, in the second case a subgroup of $[(\mathrm{SL}_2(3)\circ C_4).2\overset{2(3)}{\underset{\sqrt{-1}}{\boxtimes}}U_3(3)]_{24}$, and in the third case a subgroup of $[2.\mathrm{Co}_1]_{24}$, contradicting the maximality of G.

In the last case $C^{(\infty)}\cong U_4(2)$, the group $C^{(\infty)}B$ is already uniform fixing three lattices up to isomorphism. Since their automorphism groups are conjugate to $F_4\otimes E_6$ and $[6.U_4(3).2\overset{2}{\underset{\sqrt{-3}}{\boxtimes}}\mathrm{SL}_2(3)]_{24}$, this is a contradiction to $O_2(G)\times O_3(G)\cong Q_8$. $\square$

Proposition 6.22. *$O_2(G)$ is not isomorphic to C_4 or D_8.*

Proof. In both cases G contains a normal subgroup $N\cong C_4$. The centralizer $C:=C_G(N)$ embeds in $\mathrm{GL}_{12}(\mathbb{Q}[\sqrt{-1}])$. Let Δ denote the natural representation of G. The primitivity of G implies that $\Delta|_{C^{(\infty)}}=k\cdot\Gamma$ for some rational irreducible representation $\Gamma: C^{(\infty)}\to\mathrm{GL}_d(\mathbb{Q})$ with $d=24/k$.

Since all subgroups of $\mathrm{GL}_3(\mathbb{Q})$ are soluble, $d>3$.

If $d=4$, then $C^{(\infty)}$ is conjugate to Alt_5, and C is reducible.

If $d=6$, the possibilities for $C^{(\infty)}$ are Alt_5, $L_2(7)$ (2 matrix groups), Alt_7, or $U_4(2)$. In all cases, $O_2(G)C^{(\infty)}$ is reducible and $[G:O_2(G)C^{(\infty)}]\leq 2$; therefore G is imprimitive.

If $d=8$, then $C^{(\infty)}$ is conjugate to one of $\mathrm{SL}_2(5)$ or $\mathrm{SL}_2(9)$ (see Table 4). Since $\mathrm{Out}(C^{(\infty)})$ is a 2-group, G is reducible.

If $d=12$, then $C^{(\infty)}$ is conjugate to one of $\mathrm{SL}_2(5)$, $\mathrm{SL}_2(11)$, $L_3(3)$, $U_3(3)$, $2.M_{12}$, or Alt_{13} (see Table 4).

In the first case, $C^{(\infty)}\cong\mathrm{SL}_2(5)$, the group $C^{(\infty)}N$ is reducible. The primitivity of G implies $O_2(G)\cong D_8$ and G contains the uniform normal subgroup $C^{(\infty)}O_2(G)$ of index 2. Since the automorphism group on a normal critical lattice is conjugate to $[\mathrm{SL}_2(5)\overset{2(2)}{\underset{\infty,2}{\otimes}}2_-^{1+4'}.\mathrm{Alt}_5]_{24}$, this is a contradiction.

If $C^{(\infty)}\cong\mathrm{SL}_2(11)$, then $C^{(\infty)}N$ is irreducible with commuting algebra isomorphic to

$$\mathbb{Q}[\sqrt{11},\sqrt{-1}].$$

Applying the 2-parameter argument one gets a contradiction to $N\trianglelefteq G$.

If $C^{(\infty)}\cong L_3(3)$, then $C^{(\infty)}N$ is already uniform. Its unique r.i.m.f. supergroup is conjugate to A_{12}^2 contradicting the primitivity of G.

If $C^{(\infty)}\cong U_3(3)$, then $C^{(\infty)}N$ is reducible. The primitivity of G implies that $O_2(G)\cong D_8$ and the uniform group $C^{(\infty)}O_2(G)$ is of index 2 in G. The group $C^{(\infty)}O_2(G)$ fixes up to isomorphism three lattices and its unique primitive r.i.m.f. supergroup is conjugate to $[(\mathrm{SL}_2(3)\circ C_4).2\overset{2(3)}{\underset{\sqrt{-1}}{\boxtimes}}U_3(3)]_{24}$ contradicting $O_2(G)\cong D_8$.

If $C^{(\infty)}\cong 2.M_{12}$, then $C^{(\infty)}N$ is already uniform. Since the automorphism group of a normal critical lattice is imprimitive, one gets a contradiction.

In the last case, $C^{(\infty)}\cong\mathrm{Alt}_{13}$, the group $C^{(\infty)}N$ is uniform fixing up to isomorphism four lattices. Its unique r.i.m.f. supergroup is conjugate to A_{12}^2 and imprimitive.

If $d=24$, then $C^{(\infty)}=G^{(\infty)}$ is already irreducible. Propositions 5.1 and 5.2 yield a contradiction to the assumption on $O_2(G)$. $\square$

Proposition 6.23. *If $G\leq\mathrm{GL}_{24}(\mathbb{Q})$ is a primitive r.i.m.f. group with largest soluble normal subgroup ± 1, then G is conjugate to one of these 12 r.i.m.f. groups:* $[2.\mathrm{Co}_1]_{24}$, $[(\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2\overset{2}{\underset{\sqrt{5}}{\boxtimes}}\mathrm{Alt}_5]_{24,1}$, $[(\mathrm{SL}_2(5)\circ\mathrm{SL}_2(5)):2\overset{2}{\underset{\sqrt{5}}{\boxtimes}}\mathrm{Alt}_5]_{24,2}$, $[\mathrm{SL}_2(13)\overset{2(2)}{\Box}\mathrm{SL}_2(3)]_{24}$, $[\pm L_2(11):2]_{24}$, $[2.J_2\overset{2}{\Box}\mathrm{SL}_2(5)]_{24}$, $[\mathrm{SL}_2(7)\overset{2}{\underset{\sqrt{-7}}{\boxtimes}}L_2(7)]_{24}$, $[\pm U_4(2).2]_{24}$, A_{24} , $A_4\otimes A_6$, $A_4\otimes A_6^{(2)}$, *or* $A_4\otimes E_6$.

Proof. Let Δ denote the natural representation of G. The primitivity of G implies that $\Delta|_{C^{(\infty)}}=k\cdot\Gamma$ for some rational irreducible representation $\Gamma: C^{(\infty)}\to\mathrm{GL}_d(\mathbb{Q})$ with $d=24/k$. The assumption on the Fitting group of G implies that $C_G(G^{(\infty)})\subseteq\pm G^{(\infty)}$.

Since all subgroups of $\mathrm{GL}_3(\mathbb{Q})$ are soluble, $d>3$.

If $d=4$, then $C^{(\infty)}$ is conjugate to Alt_5, and C is reducible.

If $d=6$, one has the following possibilities for $C^{(\infty)}$: Alt_5, $L_2(7)$ (two matrix groups), Alt_7, or

$U_4(2)$. In all cases the group $O_2(G)C^{(\infty)}$ is reducible and $[G{:}O_2(G)C^{(\infty)}] \leq 2$; therefore G is imprimitive.

If $d=8$, then $C^{(\infty)}$ is conjugate to one of $\mathrm{SL}_2(5)$ or $\mathrm{SL}_2(9)$ (see Table 4). Since $\mathrm{Out}(C^{(\infty)})$ is a 2-group, G is reducible.

If $d=12$, then $C^{(\infty)}$ is conjugate to one of $\mathrm{SL}_2(5)$, $\mathrm{SL}_2(11)$, $L_3(3)$, $U_3(3)$, $2.M_{12}$, or Alt_{13} (see Table 4. In all cases one has $|\mathrm{Out}(G^{(\infty)})| = 2$ and hence G is imprimitive.

