Iterating the Sum-of-Divisors Function

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CONTENTS

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Let $\sigma^0(n) = n$ and $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$, where $m \geq 1$ and σ is the sum-of-divisors function. We say that n is (m,k)-perfect if $\sigma^m(n) = kn$. We have tabulated all $(2,k)$-perfect numbers up to 10^9 and all $(3,k)$- and $(4,k)$-perfect numbers up to $2 \cdot 10^8$. These tables have suggested several conjectures, some of which we prove here. We ask in particular: For any fixed $m \geq 1$, are there infinitely many (m,k)-perfect numbers? Is every positive integer (m,k)-perfect, for sufficiently large $m \geq 1$? In this connection, we have obtained the smallest value of m such that n is (m,k)-perfect, for $1 \leq n \leq 1000$. We also address questions concerning the limiting behaviour of $\sigma^{m+1}(n)/\sigma^m(n)$ and $(\sigma^m(n))^{1/m}$, as $m \to \infty$.

1. INTRODUCTION

All roman letters in this article denote positive integers, unless indicated otherwise, and σ denotes the sum-of-divisors function.

There is a great deal of literature concerning the iteration of the function $\sigma(n) - n$, much of it concerned with whether the iterated values eventually terminate at zero, cycle or become unbounded, depending on the value of n. See [Erdős et al. 1990; Guy 1994, p. 62] for details.

Less work has been done on iterates of σ itself. We define $\sigma^0(n) = n$ and $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$ for $m \geq 1$, and we call n (m,k)-perfect if $\sigma^m(n) = kn$. The classical *perfect* numbers are $(1,2)$-perfect. *Multiperfect* numbers are $(1,k)$-perfect, *superperfect* numbers are $(2,2)$-perfect, *multiply superperfect* numbers [Pomerance 1975] are $(2,k)$-perfect, m*-superperfect* numbers (ascribed by [Guy 1994, p. 65] to Bode; see also [Lord 1975]) are $(m,2)$-perfect.

Write $N_p = 2^{p-1}$ when $2^p - 1$ is a (Mersenne) prime. Superperfect numbers were introduced by Suryanarayana [1969], who showed there that the

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only even ones are the powers N_p. Bode and Lord, mentioned above, showed independently that an m-superperfect number can be even only if $m = 2$.

For a simple proof of these facts, we note that, since $\sigma(n) = n\sum_{d|n}(1/d)$, we have

$$\sigma(\sigma(n)) = n\sum_{d|n}\frac{1}{d}\sum_{e|\sigma(n)}\frac{1}{e}. \tag{1.1}$$

Suppose n is m-superperfect and $2^a \parallel n$ (that is, $2^a \mid n$ but $2^{a+1} \nmid n$). Then, for $m \geq 2$,

$$2 = \frac{\sigma^m(n)}{n} \geq \frac{\sigma(\sigma(n))}{n} \geq \Bigl(1+\frac{1}{2}+\cdots+\frac{1}{2^a}\Bigr)\Bigl(1+\frac{1}{2^{a+1}-1}\Bigr) = 2.$$

So as not to have a contradiction, we must have equality throughout. Thus, $m = 2$, $n = 2^a$ and $2^{a+1}-1$ is prime.

Kanold [1969] showed that an odd superperfect number must be a perfect square. This is similarly proved, using (1.1). For suppose n is superperfect, and that $\sigma(n)$ is even. Say $2^a \parallel \sigma(n)$, so that $(2^{a+1}-1) \mid n$ since n is superperfect. Then

$$2=\frac{\sigma(\sigma(n))}{n} \geq \Bigl(1+\frac{1}{2^{a+1}-1}\Bigr)\Bigl(1+\frac{1}{2}+\cdots+\frac{1}{2^a}\Bigr)=2.$$

Since we must have equality, we have both $\sigma(n) = 2^a$ and $n = 2^{a+1}-1$. This contradiction means that $\sigma(n)$ must be odd, so, if n is odd, then n is a square.

Other work on the iteration of σ has concerned whether

$$s_m = \liminf_{n\to\infty}\frac{\sigma^m(n)}{n}$$

is finite or not. See [Maier 1984], where s_3 is shown to be finite, and for the history of this problem.

In this paper, we will give particular attention to some questions raised in [Erdős et al. 1990]. The authors list the following six statements (reproduced in [Guy 1994, pp. 97–98]), with the comment: "We can neither prove nor disprove any of these statements."

(i) For any $n > 1$, $\sigma^{m+1}(n)/\sigma^m(n) \to 1$ as $m \to \infty$.
(ii) For any $n > 1$, $\sigma^{m+1}(n)/\sigma^m(n) \to \infty$ as $m \to \infty$.
(iii) For any $n > 1$, $(\sigma^m(n))^{1/m} \to \infty$ as $m \to \infty$.
(iv) For any $n > 1$, there is m with $n \mid \sigma^m(n)$.
(v) For any $n, l > 1$, there is m with $l \mid \sigma^m(n)$.
(vi) For any $n_1, n_2 > 1$, there are m_1, m_2 with $\sigma^{m_1}(n_1) = \sigma^{m_2}(n_2)$.

We will give some computational evidence to indicate that statements (ii), (iii), (iv) and (v) are true, and that statements (i) and (vi) are false.

Hausman [1982] has considered questions corresponding to some of those here for the Euler phi-function. In particular, she has completely characterised all n such that $n = k\varphi^m(n)$, where φ^m is defined analogously to σ^m.

2. TABLES OF (m,k)-PERFECT NUMBERS

Table 1 gives all $(2,k)$-perfect numbers up to 10^9. In [Cohen and te Riele 1995], we also give all $(3,k)$- and $(4,k)$-perfect numbers up to $2\cdot 10^8$. They are given in terms of increasing values of k. Corresponding lists, given as originally obtained with n increasing, are available from the authors. All the following comments arise from inspections of such lists.

Many conjectures can be made, along the lines of that in [Guy 1994, p. 48] that there are only finitely many $(1,k)$-perfect numbers for $k \geq 3$. That particular conjecture is well-supported by the list that has been accumulated by [Schroeppel 1993], showing over 2000 such numbers, which is almost three times the number that were known just three years ago, and especially by the facts that no new $(1,3)$-perfect numbers have been found in the last 350 years, nor any new $(1,4)$-perfect numbers in the last 65 years. On the other hand, if the well-known conjecture that there are infinitely many powers N_p is true, there are infinitely many $(1,2)$-perfect numbers.

There is a parallel situation with $(2,k)$-perfect numbers, of which there are families involving the powers N_p. Besides the well-known result that N_p is $(2,2)$-perfect, we know that:

k	n	k	n	k	n
1	1	8	$960 = 2^6 \cdot 3 \cdot 5$	11	$4404480 = 2^8 \cdot 3 \cdot 5 \cdot 31 \cdot 37$
2	$2 = 2$	8	$4092 = 2^2 \cdot 3 \cdot 11 \cdot 31$	11	$57669920 = 2^5 \cdot 5 \cdot 7 \cdot 11 \cdot 31 \cdot 151$
2	$4 = 2^2$	8	$16368 = 2^4 \cdot 3 \cdot 11 \cdot 31$	11	$238608384 = 2^{13} \cdot 3 \cdot 7 \cdot 19 \cdot 73$
2	$16 = 2^4$	8	$58254 = 2 \cdot 3 \cdot 7 \cdot 19 \cdot 73$	12	$2200380 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 31$
2	$64 = 2^6$	8	$61440 = 2^{12} \cdot 3 \cdot 5$	12	$8801520 = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 31$
2	$4096 = 2^{12}$	8	$65472 = 2^6 \cdot 3 \cdot 11 \cdot 31$	12	$14913024 = 2^9 \cdot 3 \cdot 7 \cdot 19 \cdot 73$
2	$65536 = 2^{16}$	8	$116508 = 2^2 \cdot 3 \cdot 7 \cdot 19 \cdot 73$	12	$35206080 = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 31$
2	$262144 = 2^{18}$	8	$466032 = 2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 73$	12	$140896000 = 2^8 \cdot 5^3 \cdot 7 \cdot 17 \cdot 37$
3	$8 = 2^3$	8	$710400 = 2^8 \cdot 3 \cdot 5^2 \cdot 37$	12	$459818240 = 2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$
3	$21 = 3 \cdot 7$	8	$983040 = 2^{16} \cdot 3 \cdot 5$	12	$775898880 = 2^8 \cdot 3 \cdot 5 \cdot 37 \cdot 43 \cdot 127$
3	$512 = 2^9$	8	$1864128 = 2^6 \cdot 3 \cdot 7 \cdot 19 \cdot 73$	13	$57120 = 2^5 \cdot 3 \cdot 5 \cdot 7 \cdot 17$
4	$15 = 3 \cdot 5$	8	$3932160 = 2^{18} \cdot 3 \cdot 5$	13	$932064 = 2^5 \cdot 3 \cdot 7 \cdot 19 \cdot 73$
4	$1023 = 3 \cdot 11 \cdot 31$	8	$4190208 = 2^{12} \cdot 3 \cdot 11 \cdot 31$	13	$3932040 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 31 \cdot 151$
4	$29127 = 3 \cdot 7 \cdot 19 \cdot 73$	8	$67043328 = 2^{16} \cdot 3 \cdot 11 \cdot 31$	13	$251650560 = 2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 31 \cdot 151$
6	$42 = 2 \cdot 3 \cdot 7$	8	$119304192 = 2^{12} \cdot 3 \cdot 7 \cdot 19 \cdot 73$	14	$217728 = 2^7 \cdot 3^5 \cdot 7$
6	$84 = 2^2 \cdot 3 \cdot 7$	8	$268173312 = 2^{18} \cdot 3 \cdot 11 \cdot 31$	14	$1278720 = 2^8 \cdot 3^3 \cdot 5 \cdot 37$
6	$160 = 2^5 \cdot 5$	9	$168 = 2^3 \cdot 3 \cdot 7$	14	$2983680 = 2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 37$
6	$336 = 2^4 \cdot 3 \cdot 7$	9	$10752 = 2^9 \cdot 3 \cdot 7$	14	$5621760 = 2^{11} \cdot 3^2 \cdot 5 \cdot 61$
6	$1344 = 2^6 \cdot 3 \cdot 7$	9	$331520 = 2^8 \cdot 5 \cdot 7 \cdot 37$	14	$14008320 = 2^{14} \cdot 3^2 \cdot 5 \cdot 19$
6	$86016 = 2^{12} \cdot 3 \cdot 7$	9	$691200 = 2^{10} \cdot 3^3 \cdot 5^2$	14	$298721280 = 2^{13} \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 17$
6	$550095 = 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 31$	9	$1556480 = 2^{14} \cdot 5 \cdot 19$	14	$955367424 = 2^{14} \cdot 3^2 \cdot 11 \cdot 19 \cdot 31$
6	$1376256 = 2^{16} \cdot 3 \cdot 7$	9	$1612800 = 2^{10} \cdot 3^2 \cdot 5^2 \cdot 7$	15	$1058148 = 2^2 \cdot 3^2 \cdot 7 \cdot 13 \cdot 17 \cdot 19$
6	$5505024 = 2^{18} \cdot 3 \cdot 7$	9	$106151936 = 2^{14} \cdot 11 \cdot 19 \cdot 31$	15	$29352960 = 2^{10} \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13$
7	$24 = 2^3 \cdot 3$	10	$480 = 2^5 \cdot 3 \cdot 5$	16	$7526400 = 2^{11} \cdot 3 \cdot 5^2 \cdot 7^2$
7	$1536 = 2^9 \cdot 3$	10	$504 = 2^3 \cdot 3^2 \cdot 7$	16	$23591520 = 2^5 \cdot 3^3 \cdot 5 \cdot 43 \cdot 127$
7	$47360 = 2^8 \cdot 5 \cdot 37$	10	$13824 = 2^9 \cdot 3^3$	16	$55046880 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 43 \cdot 127$
7	$343976 = 2^3 \cdot 19 \cdot 31 \cdot 73$	10	$32256 = 2^9 \cdot 3^2 \cdot 7$	18	$39352320 = 2^{11} \cdot 3^2 \cdot 5 \cdot 7 \cdot 61$
8	$60 = 2^2 \cdot 3 \cdot 5$	10	$32736 = 2^5 \cdot 3 \cdot 11 \cdot 31$	19	$312792480 = 2^5 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 31$
8	$240 = 2^4 \cdot 3 \cdot 5$	10	$1980342 = 2 \cdot 3^3 \cdot 7 \cdot 13^2 \cdot 31$	22	$83825280 = 2^7 \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 11$