If $d = 24$, then $G^{(\infty)}$ is already $\mathbb{Q}$-irreducible and the statement of the proposition follows from Proposition 5.1 and 5.2. $\square$

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	$\mathbb{F}_q$	field with q elements
	ζ_m	primitive m-th root of unity
	$\mathcal{Q}_{p,q}$	quaternion algebra over $\mathbb{Q}$ ramified at p and q (with Hasse invariant $\frac{1}{2}$)
	$\mathcal{Q}_{\alpha,p,q}$	quaternion algebra over $\mathbb{Q}[\alpha]$ ramified at places over p and q (with Hasse invariant $\frac{1}{2}$)
	I_n	$n \times n$-unit matrix
	G'	derived subgroup of the group G
	$G^{(\infty)}$	last term of the derived series of the group G
	$O_p(G)$	biggest normal p-subgroup of G
	$O_{p'}(G)$	biggest normal subgroup of G of order prime to p
	$O^{p'}(G)$	smallest normal subgroup of G with index prime to p
	C_n	cyclic group of order n
	D_{2n}	dihedral group of order $2n$
	Q_{2n}	generalized quaternion group of order $2n$
	QD_{2^n}	generalized quasidihedral group of order 2^n with presentation $\langle a, b \mid a^2, b^{2^{n-1}}, b^a = b^{-1+2^{n-2}} \rangle$
	$\mathrm{SL}_n^{\pm}(q)$	group of $n \times n$-matrices over $\mathbb{F}_q$ with determinant ± 1
	Alt_n	Alternating group of degree n
	2_+^{1+2n}	central product of n copies of D_8
	p_+^{1+2n}	extraspecial p-group of prime exponent $p \neq 2$
	$S_6(2), U_4(2), \ldots$	groups of Lie type in the notation of [Conway et al. 1985]
	$A \wr B$	wreath product of the group A with B
	$A \overset{C}{\curlywedge} B$	subdirect product of the groups A and B amalgamated over the common factor group C
	$A \underset{C}{\curlyvee} B$	central product of the groups A and B with identified central subgroup C
	$A \overset{C}{\diamond} B$	subcentral product of the groups A and B amalgamated over the common factor group C
	$\mathrm{Out}(G)$	the outer automorphism group of G
1.1	$\mathcal{F}(G)$	space of G-invariant quadratic forms
1.1	$\mathcal{F}_{>0}(G)$	positive definite cone in $\mathcal{F}(G)$
1.1	$\mathcal{Z}(G)$	set of G-invariant lattices
1.2	$\mathrm{Aut}(F, L)$	automorphism group of the pair (F, L)
1.2	$\mathcal{B}(\mathcal{F}, L)$	Bravais group of $\mathcal{F}$ with respect to L
1.2	$\mathcal{B}(G, L)$	Bravais group of G with respect to L
1.4	$L^{\#(F)}$	dual lattice of L (with respect to F)
1.4	$L^{\mathrm{ev}(F)}$	even sublattice of L (with respect to F)
1.4	$\det(F, L)$	determinant of L (with respect to F) $= \lvert L^{\#(F)}/L \rvert$
1.8	$A_n, \ldots, E_8$	root systems
1.8	$M_{p+1,r}^{\alpha,\beta}$	lattices of $L_2(p)$ of degree $p+1$
1.8	$A_{p-1}^{(m)}$	lattices of $L_2(p)$ of degree $p-1$ (Craig lattices)
1.8	$\pm A$	$\langle A, -I \rangle$
1.8	$N{:}H$	semidirect product of N with H
1.8	$N.H$	extension of N with H (usually nonsplit)
1.8	$A \underset{Q}{\otimes} B$	
1.9	$A \overset{2(p)}{\underset{Q}{\otimes}} B$, $A \overset{2(p)}{\underset{Q}{\boxtimes}} B$, $A \overset{2(p)}{\underset{Q}{\boxtimes}} B$, $A \overset{2(p)}{\bigcirc} B$, $A \overset{2(p)}{\square} B$, $A \overset{2(p)}{\sqsupset} B$, etc.	

TABLE 6. Notations used in this article.

B_1

$$\begin{array}{cc} & 1 \\ 2 & \\ \bar1 & 2 \end{array}$$

A_2

F_4

$$\begin{array}{cccc} 2&0&\bar1&\bar1 \\ &2&1&1 \\ &&2&1 \\ &&&2 \\ 2&&& \\ 1&2&& \\ 1&1&2& \\ 1&1&1&2 \end{array}$$

A_4

E_6

$$\begin{array}{cccccc} 2&\bar1&1&1&\bar1&1 \\ &2&\bar1&0&1&\bar1 \\ &&2&0&0&1 \\ &&&2&\bar1&1 \\ 3&&&&2&\bar1 \\ 1&3&&&&2 \\ 0&1&3&&& \\ \bar1&1&1&3&& \\ 1&0&1&\bar1&3& \\ \bar1&\bar1&1&0&1&3 \end{array}$$

$M_{6,2}$

A_6

$$\begin{array}{cccccc} 2&1&1&1&1&1 \\ &2&1&1&1&1 \\ &&2&1&1&1 \\ &&&2&1&1 \\ 4&&&&2&1 \\ 1&4&&&&2 \\ \bar1&1&4&&& \\ 2&2&1&4&& \\ 1&2&2&1&4& \\ \bar2&1&0&0&\bar1&4 \end{array}$$

$A_6^{(2)}$

E_8

$$\begin{array}{cccccccc} 2&\bar1&0&1&1&\bar1&\bar1&\bar1 \\ &2&0&\bar1&\bar1&1&1&0 \\ 4&&2&1&0&0&1&\bar1 \\ \bar2&4&&2&1&\bar1&0&\bar1 \\ 0&0&4&&2&0&\bar1&\bar1 \\ 1&\bar2&2&4&&2&0&0 \\ 1&\bar2&0&2&4&&2&0 \\ \bar2&2&\bar1&\bar2&0&4&&2 \\ \bar2&2&2&0&\bar1&0&4& \\ \bar1&\bar1&\bar2&\bar1&\bar1&0&\bar1&4 \end{array}$$

$[(\mathrm{SL}_2(5) \overset{2}{\square} \mathrm{SL}_2(5)):2]_8$

$$\begin{array}{cccccccc} 4&&&&&&& \\ 2&4&&&&&& \\ 2&0&4&&&&& \\ \bar2&\bar1&\bar2&4&&&& \\ \bar1&\bar2&\bar1&2&4&&& \\ 2&1&2&\bar1&0&4&& \\ 2&2&1&\bar2&\bar1&0&4& \\ 1&\bar1&2&0&1&0&1&4 \end{array}$$

$M_{8,3}$

$[6.U_4(3).2^2]_{12}$

$$\begin{array}{cccccccccccc} 4&0&1&\bar2&0&\bar2&0&1&\bar2&1&0&\bar2 \\ &4&0&2&1&1&\bar2&\bar2&0&0&2&0 \\ 4&&4&1&\bar2&\bar2&0&2&0&0&1&\bar2 \\ \bar2&4&&4&\bar1&1&\bar2&\bar1&2&0&2&0 \\ \bar1&1&4&&4&2&0&\bar2&0&\bar2&\bar1&2 \\ \bar1&0&0&4&&4&\bar1&\bar2&2&\bar2&\bar1&2 \\ 0&0&0&\bar1&4&&4&2&\bar2&0&\bar2&0 \\ \bar1&0&2&1&1&4&&4&\bar1&1&0&\bar1 \\ \bar1&\bar1&0&0&2&2&4&&4&\bar2&0&2 \\ 1&0&0&1&1&1&0&4&&4&2&\bar2 \\ 2&1&1&0&\bar1&0&\bar1&1&4&&4&\bar1 \\ \bar1&\bar1&0&1&0&1&0&1&0&4&&4 \\ 0&\bar1&\bar1&2&1&1&1&2&1&2&4& \\ 0&0&0&\bar1&0&\bar1&0&\bar1&1&2&1&4 \end{array}$$

$[3_+^{1+2}:\mathrm{SL}_2(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{12}$

$[\pm D_{10} \underset{\sqrt{5}}{\overset{2}{\boxtimes}} \mathrm{Alt}_5]_{12}$

$$\begin{array}{cccccccccccc} 4&\bar2&0&0&\bar2&0&0&2&\bar1&1&\bar1&0 \\ &4&\bar1&1&0&0&\bar2&0&1&\bar1&0&0 \\ 4&&4&\bar2&\bar1&0&1&1&0&2&1&0 \\ \bar1&4&&4&0&\bar2&0&\bar1&0&0&1&0 \\ 2&\bar1&4&&4&\bar1&1&\bar1&2&0&1&1 \\ \bar1&2&\bar1&4&&4&\bar2&0&0&\bar1&\bar2&0 \\ \bar2&2&\bar1&1&4&&4&0&0&1&2&1 \\ \bar1&1&1&0&1&4&&4&1&2&0&2 \\ 1&0&1&\bar1&0&1&4&&4&2&1&2 \\ 0&\bar1&\bar1&1&\bar1&\bar1&1&4&&4&2&2 \\ 0&0&1&2&0&1&\bar1&1&4&&4&2 \\ \bar1&1&1&2&1&1&1&1&2&4&&4 \\ \bar1&\bar1&0&\bar1&0&1&1&1&1&1&4& \\ 1&0&1&1&\bar1&1&1&0&1&2&1&4 \end{array}$$