TABLE 1. All $(2, k)$-perfect numbers n with $n < 10^9$

(A) $N_p \cdot 3 \cdot 7$ is $(2, 6)$-perfect.
(B) $N_p \cdot 3 \cdot 7 \cdot 19 \cdot 73$ is $(2, 8)$-perfect; for $p > 2$, $N_p \cdot 3 \cdot 5$ and $N_p \cdot 3 \cdot 11 \cdot 31$ are $(2, 8)$-perfect.
(C) For $p > 2$, $N_p \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 31$ is $(2, 12)$-perfect.

These are particular cases of the following general result.

Theorem 2.1. *Suppose that l is an odd $(2, k)$-perfect number. For any a such that $2^a \mid k\sigma(\sigma(2^a))$ and such that $\sigma(2^a)$ and $\sigma(l)$ are relatively prime, the number $2^a l$ is $(2,\ 2^{-a} k\sigma(\sigma(2^a)))$-perfect.*

Proof. Since l is odd we have $\sigma(2^a l) = \sigma(2^a)\sigma(l)$, and since $(\sigma(2^a), \sigma(l)) = 1$ we have

$$\begin{aligned}\sigma(\sigma(2^a l)) &= \sigma(\sigma(2^a))\sigma(\sigma(l)) = \sigma(\sigma(2^a))kl \\ &= 2^{-a} k\sigma(\sigma(2^a)) \cdot 2^a l. \qquad \square\end{aligned}$$

As a corollary, when $\sigma(2^a)$ is a (Mersenne) prime the condition $2^a \mid k\sigma(\sigma(2^a))$ is true and, provided $\sigma(2^a) \nmid \sigma(l)$, the number $2^a l$ is $(2, 2k)$-perfect. The statements (A), (B) and (C) above all arise from an application of this theorem to the five nontrivial examples of odd $(2, k)$-perfect numbers in Table 1. Furthermore, we may, for example, apply the more general result of Theorem 2.1 to the $(2, 4)$-perfect number $3 \cdot 7 \cdot 19 \cdot 73$, with $a = 5, 9, 13$ (but to no other values of a that we could find). In this way, we can deduce a family of $(2, k)$-perfect numbers (with varying k) that is "larger" than the set of Mersenne primes.

No other possibly infinite family of $(2,k)$-perfect numbers has been noticed, and we may conjecture that, apart from the above, there are only finitely many of these numbers for each k. We would also make the uncharacteristic conjecture that all $(2,4)$-perfect numbers are odd! Notice from Table 1 that we have found $(2,k)$-perfect numbers for all $k \le 16$, except for $k=5$, and we conjecture that there are no $(2,5)$-perfect numbers.

No pattern has been discerned in (m,k)-perfect numbers, with any $m \ge 3$, and we conjecture that there are only finitely many for each k.

Some interrelationships between the tables have been noticed. The following facts, for example, are easily verified.

(D) If n is $(2,4)$-perfect, n is odd and $7 \nmid \sigma(n)$, then n is $(4,32)$-perfect.
(E) If n is $(2,7)$-perfect, $7 \nmid n$ and $2^2 \parallel \sigma(n)$, then n is $(4,63)$-perfect.

The next result can be contrasted with the easily proved result that the equation $\sigma(2n) = 2\sigma(n)$ has no solutions.

Theorem 2.2. *The equation $\sigma(\sigma(2n)) = 2\sigma(\sigma(n))$ has infinitely many solutions.*

Proof. We need only verify that this equation is satisfied by $n = 2t$ for any t with $(2,t) = (3,\sigma(t)) = (7,\sigma(t)) = 1$, and that any prime $t \equiv 1 \pmod{21}$ satisfies these conditions. There are infinitely many such primes. □

This theorem can be generalised in various ways. For example, we have $\sigma(\sigma(2^a n)) = 2^a\sigma(\sigma(n))$ when $n = 2^a t$, where

$$(2,t) = (2^{a+1}-1, \sigma(t)) = (2^{2a+1}-1, \sigma(t)) = 1$$

and $2^{a+1}-1$ and $2^{2a+1}-1$ are primes. The latter is the case for $a=1$ (as in the proof), and $a = 2$, 6, 30.

3. IS EVERY NUMBER (m,k)-PERFECT?

In support of statement (iv) from the Introduction, that all numbers n are (m,k)-perfect for m large enough, we have successfully tested all values of n up to 1000. In this connection, it is convenient to define

$$\widetilde{m}(n) = \inf\left\{ m \ge 1 : \frac{\sigma^m(n)}{n} \text{ is an integer} \right\},$$
$$\tilde{k}(n) = \frac{\sigma^{\widetilde{m}(n)}(n)}{n}.$$

(If $\widetilde{m}(n)$ is infinite, we understand $\tilde{k}(n)$ to be infinite also.)

Representative values of $\widetilde{m}$ and $\tilde{k}$ are given in Table 2. A more complete version of this listing [Cohen and te Riele 1995, Table 4] gives the data for all $n \le 400$.

We will comment on the more computationally difficult cases later; they tend to be those for which $\widetilde{m}(n) > n$. There are fourteen such cases up to $n = 400$, namely $n = 3$, 11, 29, 53, 58, 59, 67, 101, 109, 131, 149, 173, 202, 239.

The values of $\tilde{k}(n)$ of course become extremely large, the largest observed value in Table 2 being $\tilde{k}(389) \approx 5\cdot 10^{232}$ and the largest for $n \le 1000$ being $\tilde{k}(659) \approx 1.5\cdot 10^{1183}$. It is interesting then that the following theorem allows us to predict exact values of $\widetilde{m}(n)$ and $\tilde{k}(n)$ in many cases, making use of earlier values.

Theorem 3.1. *Suppose there are integers n, $t \ge 2$, a and M such that $\widetilde{m}(n)$ is finite, $t \mid \tilde{k}(n)$,*

$$\sigma^{M+a}(n) = \sigma^M(tn), \tag{3.1}$$

and $M < \widetilde{m}(n) - a$. Then $\widetilde{m}(tn) \le \widetilde{m}(n) - a$. If $\widetilde{m}(tn) = \widetilde{m}(n) - a$, then $\tilde{k}(tn) = \tilde{k}(n)/t$. If $\widetilde{m}(tn) < \widetilde{m}(n) - a$, then $\widetilde{m}(tn) < M$ and $\tilde{k}(tn) < \alpha\tilde{k}(n)/t$, where