$[\mathrm{SL}_2(5) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{12}$

$[\pm 3.\mathrm{Alt}_6.2^2]_{12}$

$$\begin{array}{cccccccccccc} 8&1&2&4&0&3&\bar4&\bar3&\bar2&\bar2&\bar3&2 \\ &8&0&\bar1&2&\bar1&\bar3&3&\bar2&1&\bar4&4 \\ 4&&8&\bar1&2&4&\bar3&2&3&\bar4&4&0 \\ \bar1&4&&8&\bar2&0&\bar3&\bar1&\bar4&\bar2&1&\bar2 \\ \bar2&1&4&&8&4&\bar3&2&4&\bar2&0&4 \\ 0&\bar2&1&4&&8&\bar4&\bar2&2&\bar4&\bar1&1 \\ 1&2&0&\bar2&4&&8&\bar1&1&2&3&\bar3 \\ 0&\bar1&0&\bar1&0&4&&8&1&1&\bar1&0 \\ 1&\bar1&1&2&0&\bar1&4&&8&\bar2&1&2 \\ 0&0&1&0&0&1&0&4&&8&2&2 \\ \bar1&1&0&\bar1&0&0&1&\bar1&4&&8&\bar2 \\ 2&0&0&0&2&0&1&0&0&4&&8 \\ \bar1&0&2&2&\bar1&0&2&0&1&1&4& \\ 1&0&\bar1&\bar1&1&1&1&1&2&2&1&4 \end{array}$$

$[L_2(7) \overset{2(2)}{\otimes} D_8]_{12}$

$[L_2(7) \overset{2(2)}{\boxtimes} D_8]_{12}$

$$\begin{array}{cccccccccccc} 8&\bar2&\bar4&4&\bar2&\bar3&1&3&4&\bar4&4&4 \\ &8&4&\bar2&\bar3&4&3&\bar4&\bar1&2&\bar1&\bar1 \\ 2&&8&\bar1&1&3&3&\bar3&0&1&\bar4&\bar4 \\ 1&2&&8&\bar2&0&3&0&0&0&4&3 \\ 1&1&2&&8&1&\bar4&0&0&0&\bar2&\bar1 \\ 1&1&1&2&&8&2&\bar4&\bar1&2&\bar1&\bar2 \\ 1&1&1&1&2&&8&\bar3&1&1&\bar1&\bar2 \\ 1&1&1&1&1&2&&8&4&\bar2&0&3 \\ 1&1&1&1&1&1&2&&8&\bar4&0&0 \\ 1&1&1&1&1&1&1&2&&8&0&\bar1 \\ 1&1&1&1&1&1&1&1&2&&8&4 \\ 1&1&1&1&1&1&1&1&1&2&&8 \\ 1&1&1&1&1&1&1&1&1&1&2& \\ 1&1&1&1&1&1&1&1&1&1&1&2 \end{array}$$

A_{12}

$[2.\mathrm{Co}_1]_{24}$

$$\begin{array}{cccccccccccccccccccccccc} 4&2&1&1&1&1&1&2&1&2&1&2&1&1&2&1&2&2&2&2&2&1&2&1 \\ &4&2&1&1&2&1&1&2&2&1&1&1&1&2&1&1&2&2&2&2&1&2&1 \\ 4&&4&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&1 \\ 1&4&&4&2&2&2&2&2&2&2&2&2&2&2&2&2&2&1&1&1&2&2&0 \\ 1&1&4&&4&2&2&2&2&2&2&2&2&2&2&2&2&1&1&1&1&1&1&0 \\ 1&2&2&4&&4&2&2&2&2&2&2&2&1&2&1&2&2&2&2&1&1&2&1 \\ 1&2&0&0&4&&4&2&2&2&2&2&2&2&2&1&2&2&2&2&2&2&2&1 \\ 2&1&1&1&1&4&&4&2&2&2&2&2&2&2&2&2&2&1&2&2&2&2&1 \\ 2&1&1&1&1&2&4&&4&2&2&2&2&2&2&2&1&2&2&2&2&2&2&1 \\ 1&2&1&2&2&2&1&4&&4&1&1&2&2&2&2&2&2&2&2&1&2&2&1 \\ 1&2&1&2&2&0&1&2&4&&4&2&2&2&1&2&2&2&2&1&2&1&1&0 \\ 2&1&2&1&1&2&2&1&1&4&&4&2&2&2&2&2&1&2&2&2&2&2&0 \\ 2&2&1&2&2&2&2&2&2&2&4&&4&2&2&2&1&1&1&2&2&2&2&1 \\ 2&1&1&1&1&2&2&1&1&2&2&4&&4&2&2&1&1&2&2&1&2&2&0 \\ 2&1&2&1&1&1&2&1&1&2&2&1&4&&4&2&2&1&1&2&2&2&2&1 \\ 1&1&2&1&1&1&2&1&1&2&2&2&2&4&&4&2&1&2&2&2&2&1&1 \\ 2&1&2&2&0&2&2&1&0&2&2&2&2&2&4&&4&2&2&1&2&2&1&0 \\ 2&1&1&1&1&1&2&1&0&1&1&1&1&1&2&4&&4&2&2&2&1&2&1 \\ 2&2&2&2&0&2&2&1&1&2&2&2&1&2&2&1&4&&4&2&2&2&2&1 \\ 2&1&1&1&1&1&1&1&0&2&1&1&2&1&2&1&1&4&&4&2&2&2&2 \\ 2&2&1&1&1&1&1&1&1&1&1&1&2&1&1&1&1&2&4&&4&2&2&1 \\ 2&1&1&1&1&2&2&1&1&2&2&2&2&1&2&0&2&2&1&4&&4&2&1 \\ 2&1&1&2&0&2&2&1&0&2&2&1&2&1&2&1&2&2&1&2&4&&4&1 \\ 2&1&2&1&1&1&1&1&0&2&1&1&2&1&2&2&1&2&1&1&2&4&&4 \\ 2&0&2&1&1&1&2&1&1&1&1&2&2&2&2&2&1&1&1&1&1&2&4& \\ 2&1&1&2&0&2&1&1&0&2&2&2&1&1&2&1&2&2&1&2&2&1&0&4 \end{array}$$

$[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2 \underset{\sqrt{5}}{\overset{2}{\boxtimes}} \mathrm{Alt}_5]_{24,2}$

$[6.U_4(3).2 \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$

$$\begin{array}{cccccccccccccccccccccccc} 4&2&1&0&2&1&1&1&2&2&1&2&2&1&1&2&1&1&1&1&1&1&0&1 \\ &4&1&0&2&1&2&0&2&1&1&2&1&2&1&1&1&1&1&1&1&1&1&1 \\ 4&&4&1&1&2&2&2&2&1&1&0&2&1&2&1&1&2&1&1&2&1&2&2 \\ 2&4&&4&0&1&1&1&0&0&1&0&2&1&1&2&1&1&1&1&1&1&2&1 \\ 2&2&4&&4&2&1&1&2&2&1&1&1&1&1&1&1&1&0&1&1&2&1&1 \\ 2&2&2&4&&4&2&2&1&1&1&0&2&1&1&1&1&1&1&2&2&2&2&2 \\ 2&2&2&1&4&&4&1&1&1&2&1&2&2&1&1&0&2&1&2&2&1&2&2 \\ 2&2&1&2&2&4&&4&1&2&1&0&2&1&1&1&2&2&1&2&2&1&2&1 \\ 2&2&2&2&2&2&4&&4&2&1&1&1&1&2&1&1&1&0&1&1&1&1&1 \\ 1&2&2&2&2&2&2&4&&4&1&1&1&0&1&1&2&1&0&1&1&1&1&1 \\ 1&1&1&0&2&1&2&1&4&&4&2&2&1&0&1&0&2&2&2&2&0&1&1 \\ 2&1&2&2&2&2&2&2&1&4&&4&1&1&1&1&1&1&2&1&1&1&0&0 \\ 2&2&1&2&2&1&2&2&1&2&4&&4&1&1&2&1&2&1&2&2&1&1&2 \\ 1&1&1&1&1&1&2&1&2&1&1&4&&4&2&2&1&1&2&1&1&2&1&1 \\ 1&1&1&1&1&1&2&1&2&1&1&2&4&&4&2&2&1&1&0&1&2&1&1 \\ 1&2&0&1&1&2&2&1&2&1&1&2&2&4&&4&2&1&2&1&1&2&1&1 \\ 2&2&2&2&2&2&1&2&0&2&2&0&1&0&4&&4&1&1&1&1&2&1&0 \\ 1&1&1&1&1&1&2&1&1&1&1&2&2&2&0&4&&4&1&2&2&0&2&0 \\ 1&2&1&0&2&1&2&1&2&0&1&2&2&2&1&2&4&&4&1&1&1&1&1 \\ 1&2&0&1&1&2&2&1&1&1&1&2&2&2&1&2&2&4&&4&2&1&2&0 \\ 1&2&1&1&1&1&2&1&2&0&1&2&2&2&0&2&2&2&4&&4&1&2&0 \\ 1&1&1&1&1&1&2&1&2&1&1&2&2&1&1&1&2&2&1&4&&4&1&1 \\ 1&2&1&1&1&1&2&0&2&1&1&1&2&2&1&1&2&2&1&2&4&&4&1 \\ 1&2&1&0&1&1&2&1&2&1&1&2&1&2&0&1&2&2&2&2&2&4&&4 \\ 1&1&1&1&1&1&2&1&2&2&1&2&2&2&0&2&1&2&1&2&2&2&4& \\ 1&2&1&1&1&1&1&1&1&1&1&1&2&2&1&2&2&2&2&1&2&2&2&4 \end{array}$$

$[(\pm3).\mathrm{PGL}_2(9) \overset{2(2)}{\circ} \mathrm{SL}_2(3)]_{24}$

$[\mathrm{Sp}_4(3) \underset{\sqrt{-3}}{\overset{2}{\boxtimes}} (3_+^{1+2}:\mathrm{SL}_2(3))]_{24}$

$$\begin{array}{cccccccccccccccccccccccc} 4&1&2&1&1&2&2&1&1&1&1&2&2&2&2&1&1&1&1&2&1&1&2&1 \\ &4&2&0&1&2&1&1&0&1&1&0&1&1&0&1&1&1&1&1&1&1&1&1 \\ 4&&4&1&2&1&1&1&1&2&1&1&1&1&1&2&2&2&1&1&0&0&1&1 \\ 1&4&&4&2&1&1&2&2&1&1&2&1&1&1&1&1&1&1&1&1&1&1&1 \\ 2&1&4&&4&2&0&1&2&1&1&1&0&0&1&1&1&1&1&0&1&1&0&1 \\ 1&2&0&4&&4&1&0&1&1&0&1&1&1&1&0&1&1&0&1&1&1&1&0 \\ 1&0&1&0&4&&4&1&1&2&1&2&2&2&2&1&1&1&0&2&0&0&2&0 \\ 1&0&1&0&0&4&&4&1&1&2&1&1&1&0&2&1&1&2&1&2&2&1&2 \\ 0&1&0&1&1&0&4&&4&1&2&1&1&1&2&1&2&1&1&1&1&1&0&1 \\ 0&0&0&1&0&1&2&4&&4&1&1&1&1&1&2&2&2&1&1&0&0&1&0 \\ 1&0&1&0&0&0&0&1&4&&4&1&2&1&1&1&1&1&2&1&2&2&1&2 \\ 0&1&0&1&\bar1&0&0&0&\bar1&4&&4&2&2&2&2&1&1&0&2&1&1&2&0 \\ 0&0&1&0&0&0&1&1&2&\bar1&4&&4&2&2&1&2&1&1&2&1&1&2&1 \\ 0&1&0&1&\bar1&\bar1&0&0&0&0&1&4&&4&2&1&1&2&1&2&0&1&2&0 \\ 0&0&0&1&1&0&0&0&\bar1&0&0&0&4&&4&1&1&1&0&2&0&1&0&0 \\ \bar1&0&0&0&0&0&1&1&0&0&0&0&0&4&&4&2&2&1&1&1&1&1&1 \\ 0&0&0&1&\bar1&1&0&0&\bar1&0&0&0&0&1&4&&4&2&1&1&1&0&1&1 \\ 0&0&0&1&0&1&0&0&1&0&1&0&0&\bar1&0&4&&4&1&1&0&1&1&1 \\ 1&0&1&0&0&0&\bar1&\bar1&0&0&0&1&\bar1&0&1&\bar1&4&&4&0&2&2&1&2 \\ 1&0&1&0&0&0&\bar1&\bar1&0&1&0&0&1&0&0&\bar1&0&4&&4&0&1&0&1 \\ 0&0&1&0&0&0&\bar1&0&0&1&0&0&0&0&1&0&2&0&4&&4&2&1&2 \\ 0&1&0&1&0&1&0&0&\bar1&0&\bar1&0&2&1&0&0&\bar1&0&\bar1&4&&4&0&2 \\ 0&1&0&1&0&\bar1&0&0&1&0&0&0&0&1&0&0&0&\bar1&1&0&4&&4&0 \\ 0&0&0&1&\bar1&\bar1&0&0&1&0&1&0&0&1&0&0&0&1&\bar1&0&0&4&&4 \\ 0&0&1&0&0&2&1&1&0&1&0&\bar1&\bar1&0&1&1&0&0&0&0&\bar1&0&4& \\ 0&0&0&1&\bar1&1&0&0&\bar1&2&0&0&0&\bar1&0&0&1&0&1&0&0&0&1&4 \end{array}$$

$[3.S_6 \overset{2(2)}{\otimes} D_8]_{24}$

TABLE 7. The invariant forms of the primitive r.i.m.f. subgroups of $GL_d(\mathbb{Q})$ with d dividing 24 that are not tensor products of forms of smaller dimension. For compactness, we write $-x$ as $\bar{x}$.

$[6.L_3(4).2 \overset{2(2)}{\otimes} D_8]_{24}$

8 4 4 4 2 4 4 2 2 4 0 4 2 2 4 2 4 4 4 1 2 1 3 1
  8 4 4 4 4 4 4 2 4 4 4 2 2 4 4 4 4 4 3 4 2 2 2
8 8 4 4 3 3 2 4 3 2 3 3 3 4 4 4 2 3 1 4 1 2 1
3 8 8 2 4 4 4 4 4 4 4 2 2 4 4 4 4 4 2 3 3 4 3
4 1 8 8 3 3 4 4 3 4 3 2 3 0 4 4 2 3 3 2 1 2 1
3 3 4 8 8 4 2 3 4 4 4 2 2 3 3 4 4 4 2 2 2 3 4
3 3 2 3 8 8 2 3 4 4 4 1 2 3 3 4 4 4 3 3 2 4 4
4 2 3 1 3 8 8 4 2 2 2 3 1 0 4 4 0 2 2 3 2 3 2
4 4 3 2 3 2 8 8 3 4 3 2 3 0 4 4 2 3 1 3 3 3 3
4 3 4 2 2 4 3 8 8 4 4 3 4 3 3 4 4 4 2 3 4 2 2
2 3 2 4 3 3 4 3 8 8 4 0 2 2 4 4 4 4 3 2 3 1 3
4 3 4 2 4 3 4 4 4 8 8 1 2 3 3 4 4 $\underline{4}$ 4 2 1 2 1
2 3 2 3 4 3 4 2 4 3 8 8 3 3 4 1 1 $\bar{1}$ 3 1 3 2 3
3 3 4 2 1 3 3 3 1 3 2 8 8 3 3 2 2 2 1 3 2 1 1
4 4 2 2 3 3 4 2 4 4 2 3 8 8 4 2 4 3 2 2 2 1 2
3 2 4 4 2 2 3 4 3 3 2 1 1 8 8 4 4 3 3 4 4 3 4
4 3 3 4 2 1 4 3 4 2 2 3 4 2 8 8 4 4 0 3 2 1 2
2 3 4 3 2 3 1 4 3 2 0 4 2 2 4 8 8 4 2 2 4 2 4
2 4 4 3 0 2 4 4 3 2 2 4 4 2 3 4 8 8 1 3 3 3 1
4 2 3 3 3 3 4 1 2 2 4 4 4 1 4 2 3 8 8 2 3 2 3
4 3 2 4 3 3 4 2 4 2 2 4 4 1 4 3 3 4 8 8 3 4 3
2 4 2 2 3 3 4 3 4 2 4 2 4 1 4 2 4 2 4 8 8 2 4
4 3 2 3 4 2 2 2 3 4 2 3 4 2 4 4 2 4 2 1 8 8 4
4 2 4 3 3 4 2 4 3 4 1 3 4 2 2 4 4 2 4 3 4 8 8
4 2 2 3 3 2 4 3 4 3 4 1 2 3 2 1 1 2 2 0 3 2 8
4 3 3 2 4 4 2 4 2 3 4 1 1 4 1 2 1 3 1 2 2 2 2 8

$[(\mathrm{SL}_2(3) \circ C_4).2 \overset{2(3)}{\underset{\sqrt{-1}}{\boxtimes}} U_3(3)]_{24}$