$$\alpha = \frac{\sigma^{M+a}(n)}{\sigma^{\widetilde{m}(n)}(n)} < 1.$$

n	$\widetilde{m}$	$\tilde{k}$	
1	1	1	
2	2	2	2
3	4	5	5
4	2	2	2
5	5	24	$2^3 3$
6	1	2	2
7	5	24	$2^3 3$
8	2	3	3
9	7	168	$2^3 3\cdot 7$
10	4	12	$2^2 3$
11	15	1.8E006	$2^6 3^2 5\cdot 7^2 13$
12	3	10	$2\cdot 5$
13	13	84480	$2^9 3\cdot 5\cdot 11$
14	3	12	$2^2 3$
15	2	4	2^2
16	2	2	2
17	13	92520	$2^3 3^2 5\ldots$
18	4	20	$2^2 5$
19	12	62720	$2^8 5\cdot 7^2$
20	5	84	$2^2 3\cdot 7$
21	2	3	3
22	13	49920	$2^8 3\cdot 5\cdot 13$
23	16	6.5E006	$2^9 11\cdot 13\ldots$
24	2	7	7
25	17	881280	$2^7 3^4 5\cdot 17$
26	4	28	$2^2 7$
27	9	3360	$2^5 3\cdot 5\cdot 7$
28	1	2	2
29	78	5.1E047	$2^{29} 3^5 5^2 7^2 13\cdot 19\ldots$
30	7	728	$2^3 7\cdot 13$
31	10	912	$2^4 3\cdot 19$
32	4	18	$2\cdot 3^2$
33	17	1.9E007	$2^{13} 19\ldots$
34	11	46260	$2^2 3^2 5\ldots$
35	6	144	$2^4 3^2$
36	5	42	$2\cdot 3\cdot 7$
37	28	3.0E013	$2^{14} 5\cdot 7\cdot 11\cdot 13\ldots$
38	22	3.8E010	$2^5 3^4 5\cdot 7^2 13\ldots$
39	4	30	$2\cdot 3\cdot 5$
40	7	663	$3\cdot 13\cdot 17$
41	39	3.4E022	$2^{20} 3\cdot 5\cdot 7\ldots$
42	2	6	$2\cdot 3$
43	16	4.5E006	$2^6 3\cdot 5\ldots$
44	16	1.4E007	$2^{11} 3\cdot 19\ldots$
45	16	8.2E006	$2^7 3^3 7\cdot 11\ldots$
46	10	19224	$2^3 3^3\ldots$
47	32	3.8E015	$2^{12} 3^4 7\cdot 13\ldots$
48	5	105	$3\cdot 5\cdot 7$
49	13	92928	$2^8 3\cdot 11^2$
50	17	1.8E007	$2^9 3^4 5\cdot 7\cdot 13$
51	9	5120	$2^{10} 5$
52	3	5	5
53	58	4.2E033	$2^{20} 3^4 5^4 7^4 13\cdot 17\cdot 19\ldots$
54	11	100620	$2^2 3^2 5\cdot 13\ldots$
55	19	8.2E008	$2^{24} 7^2$
56	5	182	$2\cdot 7\cdot 13$
57	13	271852	$2^2 7^2 19\ldots$
58	67	3.9E042	$2^{21} 3^2 5^2 7^2 11\cdot 19^2\ldots$
59	97	1.2E064	$2^{56} 3^6 5\cdot 7^3 11\cdot 17\cdot 19\ldots$
60	2	8	2^3
61	23	2.7E011	$2^{10} 3^3 5\cdot 7\ldots$
62	5	96	$2^5 3$
63	16	5.7E006	$2^3 3^2 5\cdot 7\cdot 11^2 19$
64	2	2	2
65	4	24	$2^3 3$
66	8	1078	$2\cdot 7^2 11$
67	101	9.4E066	$2^{21} 3^{10} 7^4 11\cdot 13\cdot 17\cdot 19^3\ldots$
68	21	4.6E010	$2^{16} 3^2 5\cdot 61\ldots$
69	19	3.2E009	$2^{13} 7^3 13\ldots$
70	11	26624	$2^{11} 13$
71	50	8.0E027	$2^{11} 3^3 5\cdot 7^5 11\cdot 13\ldots$
72	4	28	$2^2 7$
73	20	8.5E008	$2^9 7^2\ldots$
74	20	2.0E009	$2^6 3^2 11\cdot 19\ldots$
75	23	5.6E010	$2^4 3^2 17\cdot 19\ldots$
76	14	4.2E006	$2^4 3\cdot 7\cdot 11\ldots$
77	21	4.5E010	$2^{14} 3^3 5\cdot 11^2 13^2$
78	10	14080	$2^8 5\cdot 11$
79	36	6.0E018	$2^9 3^5 5^2 7^3 11\cdot 13^2 19\ldots$
80	5	124	$2^2\ldots$
81	15	2.2E006	$2^3 7\cdot 13\ldots$
82	42	1.2E024	$2^{16} 3^6 5^4 7^3 11\cdot 13\cdot 19^2\ldots$
83	26	3.6E012	$2^{13} 3^6 5\cdot 13^2 17\ldots$
84	2	6	$2\cdot 3$
85	36	2.1E017	$2^{26} 3^3 5^2 7^2 19\ldots$
86	17	1.7E008	$2\cdot 3^3 13\ldots$
87	43	2.4E023	$2^{20} 3\cdot 5^3 7\cdot 11\cdot 13^3\ldots$
88	8	4158	$2\cdot 3^3 7\cdot 11$
89	13	6.1E005	$2^8 3\cdot 5\cdot 7\ldots$
90	7	1008	$2^4 3^2 7$
91	17	1.8E007	$2^{13} 3^3 5\cdot 17$
92	14	1.6E006	$2^7 11\cdot 13\ldots$
93	10	5824	$2^6 7\cdot 13$
94	54	5.8E031	$2^{39} 3^2 5\cdot 7\cdot 11^2 19\ldots$
95	19	3.3E008	$2^{11} 3^2 11\cdot 13\ldots$