A_{24}

2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
  2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
8 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
4 8 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
3 4 8 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
3 4 4 8 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
3 4 4 3 8 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
2 3 4 4 2 8 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
4 3 4 2 4 4 8 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
1 4 4 4 4 4 4 8 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1
2 4 4 4 2 3 4 4 8 2 1 1 1 1 1 1 1 1 1 1 1 1 1
4 4 4 3 3 4 4 3 4 8 2 1 1 1 1 1 1 1 1 1 1 1 1
3 4 4 4 4 2 3 4 4 3 8 2 1 1 1 1 1 1 1 1 1 1 1
3 4 3 4 4 3 4 4 4 1 2 8 2 1 1 1 1 1 1 1 1 1 1
3 3 4 3 4 4 4 4 2 4 2 3 8 2 1 1 1 1 1 1 1 1 1
3 4 4 4 1 4 4 4 4 4 3 3 4 8 2 1 1 1 1 1 1 1 1
4 4 4 3 3 4 4 3 3 4 3 1 2 4 8 2 1 1 1 1 1 1 1
3 2 4 3 2 4 4 2 3 4 3 1 4 4 4 8 2 1 1 1 1 1 1
4 4 4 4 2 4 4 4 4 4 4 3 4 4 2 4 8 2 1 1 1 1 1
2 3 4 3 4 2 2 4 2 3 4 3 3 3 2 4 3 8 2 1 1 1 1
3 2 4 3 4 4 2 4 2 4 4 1 4 3 3 4 3 4 8 2 1 1 1
3 4 3 3 4 1 2 2 4 3 4 4 4 3 1 3 2 3 3 8 2 1 1
3 2 2 3 4 1 4 4 4 1 4 4 3 3 3 3 2 4 3 4 8 2 1
4 3 4 2 4 4 4 4 2 2 3 3 4 2 4 2 3 2 4 2 4 8 2
1 4 4 4 3 4 2 3 3 1 4 4 2 3 2 3 3 4 2 3 2 1 8
3 3 4 4 4 4 4 3 3 4 4 4 4 4 2 4 4 4 4 4 2 2 4 8

$[2.J_2 \overset{2}{\square} \mathrm{SL}_2(5)]_{24}$

$[\mathrm{SL}_2(5) \overset{2(2)}{\underset{\infty,2}{\otimes}} 2_-^{1+4'}.\mathrm{Alt}_5]_{24}$

6 3 3 3 3 3 1 2 2 2 3 2 1 2 1 1 3 1 1 1 1 3 2 2
  6 1 3 1 1 2 1 1 1 0 1 1 1 2 2 1 2 2 1 1 1 1 1
8 6 1 3 3 2 1 2 2 3 3 2 1 0 1 3 2 0 2 2 3 2 1
1 8 6 1 2 1 1 1 1 1 1 2 1 1 1 1 1 1 2 2 2 1 1
1 2 8 6 3 3 1 1 2 3 2 2 2 2 1 3 2 2 2 0 3 2 3
4 2 2 8 6 1 1 2 2 3 2 0 2 2 1 3 3 1 2 0 3 2 2
3 3 1 3 8 6 1 1 2 2 2 3 3 3 3 2 3 3 2 1 2 3 2
4 2 2 4 3 8 6 3 3 3 3 3 1 2 1 1 1 1 3 3 2 1 3
3 2 4 1 2 1 8 6 3 2 3 3 1 2 2 3 2 0 3 3 2 3 3
4 2 2 2 0 2 1 8 6 2 3 3 3 2 2 2 2 1 3 0 3 2 3
4 2 2 2 2 4 1 4 8 6 2 2 1 2 2 3 2 2 3 2 3 1 2
3 4 2 1 1 1 4 1 1 8 6 3 1 1 0 1 2 0 3 3 2 2 3
3 3 1 3 2 2 2 2 0 1 8 6 1 2 2 2 1 1 3 $\underline{3}$ 2 2 3
3 3 1 2 2 3 2 2 3 1 4 8 6 3 3 2 3 3 1 $\bar{1}$ 2 3 2
2 4 4 4 3 4 2 4 4 2 3 3 8 6 3 2 3 3 2 0 2 3 3
4 2 2 4 2 2 1 4 2 1 3 0 4 8 6 3 3 3 2 1 1 3 1
3 3 1 2 4 0 2 3 2 1 2 4 3 3 8 6 2 1 2 1 3 3 2
3 1 3 2 2 0 1 3 2 2 1 2 3 3 4 8 6 3 3 1 1 3 2
2 2 2 2 3 4 1 0 0 1 3 1 4 4 1 1 8 6 2 0 0 0 2
3 1 3 3 4 3 1 0 2 2 1 1 3 2 2 4 3 8 6 3 1 1 3
3 1 3 2 1 3 1 2 3 2 2 4 3 0 2 4 1 $\underline{2}$ 8 6 0 1 0
2 4 1 2 2 1 1 2 1 2 3 0 2 4 3 1 2 $\bar{1}$ 0 $\underline{8}$ 6 2 2
3 1 3 0 2 2 1 3 3 2 2 1 3 2 1 2 2 4 2 $\bar{1}$ 8 6 2
3 1 3 3 1 2 1 2 0 2 4 2 3 3 1 2 3 2 4 1 4 8 6
3 3 1 0 4 2 2 3 3 1 4 2 3 2 2 1 2 2 1 2 4 2 8
4 2 3 2 3 2 3 2 2 1 3 3 1 2 3 1 1 1 1 4 1 1 3 8

$[(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)):2 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} \mathrm{Alt}_5]_{24,1}$

$[\mathrm{SL}_2(5) \overset{2}{\underset{\infty,3}{\boxtimes}} (\pm 3_+^{1+2}).\mathrm{GL}_2(3)]_{24}$

8 2 2 2 4 4 2 4 2 1 2 4 2 1 $\underline{1}$ 2 2 2 1 2 4 2 2 2
  8 4 4 2 1 2 4 4 2 2 2 1 4 $\bar{1}$ 0 2 2 2 2 2 0 0 1
4 8 2 4 1 2 4 2 2 2 1 2 4 1 2 2 2 1 2 4 2 2 $\underline{2}$
2 4 8 1 2 $\underline{2}$ 4 2 2 4 4 1 4 1 4 0 0 2 2 1 0 0 $\bar{1}$
1 1 4 8 2 $\bar{2}$ 2 0 1 1 2 2 2 0 1 4 0 2 1 2 1 4 0
2 2 1 4 8 1 2 0 1 4 4 1 2 2 2 2 2 $\underline{1}$ 2 2 4 $\underline{4}$ 2
1 2 1 1 4 8 4 4 1 2 1 2 1 1 2 0 4 $\bar{1}$ 2 4 2 $\bar{1}$ 2
1 1 1 0 1 4 8 2 2 4 2 2 2 $\underline{0}$ 4 2 2 0 1 2 1 $\underline{1}$ 0
1 1 1 2 0 0 4 8 1 1 2 0 $\underline{2}$ $\bar{1}$ 0 1 4 1 $\underline{2}$ 4 1 $\bar{1}$ 1
1 2 1 1 1 1 2 4 8 2 1 1 $\bar{2}$ 2 4 2 2 2 $\bar{1}$ 1 0 0 2
2 1 0 1 0 1 2 2 4 8 2 2 4 2 4 1 1 1 2 1 2 2 1
1 2 1 1 2 1 0 1 0 4 8 2 2 1 2 0 $\underline{0}$ 1 4 2 2 2 1
1 2 1 1 1 1 2 2 2 1 4 8 2 4 0 2 $\bar{2}$ 1 4 0 2 2 4
1 1 1 2 1 1 1 0 0 0 $\underline{0}$ 4 8 1 0 0 $\underline{0}$ 1 4 2 2 2 1
2 1 1 2 1 1 1 0 0 1 $\bar{1}$ 2 4 8 2 1 $\bar{1}$ 2 2 1 4 2 4
1 1 2 1 1 1 1 1 0 1 1 1 1 4 8 2 2 2 0 1 2 2 0
1 2 1 0 1 2 0 0 1 1 1 1 1 1 4 8 2 4 0 2 1 4 2
1 2 1 0 1 1 1 2 $\underline{2}$ 1 1 0 0 1 2 4 8 0 $\underline{0}$ 4 1 1 0
1 2 1 2 1 0 1 1 $\bar{1}$ 1 0 2 2 1 0 0 4 8 $\bar{1}$ 0 0 2 2
2 2 1 2 1 0 2 1 1 1 1 1 1 1 1 1 1 4 8 1 4 2 2
1 1 1 2 0 1 1 0 0 1 0 2 2 2 1 0 2 1 4 8 1 1 2
1 2 1 1 2 1 0 1 0 2 1 1 1 0 1 1 1 1 0 4 8 2 4
2 2 0 1 2 1 0 1 1 2 1 1 1 0 1 1 1 0 0 2 4 8 1
1 1 2 1 0 2 1 1 0 1 1 1 1 2 1 1 1 1 2 1 0 4 8
1 1 2 2 0 1 1 0 0 0 0 2 2 1 1 0 2 1 2 1 0 1 4
1 1 1 1 1 1 1 1 0 1 0 2 2 0 0 0 2 1 1 2 1 0 2 4