96	4	62	$2\ldots$
97	43	3.4E023	$2^8 3^9 7^2 13\cdot 19\ldots$
98	3	6	$2\cdot 3$
99	18	7.2E007	$2^{10} 13\ldots$
100	20	1.3E008	$2^6 3^6 7\cdot 13\ldots$
101	120	3.7E079	$2^{45} 3^{13} 5^3 7^5 13^2 17^2 19^2\ldots$
102	35	1.7E017	$2^{25} 3^2 5^3 7^2 19\ldots$
103	65	3.4E040	$2^{24} 3^9 5^2 7^3 11\cdot 19^2\ldots$
104	10	6096	$2^4 3\ldots$
105	12	87552	$2^9 3^2 19$
106	54	3.7E030	$2^{17} 3^3 5^2 7^2 11\cdot 17^3 19\ldots$
107	64	5.8E036	$2^{17} 3^8 5^3 7\cdot 11^2 13\cdot 19\ldots$
108	13	491400	$2^3 3^3 5^2 7\cdot 13$
....			
348	22	2.8E011	$2^9 3\cdot 5\ldots$
349	188	3.5E140	$2^{86} 3^{29} 5^7 7^8 11^4 13^2 17^3 19^7\ldots$
350	16	3.7E007	$2^7 3^6 13\ldots$
351	19	1.7E009	$2^7 3\cdot 5\cdot 11\cdot 19\ldots$
352	5	93	$3\ldots$
353	263	1.4E201	$2^{74} 3^{27} 5^{13} 7^{10} 11^3 13^4 17\cdot 19^7\ldots$
354	69	3.4E041	$2^{39} 3^4 7^2 11\cdot 13\cdot 19^2\ldots$
355	42	1.6E024	$2^{16} 3^7 5\cdot 7^3 11\cdot 19\ldots$
356	9	9568	$2^5 13\ldots$
357	10	5120	$2^{10} 5$
358	74	2.8E048	$2^{22} 3^8 5\cdot 7^4 19^3\ldots$
359	166	1.1E120	$2^{47} 3^{15} 5^7 7^{10} 11^2 13^4 17\cdot 19^7\ldots$
360	8	4369	$17\ldots$
361	19	1.5E008	$2^7 3^2 7^2\ldots$
362	53	7.6E032	$2^{23} 3^4 5^2 7^2 11\cdot 13\cdot 19^2\ldots$
363	10	12544	$2^8 7^2$
364	13	551880	$2^3 3^3 5\cdot 7\ldots$
365	42	1.7E024	$2^{15} 3^6 5^3 7^2 13\cdot 17\ldots$
366	15	1.0E007	$2^{10} 5\ldots$
367	146	1.5E105	$2^{65} 3^{21} 5^3 7^6 11^5 13^3 17\cdot 19^4\ldots$
368	15	1.0E007	$2^7 3\cdot 5\ldots$
369	35	5.1E020	$2^{18} 3^4 5^2 7\ldots$
370	7	768	$2^8 3$
371	34	4.1E018	$2^{15} 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\ldots$
372	7	1530	$2\cdot 3^2 5\cdot 17$
373	145	4.3E103	$2^{56} 3^{28} 5^4 7^6 11\cdot 13^4 17\cdot 19\ldots$
374	32	2.2E017	$2^{13} 3^3 5\ldots$
375	25	5.0E012	$2^{19} 3^2 5\cdot 7\cdot 13^2\ldots$
376	64	5.6E037	$2^{16} 3^5 5^2 7\cdot 11\cdot 13\cdot 19\ldots$
377	68	9.3E043	$2^{40} 3^5 5^4 7^4 11\cdot 13^3 19^3\ldots$
378	7	2912	$2^5 7\cdot 13$
379	67	6.4E041	$2^{25} 3^{13} 5\cdot 7^2 19^2\ldots$
380	15	4.3E007	$2^8 3^3 7^2\ldots$
381	9	3072	$2^{10} 3$
382	99	8.3E066	$2^{38} 3^{18} 5^5 7^2 11^4 13^6 17^3 19\ldots$
383	250	9.4E191	$2^{88} 3^{24} 5^{13} 7^{10} 11^2 13^4 17^2 19^3\ldots$
384	6	341	$11\ldots$
385	14	948024	$2^3 3^4 7\cdot 11\cdot 19$
386	81	4.3E053	$2^{33} 3^6 5^4 7^5 13\cdot 17^2 19\ldots$
387	28	6.8E014	$2^{17} 3^3 5\cdot 7\cdot 11^2 19^2\ldots$
388	66	1.9E041	$2^{23} 3^5 5^4 7^2 13\cdot 17^3\ldots$
389	296	4.9E232	$2^{92} 3^{30} 5^{15} 7^3 11\cdot 17^3 19^6\ldots$
390	12	389120	$2^{12} 5\cdot 19$
391	34	3.9E018	$2^{15} 3^2 5\cdot 7^2 11\cdot 13\ldots$
392	20	2.9E009	$2^{12} 3^3 19^2\ldots$
393	205	9.1E153	$2^{65} 3^{13} 5^{12} 7^{10} 11^3 17^2 19^2\ldots$
394	47	2.8E028	$2^{10} 3^8 5\cdot 7^2 13\cdot 17\ldots$
395	63	1.2E039	$2^{42} 3^7 7\cdot 13\cdot 19\ldots$
396	10	22320	$2^4 3^2 5\ldots$
397	124	8.0E082	$2^{43} 3^{10} 5^4 7^3 13^2 19\ldots$
398	37	6.5E018	$2^{18} 3^3 5^5 19\ldots$
399	5	57	$3\cdot 19$
400	7	81	3^4