$[L_2(7) \overset{2(2)}{\otimes} F_4]_{24}$

$[L_2(7) \overset{2(2)}{\boxtimes} F_4]_{24}$

8 3 1 3 4 4 2 4 4 4 4 3 3 2 4 2 3 2 3 2 2 2 2 2
  8 2 4 2 2 2 2 1 2 4 1 4 1 2 1 2 4 4 1 4 4 1 4
C 8 1 2 2 2 2 4 2 2 3 1 4 2 4 1 2 1 4 2 2 4 1
6 C 8 3 3 2 3 2 3 1 1 3 1 3 3 4 $\underline{2}$ 1 1 2 2 1 2
4 6 C 8 4 4 4 4 4 2 3 2 2 0 2 3 $\bar{1}$ 2 2 1 1 1 2
5 6 4 C 8 4 4 4 4 2 3 2 2 4 2 2 2 3 2 0 1 1 2
6 4 6 1 C 8 2 0 2 1 1 1 2 0 3 4 0 2 2 $\underline{1}$ 1 1 3
6 2 6 6 2 C 8 4 4 2 3 2 4 4 2 0 1 2 3 $\bar{1}$ 1 2 2
4 4 6 6 4 6 C 8 4 4 3 1 4 4 1 1 1 2 4 $\underline{1}$ 1 4 1
6 6 6 5 6 2 5 C 8 2 $\underline{3}$ 2 0 4 $\underline{2}$ 1 1 0 3 $\bar{1}$ 0 2 1
4 4 6 4 4 6 6 2 C 8 $\bar{1}$ 0 3 2 $\bar{1}$ 2 $\underline{4}$ 4 3 $\underline{4}$ $\underline{4}$ 3 2
5 3 4 4 6 6 4 2 4 C 8 1 1 3 3 1 $\bar{1}$ 0 1 $\bar{1}$ $\bar{1}$ 1 1
6 6 6 6 4 6 6 6 6 4 C 8 0 2 4 1 2 2 0 2 2 0 2
4 1 4 6 4 6 5 6 4 6 5 C 8 2 2 2 1 3 4 1 3 4 3
5 5 5 4 3 6 3 5 6 6 5 6 C 8 2 0 2 1 2 0 0 3 0
4 6 6 6 4 6 2 3 5 6 6 4 6 C 8 3 1 0 2 1 1 2 2
6 5 6 3 6 4 4 4 6 4 2 2 6 6 C 8 2 2 1 4 2 1 2
5 6 4 6 2 6 4 4 4 6 4 2 6 6 6 C 8 4 3 4 4 1 2
4 6 4 6 2 4 3 6 4 4 6 5 6 6 6 6 C 8 3 4 2 1 2
4 4 4 3 6 4 0 5 2 6 4 6 4 6 4 5 6 C 8 2 1 4 0
6 6 4 5 6 4 4 6 0 6 2 4 4 5 5 6 4 6 C 8 4 $\underline{1}$ 0
5 4 4 6 3 4 2 3 3 3 4 6 2 2 1 0 2 4 2 C 8 $\bar{1}$ 4
6 6 5 6 3 5 6 4 4 6 4 4 4 5 5 6 4 1 5 4 C 8 1
2 4 4 6 3 4 5 6 4 5 4 4 6 3 3 5 5 2 3 2 6 C 8
2 6 6 3 5 3 4 6 6 2 4 4 6 6 6 5 6 6 4 1 3 3 C
6 6 6 6 2 6 6 4 5 4 6 2 5 6 4 6 2 1 3 0 6 5 2 C

$[\mathrm{SL}_2(13) \overset{2(2)}{\square} \mathrm{SL}_2(3)]_{24}$ (C = 12)

$[\mathrm{SL}_2(7) \overset{2}{\underset{\sqrt{-7}}{\boxtimes}} L_2(7)]_{24}$

4 1 1 2 1 2 2 1 1 2 2 2 2 2 2 2 2 2 2 2 1 2 2 2
  4 2 2 1 2 1 1 2 1 1 2 1 2 1 2 2 1 1 2 2 1 2 2
4 4 1 2 1 2 1 2 2 2 1 1 1 2 2 2 1 2 1 2 2 1 2
1 4 4 1 2 1 1 2 2 2 2 2 2 1 1 2 2 1 2 1 2 1 2
0 1 4 4 1 1 2 2 2 1 1 2 1 2 2 2 2 2 1 1 2 1 1
1 0 1 4 4 2 2 1 1 1 2 2 2 2 1 2 0 1 2 2 1 2 2
1 1 1 1 4 4 2 2 1 2 1 2 1 2 1 2 1 2 1 2 1 1 2
0 1 2 1 1 4 4 2 1 1 1 1 2 2 1 2 1 2 1 1 1 0 0
1 0 1 2 2 2 4 4 1 2 1 2 1 2 1 2 2 2 1 1 1 0 1
0 1 2 1 1 2 1 4 4 2 1 2 2 2 2 2 2 1 2 1 2 1 2
1 1 1 1 0 1 1 2 4 4 2 2 1 2 1 2 2 2 2 1 2 1 2
1 1 1 1 2 1 1 1 0 4 4 1 2 1 1 2 1 1 2 1 2 2 2
1 1 1 1 2 2 2 2 1 1 4 4 1 2 1 2 2 1 2 1 1 1 2
1 1 1 1 0 2 2 1 1 0 1 4 4 2 2 2 1 1 2 1 1 1 1
1 1 1 1 1 2 2 1 0 1 2 2 4 4 2 2 1 2 1 1 1 1 1
2 1 0 1 1 1 2 0 1 1 2 1 2 4 4 2 2 2 2 2 1 2 1
2 1 0 1 2 1 2 1 0 1 2 1 2 2 4 4 2 2 2 2 2 2 1
1 2 1 0 1 1 1 1 1 0 1 2 1 1 1 4 4 2 2 1 2 1 1
1 1 1 1 1 1 2 0 0 0 1 2 1 1 1 1 4 4 1 2 2 1 1
1 2 1 0 1 1 0 1 2 0 1 1 0 0 0 2 1 4 4 2 1 2 2
0 1 2 1 1 1 1 1 1 0 1 1 1 1 0 2 1 1 4 4 1 2 2
1 2 1 0 2 0 0 0 0 2 1 0 1 1 1 1 1 2 1 4 4 1 2
1 1 1 1 2 0 1 0 1 0 1 0 0 1 0 1 2 2 2 2 4 4 2
1 1 1 1 2 1 2 0 0 0 2 1 1 2 1 1 2 1 1 1 2 4 4
1 0 1 2 2 1 2 1 0 1 2 1 1 1 2 0 1 0 1 1 1 2 4
1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 2 1 1 2 0 1 2 1 4

$[6.\mathrm{Alt}_7:2]_{24}$

TABLE 7 (continued)

$[3.M_{10} \overset{2(2)}{\underset{\sqrt{-3}}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$