TABLE 2. Every $n \leq 1000$ is (m,k)-perfect for some m, k. This table shows, for $n \leq 108$ and $348 \leq n \leq 400$, the least such value of m, called $\widetilde{m}$, and the (approximate) corresponding value of k, called $\tilde{k}$. The prime factors of $\tilde{k}(n)$ less than 20 are also given.

Proof. By definition, we have $\sigma^{\widetilde{m}(n)}(n) = \tilde{k}(n)n$, so

$$\begin{aligned}\sigma^{\widetilde{m}(n)-a}(tn) &= \sigma^{\widetilde{m}(n)-a-M}(\sigma^M(tn)) \\ &= \sigma^{\widetilde{m}(n)-a-M}(\sigma^{M+a}(n)) \\ &= \sigma^{\widetilde{m}(n)}(n) = \frac{\tilde{k}(n)}{t} \cdot tn.\end{aligned}$$

This shows that $\widetilde{m}(tn) \le \widetilde{m}(n) - a$, and that if $\widetilde{m}(tn) = \widetilde{m}(n) - a$ then $\tilde{k}(tn) = \tilde{k}(n)/t$. Suppose $\widetilde{m}(tn) < \widetilde{m}(n) - a$. Then, by definition, $n \nmid \sigma^j(n)$ for $j = M+a, \ldots, \widetilde{m}(n)-1$, so, by (3.1), $tn \nmid \sigma^j(tn)$ for $j = M, \ldots, \widetilde{m}(n) - a$. Therefore, $\widetilde{m}(tn) < M$. Then

$$\begin{aligned}\tilde{k}(tn) &= \frac{\sigma^{\widetilde{m}(tn)}(tn)}{tn} \\ &< \frac{\sigma^M(tn)}{tn} = \frac{\sigma^{M+a}(n)}{tn} \\ &= \frac{\sigma^{\widetilde{m}(n)}(n)}{tn} \cdot \frac{\sigma^{M+a}(n)}{\sigma^{\widetilde{m}(n)}(n)} = \alpha\frac{\tilde{k}(n)}{t}.\end{aligned}$$

Clearly, $\alpha < 1$, completing the proof. □

In fact, this number α would be expected to be quite small. For we have, extending (1.1),

$$\sigma^m(n) = n \prod_{j=0}^{m-1} \sum_{d \mid \sigma^j(n)} \frac{1}{d} \quad \text{for } m \ge 1,$$

so that if $\sigma^j(n)$ is even for $j = M + a, \ldots, \widetilde{m}(n)$, then

$$\begin{aligned}\frac{\sigma^{\widetilde{m}(n)}(n)}{\sigma^{M+a}(n)} &= \frac{\sigma^{\widetilde{m}(n)-M-a}(\sigma^{M+a}(n))}{\sigma^{M+a}(n)} \\ &\ge \left(1 + \tfrac{1}{2}\right)^{\widetilde{m}(n)-M-a}.\end{aligned}$$

Then $\alpha \le \left(\frac{2}{3}\right)^{\widetilde{m}(n)-M-a}$.

Many instances of Theorem 3.1 may be observed in Table 2. For example:

(a) $\sigma^4(5) = \sigma^3(10)$, $\widetilde{m}(10) = \widetilde{m}(5) - 1$ and $\tilde{k}(10) = \frac{1}{2}\tilde{k}(5)$;
(b) $\sigma^3(7) = \sigma(14)$, $\widetilde{m}(14) = \widetilde{m}(7) - 2$ and $\tilde{k}(14) = \frac{1}{2}\tilde{k}(7)$;
(c) $\sigma^6(9) = \sigma^4(36)$, $\widetilde{m}(36) = \widetilde{m}(9) - 2$ and $\tilde{k}(36) = \frac{1}{4}\tilde{k}(9)$;
(d) $\sigma^4(13) = \sigma(78)$, $\widetilde{m}(78) = \widetilde{m}(13) - 3$ and $\tilde{k}(78) = \frac{1}{6}\tilde{k}(13)$.

In each case, the other conditions of Theorem 3.1 must also be verified. It is easy to find solutions of (3.1), and we have done this for $n \le 500$, $M + a \le 30$ and $t \le 150$. There are a great many solutions, though not all satisfy the other conditions of the theorem. In all acceptable cases, we confirmed that, in the notation of the theorem, $\widetilde{m}(tn) = \widetilde{m}(n) - a$. Here are some of those examples, giving extensions of Table 2:

(e) $\sigma^{10}(101) = \sigma^6(2020)$, $\widetilde{m}(2020) = \widetilde{m}(101) - 4$ and $\tilde{k}(2020) = \frac{1}{20}\tilde{k}(101)$;
(f) $\sigma^{10}(233) = \sigma^8(2330)$, $\widetilde{m}(2330) = \widetilde{m}(233) - 2$ and $\tilde{k}(2330) = \frac{1}{10}\tilde{k}(233)$;
(g) $\sigma^{11}(394) = \sigma^{10}(6698)$, $\widetilde{m}(6698) = \widetilde{m}(394) - 1$ and $\tilde{k}(6698) = \frac{1}{17}\tilde{k}(394)$;
(h) $\sigma^8(197) = \sigma^2(29550)$, $\widetilde{m}(29550) = \widetilde{m}(197) - 6$ and $\tilde{k}(29550) = \frac{1}{150}\tilde{k}(197)$.