8 1 2 2 2 1 4 2 4 4 0 1 3 1 2 4 2 2 1 2 2 2 4 $\bar{1}$
8 2 4 4 2 1 2 1 0 2 4 2 2 2 2 2 1 4 2 3 1 0 2
A 8 2 4 2 2 2 2 4 2 4 2 2 2 4 1 2 0 3 4 4 0 2
1 A 8 2 1 2 2 4 1 1 3 1 1 3 2 4 1 2 4 2 3 2 1
1 0 A 8 2 0 0 3 2 2 2 4 4 4 3 1 1 2 0 3 4 1 4
5 $\bar{1}$ 2 A 8 2 4 0 2 4 4 0 2 2 1 2 2 4 2 2 1 1 4
1 3 0 2 A 8 4 2 3 0 2 0 0 1 2 4 2 1 4 2 2 2 0
2 5 0 1 3 A 8 1 4 2 4 2 2 2 1 3 4 2 2 2 0 1 2
0 1 1 3 $\bar{1}$ 2 A 8 2 2 1 2 1 3 4 2 2 0 2 4 4 4 2
2 3 3 1 0 0 2 A 8 2 2 4 2 2 2 2 4 2 1 2 2 2 0
3 2 $\bar{1}$ 3 1 1 5 1 A 8 2 1 3 1 2 1 4 3 2 4 2 2 3
2 0 0 1 3 0 $\bar{2}$ 0 $\bar{1}$ A 8 2 0 4 2 1 2 2 4 4 1 0 2
$\bar{1}$ 0 0 1 5 3 1 0 2 3 A 8 4 2 1 2 3 2 0 2 2 3 0
0 3 0 0 0 3 1 0 $\bar{1}$ 3 0 A 8 2 2 1 2 4 0 1 2 2 4
2 0 5 1 0 0 2 0 1 0 0 3 A 8 3 1 1 2 1 2 1 1 2
0 0 3 0 0 3 $\bar{1}$ 0 $\bar{2}$ 3 0 5 3 A 8 0 0 1 4 2 2 0 1
3 1 2 0 $\bar{2}$ $\bar{1}$ 0 1 0 0 $\bar{1}$ $\bar{2}$ 1 $\bar{1}$ A 8 2 2 2 1 2 4 0
1 0 0 2 0 0 $\bar{1}$ 0 1 $\bar{3}$ 0 0 3 0 2 A 8 2 0 4 0 4 1
3 2 1 3 $\bar{1}$ 1 0 $\bar{1}$ 0 0 $\bar{2}$ $\bar{1}$ $\bar{1}$ $\bar{2}$ 5 1 A 8 1 1 0 1 2
0 2 0 0 0 1 3 $\bar{1}$ 0 1 0 $\bar{1}$ 0 $\bar{2}$ 3 $\bar{1}$ 0 A 8 2 3 0 0
2 0 0 1 $\bar{3}$ 0 $\bar{2}$ 0 $\bar{1}$ 0 $\bar{3}$ 3 0 3 $\bar{1}$ 3 1 $\bar{1}$ A 8 2 2 2
$\bar{1}$ 0 0 1 3 0 0 $\bar{3}$ 0 0 0 $\bar{3}$ 0 0 1 3 2 2 0 A 8 2 2
$\bar{3}$ 2 0 $\bar{3}$ 0 1 0 1 0 1 0 1 0 2 0 1 0 0 $\bar{1}$ 2 A 8 1
0 1 0 0 0 2 0 $\bar{2}$ $\bar{3}$ 2 0 1 0 $\bar{1}$ 3 1 3 5 $\bar{2}$ 1 0 A 8
3 0 1 3 $\bar{1}$ 0 0 1 0 1 $\bar{2}$ $\bar{1}$ $\bar{1}$ $\bar{2}$ 0 0 0 0 $\bar{1}$ $\bar{2}$ 0 3 A
0 1 $\bar{1}$ 0 $\bar{1}$ 2 0 2 $\bar{3}$ 0 1 $\bar{1}$ $\bar{2}$ 1 0 1 0 0 2 1 3 0 3 A

$[\mathrm{Alt}_5 \overset{2}{\underset{\sqrt{5}}{\boxtimes}} (C_3 \overset{2(2)}{\boxtimes} D_8)]_{24}$ (A = 10)

$[3.M_{10} \overset{2(2)}{\boxtimes} D_8]_{24}$ (G = 16)

G 6 8 5 6 4 4 8 5 5 6 5 2 5 5 1 1 1 4 4 8 1 1 5
G 6 3 0 3 6 6 8 0 8 8 4 3 3 0 3 8 4 3 0 0 5 8
4 G 1 6 5 8 4 1 1 6 1 1 1 4 5 5 5 5 5 4 5 2 1
2 4 G 5 6 6 6 1 6 0 6 4 3 6 5 8 0 4 4 3 2 5 3
2 2 4 G 4 0 0 0 8 8 0 5 5 4 8 5 0 3 4 6 8 4 0
2 2 2 4 G 5 8 5 8 4 2 6 6 1 4 4 6 $\bar{3}$ 5 4 4 5 2
2 2 2 2 4 G 4 5 1 3 5 6 1 4 5 5 5 6 6 0 5 3 6
2 2 2 2 2 4 G 5 4 3 5 2 6 5 0 4 5 0 6 8 0 0 6
2 2 2 2 2 2 4 G 4 5 8 8 5 5 0 0 8 5 5 4 4 5 8
2 2 2 2 2 2 2 4 G 5 4 6 6 5 8 5 0 0 4 5 8 4 1
2 2 2 2 2 2 2 2 4 G 5 5 8 4 3 0 3 3 3 3 5 4 5
1 1 1 1 1 2 1 0 0 4 G 4 6 6 0 3 0 5 0 4 4 2 8
2 2 2 1 1 1 1 0 0 2 4 G 5 6 5 1 4 $\bar{1}$ 5 2 6 4 8
2 1 1 2 1 1 0 0 1 2 2 4 G 8 3 3 2 0 $\bar{1}$ 3 5 0 1
1 2 1 2 1 2 0 1 0 2 2 2 4 G 5 5 0 6 2 6 4 $\bar{1}$ 6
1 2 1 1 2 1 0 0 1 2 2 2 2 4 G 8 4 5 8 5 8 8 $\bar{3}$
1 1 2 1 1 2 0 0 1 2 2 2 2 2 4 G 3 4 5 1 5 3 $\bar{3}$
1 1 2 1 2 1 0 1 0 2 2 2 2 2 2 4 G 5 6 0 0 6 0
1 1 1 2 2 1 1 0 0 2 2 2 2 2 2 2 4 G 5 4 3 6 2
2 2 2 2 2 2 2 2 2 2 1 2 1 1 1 2 1 4 G 6 4 8 2
2 2 2 2 2 2 2 2 2 2 1 1 1 2 1 1 1 2 4 G 5 3 5
2 1 1 1 1 1 0 1 0 2 2 2 2 2 2 2 2 1 2 4 G 4 1
2 2 1 1 1 1 1 1 0 1 2 1 2 1 0 1 1 1 1 2 4 G $\bar{1}$
1 1 2 1 1 1 1 1 0 2 2 1 1 0 1 2 1 2 1 1 1 4 G
1 2 2 1 1 0 1 1 0 1 2 1 1 1 0 2 1 2 1 1 2 2 4
2 1 1 1 1 1 2 1 0 2 2 1 1 1 0 1 1 1 2 2 2 1 1 4

$[(\pm L_3(3)).2 \overset{2}{\square} C_3]_{24}$

$[(\pm D_{78}).C_{12}]_{24}$

6 2 0 2 0 1 1 0 $\bar{1}$ 2 0 0 1 2 1 1 0 1 0 1 2 0 0 1
6 0 0 1 1 2 1 1 1 0 2 2 1 0 1 1 1 2 2 2 1 2 2
8 6 3 3 1 0 0 3 0 2 1 0 2 2 0 2 2 3 1 1 2 1 2
4 8 6 1 1 1 0 1 0 2 1 $\bar{1}$ 3 2 2 2 2 2 0 2 0 0 0
3 4 8 6 2 1 1 2 0 0 1 1 2 0 0 2 2 3 2 0 2 1 2
3 4 4 8 6 2 1 2 2 0 3 1 0 0 3 2 3 0 1 1 $\bar{1}$ 1 2
2 4 2 2 8 6 2 0 2 0 2 3 1 1 1 2 2 2 1 1 1 2 2
2 3 3 3 3 8 6 1 3 2 1 2 0 1 1 2 2 2 0 0 2 1 1
3 2 4 4 1 4 8 6 0 2 2 0 1 2 2 1 2 1 2 0 2 1 1
3 2 4 2 4 3 2 8 6 1 1 3 0 1 0 2 1 0 2 0 1 2 1
0 2 3 3 4 3 3 3 8 6 2 0 0 2 2 2 0 1 1 1 2 1 $\bar{1}$
3 2 1 2 4 4 1 2 3 8 6 2 1 0 1 2 2 1 1 2 1 2 0
3 2 2 1 4 3 2 4 3 4 8 6 0 0 $\bar{1}$ 1 1 2 2 1 3 3 2
2 3 2 3 3 2 3 4 4 3 3 8 6 1 1 1 0 2 1 1 1 $\bar{1}$ 1
3 4 4 2 2 3 4 2 3 2 4 3 8 6 0 2 2 1 2 2 3 3 $\bar{1}$
2 2 3 3 4 4 4 3 4 3 3 3 3 8 6 1 1 0 0 1 $\bar{2}$ $\bar{1}$ 1
3 4 2 4 2 2 2 1 3 4 2 3 4 3 8 6 2 2 0 0 1 2 2
4 3 4 3 3 3 3 2 2 3 3 0 3 2 3 8 6 2 1 2 1 2 0
4 1 1 2 2 3 2 2 2 4 4 2 2 3 2 4 8 6 0 2 2 1 2
3 1 2 2 2 2 4 4 3 2 4 3 2 2 1 3 4 8 6 0 3 3 $\bar{1}$
3 2 2 4 4 3 2 4 3 4 2 3 1 3 2 3 4 4 8 6 1 2 0
3 3 4 3 3 4 3 2 2 4 3 2 3 3 2 4 3 3 3 8 6 3 0
2 3 2 3 3 2 3 4 3 2 3 4 3 2 4 2 3 3 3 0 8 6 0
1 0 2 2 3 2 2 4 3 2 3 2 0 4 1 2 0 1 1 0 2 8 6
2 2 2 4 1 3 4 1 4 2 2 4 4 3 4 2 4 2 2 3 3 0 8
3 2 1 3 1 2 1 2 0 4 3 1 0 $\bar{1}$ 2 3 1 2 3 2 1 2 $\bar{1}$ 8

$[\pm 3.\mathrm{PGL}_2(9) \overset{2}{\underset{\sqrt{5}}{\boxtimes}} D_{10}]_{24}$

$[\pm U_4(2).2]_{24}$

6 3 3 2 3 3 3 2 2 1 3 3 2 3 1 2 3 3 2 2 0 3 3 2
6 3 3 3 3 3 3 1 1 2 2 3 3 1 1 3 3 1 2 0 3 0 3
4 6 3 3 3 2 2 1 1 2 3 3 3 2 2 3 3 2 0 1 3 1 2
1 4 6 3 2 3 3 2 1 3 3 3 3 1 1 2 3 1 2 1 3 $\bar{1}$ 2
1 1 4 6 3 3 3 2 1 3 2 2 3 1 1 3 2 1 2 1 3 0 3
1 1 1 4 6 2 3 1 1 3 2 3 3 2 1 3 3 1 0 1 3 1 3
1 1 1 1 4 6 3 2 1 3 3 3 3 1 2 3 3 1 2 0 2 0 3
1 1 1 0 2 4 6 1 0 3 3 3 2 1 1 3 3 1 0 1 3 $\bar{1}$ 3
2 1 0 1 2 2 4 6 2 0 2 0 1 3 3 0 1 1 2 0 1 1 2
2 0 1 1 2 2 2 4 6 1 0 1 1 2 1 0 1 1 1 2 0 2 0
1 1 0 1 2 2 2 1 4 6 3 3 3 1 0 2 3 1 2 1 3 1 2
2 1 0 1 1 2 2 2 2 4 6 3 3 2 1 3 3 1 1 1 3 1 3
1 1 1 2 1 1 2 2 1 2 4 6 3 0 1 3 3 2 0 1 2 $\bar{1}$ 3
1 1 1 1 2 2 2 1 2 2 2 4 6 1 2 3 2 2 2 1 3 0 2
2 1 1 0 1 1 2 1 2 1 1 1 4 6 3 0 1 0 2 $\bar{1}$ 1 3 2
1 1 1 2 2 1 2 2 1 1 2 2 1 4 6 0 1 3 2 0 0 1 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 4 6 2 0 0 1 3 0 3
2 1 1 1 1 2 2 2 1 2 2 2 1 1 1 4 6 0 0 0 3 0 2
1 1 0 1 2 1 2 2 2 1 1 1 2 2 2 1 4 6 2 2 0 1 1
0 1 1 2 1 1 1 1 2 1 1 1 1 2 1 0 2 4 6 0 1 2 1
1 2 0 1 2 2 2 1 2 2 1 2 1 2 2 1 2 1 4 6 1 0 $\bar{1}$
2 1 1 1 1 1 2 2 1 2 1 1 1 2 2 2 2 1 2 4 6 1 3
1 1 0 2 1 1 2 2 2 2 2 1 1 1 1 1 2 2 1 1 4 6 0
1 1 1 1 2 1 2 2 1 1 2 1 1 2 2 1 2 1 2 2 2 4 6
1 0 1 1 1 0 0 1 1 1 1 1 1 0 2 0 1 1 0 0 1 0 4
1 1 0 2 1 0 0 1 1 1 1 1 0 1 1 0 1 1 1 0 2 1 2 4

$[\mathrm{SL}_2(7) \circ \tilde{S}_4]_{24}$

$[\mathrm{SL}_2(7) \overset{2}{\circ} Q_{16}]_{24}$

4 0 1 1 1 1 1 1 0 1 1 0 1 1 1 1 0 1 1 1 0 $\bar{1}$ 1 0
4 1 1 1 1 1 1 1 1 0 1 1 0 1 1 1 1 $\bar{1}$ 0 1 1 1 1
C 4 0 1 0 1 1 1 1 1 $\bar{1}$ 0 1 1 0 $\bar{1}$ 1 0 1 1 1 1 1
1 C 4 1 1 1 1 1 1 0 0 0 1 0 1 1 1 1 1 1 0 0 1
4 6 C 4 1 1 0 1 1 1 1 0 0 1 0 0 1 1 0 0 0 0 1
6 2 1 C 4 1 1 0 1 0 1 1 1 0 1 1 0 1 0 1 0 1 $\bar{1}$
3 3 3 3 C 4 1 1 1 $\bar{1}$ 1 1 1 0 $\bar{1}$ 0 1 0 1 1 1 1 1
4 4 3 6 6 C 4 1 1 0 0 1 0 0 1 0 1 1 1 0 1 0 0
1 6 4 2 2 4 C 4 0 1 1 $\bar{1}$ 0 1 0 1 0 1 0 0 1 0 1
2 2 1 6 2 3 3 C 4 0 0 1 0 1 0 0 1 0 $\bar{1}$ 0 0 $\bar{1}$ 1
3 4 3 3 0 1 2 6 C 4 1 0 0 1 1 1 1 1 0 1 0 $\bar{1}$ 1
2 4 6 0 2 1 2 2 3 C 4 1 1 1 1 1 1 1 0 1 1 0 1
2 6 3 3 2 1 4 2 6 4 C 4 1 1 1 1 1 0 1 0 0 1 1
4 4 4 3 3 2 3 $\bar{1}$ 2 6 4 C 4 1 0 1 1 1 1 1 1 1 0
6 3 2 4 0 3 3 2 4 4 6 6 C 4 1 1 1 0 0 1 0 0 1
6 1 2 3 2 3 2 1 4 4 2 6 4 C 4 1 1 1 1 1 0 1 1
6 2 2 2 2 1 1 4 6 2 2 2 6 6 C 4 0 0 0 1 1 0 0
3 6 4 4 6 4 3 6 6 2 4 1 2 0 2 C 4 1 0 1 0 0 1
4 2 1 6 2 3 2 6 2 1 4 2 6 1 2 4 C 4 1 $\bar{1}$ 1 0 1
2 6 6 1 0 1 6 3 6 6 6 6 2 4 4 3 C 4 1 1 1 1
3 3 1 1 6 3 4 0 2 3 6 6 6 2 4 2 2 4 C 4 1 0 0
4 6 6 3 6 6 6 2 3 2 4 1 0 1 1 6 1 4 2 C 4 0 1
6 4 6 4 4 6 4 3 2 4 1 2 3 4 4 6 4 3 1 6 C 4 0
6 2 6 0 2 2 1 2 2 4 3 2 4 4 2 3 2 2 0 3 4 C 4
2 0 2 4 0 0 1 6 4 1 2 1 4 2 6 2 6 2 2 $\bar{1}$ 3 0 C
4 2 1 2 3 2 2 3 6 0 6 2 2 6 4 3 2 0 4 2 1 3 3 C

$[\mathrm{SL}_2(11) \overset{2(2)}{\underset{\sqrt{-11}}{\boxtimes}} \mathrm{SL}_2(3)]_{24}$ (C = 12)

6
3 6
2 2 6
2 3 3 6
1 1 2 1 6
2 3 3 3 2 6
2 1 2 2 3 2 6
3 1 3 2 1 2 2 6
3 3 1 2 2 0 2 2 6
3 1 1 1 2 1 3 2 3 6
1 2 1 1 2 2 1 1 2 2 6
2 2 1 3 2 1 1 2 3 2 2 6
2 2 3 3 2 3 3 3 2 2 0 0 6
3 2 3 2 3 3 2 2 1 1 2 1 2 6
1 1 3 2 2 2 3 2 2 2 1 1 3 1 6
2 1 1 1 2 1 1 2 2 3 3 2 2 1 2 6
1 2 2 2 3 3 2 1 2 3 3 1 3 2 2 2 6
1 2 3 3 3 2 1 1 2 2 2 3 2 2 2 3 3 6
0 0 0 1 2 0 1 1 1 1 2 2 1 0 $\bar{1}$ 3 1 1 6
2 2 2 2 1 1 1 3 1 1 2 2 2 3 1 2 1 1 2 6
3 2 2 2 1 3 2 3 1 2 1 1 3 2 3 3 1 1 0 2 6
2 2 1 0 1 1 2 1 1 1 2 0 0 1 0 1 1 0 2 2 0 6
2 1 1 1 1 0 2 1 1 1 1 1 0 1 0 0 1 0 2 2 1 3 6
2 3 0 3 1 1 1 1 2 0 0 2 1 1 0 0 0 0 2 2 1 2 2 6

$[\pm L_2(11):2]_{24}$

TABLE 7 (continued)