In (g), for example, where $6698 = 17 \cdot 394$, it is clear that we need to know at least the small prime factors of $\tilde{k}(n)$ for each n in order that the condition $t \mid \tilde{k}(n)$ might be checked. These small prime factors, namely those less than 20, have been included in Table 2.

There is no reason, in (3.1), why a cannot in fact be zero or negative (provided $M + a > 0$). We found one instance of this in the above search: $\sigma^8(404) = \sigma^8(808)$, from which, as in Theorem 3.1, we could verify that $\widetilde{m}(808) = \widetilde{m}(404)$ and

$$\tilde{k}(808) = \tfrac{1}{2}\tilde{k}(404).$$

This led us to seek solutions of the equation

$$\sigma^m(tn) = \sigma^m(n) \tag{3.2}$$

over a much larger range. For $t \le 4$, $m \le 12$ and $n \le 10^5$, the solutions are listed in Table 3. Note that for any pair (m_0, n) that satisfies (3.2) for some t, we also have the solutions (m, n) for all $m \ge m_0$.

Following on from this, can it be proved that the equation $\sigma(\sigma(2n)) = \sigma(\sigma(n))$ has no solutions?

$(2,8,\ 404)$	$(2,6,\ 6938)$	$(2,7,15488)$	$(2,8,20800)$
$(2,4,21086)$	$(2,4,25056)$	$(2,8,27712)$	$(2,4,31840)$
$(2,4,33376)$	$(2,4,35872)$	$(2,6,47166)$	$(2,4,67320)$
$(2,6,69626)$	$(2,4,79880)$	$(2,4,84120)$	$(2,4,84744)$
$(2,4,86904)$	$(2,4,87768)$	$(2,4,95064)$	$(2,4,95896)$
	$(3,10,633)$	$(3,6,52491)$	

TABLE 3. Solutions (t,m,n) of $\sigma^m(tn)=\sigma^m(n)$ with $t\le 4$, $m\le 12$, and $n\le 10^5$.

We can prove only that for any n satisfying this equation we must have $2^a \parallel n$, with $\sigma(2^{a+1}-1)\ge 2^{a+2}$. This condition is satisfied by $a=11$, 23, 35, 39, 47, …. For these five smallest possible values of a, we have checked each $n=2^a l$, with $l<10^4$ odd, and found no solutions.

4. DISCUSSION OF THE SIX STATEMENTS

The preceding section has been largely concerned with statement (iv) of the six by [Erdős et al. 1990] given in the Introduction. This was also posed by Carl Pomerance as unsolved problem 94:13 at the Western Number Theory Conference in December 1994 at San Diego. The following slightly edited comment accompanied the problem: "It is inconceivable that the conjecture is false. Each (odd part of) n divides $2^{rs}-1$ for a suitable s and all r, and $\sigma(2^{rs-1})=2^{rs}-1$. As m increases, $\sigma^m(n)$ increases quite rapidly, and so does the power of 2 it contains, albeit very erratically. How can the sequence of exponents of 2 avoid all members of the arithmetic progression $rs-1$?"

We observe next that Theorem 3.1 shows some relationship between statements (iv) and (vi) in the Introduction, in that a value for m for which $tn\,|\,\sigma^m(tn)$ may be inferred from a suitable solution of $\sigma^{m_1}(n)=\sigma^{m_2}(tn)$. If we write n_1, n_2, for n, tn, respectively, in Theorem 3.1 then clearly we have

$$n_1\tilde{k}(n_1)=n_2\tilde{k}(n_2). \tag{4.1}$$

Furthermore, given n_1 and n_2, if we notice that (4.1) is satisfied then we have a solution of the equation $\sigma^{m_1}(n_1)=\sigma^{m_2}(n_2)$, namely $m_1=\widetilde{m}(n_1)$ and $m_2=\widetilde{m}(n_2)$. This demonstrates a relationship between the two statements in the reverse direction. We have observed from Table 2 the following nine instances of pairs (n_1,n_2) that satisfy (4.1), but in which n_2 is *not* a multiple of n_1: $(7,24)$, $(9,168)$, $(10,12)$, $(14,24)$, $(18,120)$, $(36,168)$, $(62,96)$, $(72,336)$ and $(341,384)$.

While Table 2 and the further computations for $n\le 1000$ support the truth of statement (iv), we do not believe that statement (vi) is true. The figure below shows how sequences $\{\sigma^i(n)\}_{i=1}^{\infty}$, for any n in the figure, merge into the sequence 480, 1512, 4800, …. (For example, $\sigma(45)=78$, $\sigma^2(45)=168$, $\sigma^3(45)=480$, ….)

$$\left.\begin{array}{r} \left.\begin{array}{r} \left.\begin{array}{r} \left.\begin{array}{r}2,3,4,7,8,15\\9,13,14\\23\end{array}\right\}24 \\ 37,38\\59\end{array}\right\}60 \\ 45,78\\92\\123\\143\\167\end{array}\right\}168 \\ \left.\begin{array}{r}22,36,91,112\\169,183\\175\end{array}\right\}248 \\ 148,266\end{array}\right\}480,1512,4800,\ldots$$

No other values of $n\le 200$ are such that the sequence $\{\sigma^i(n)\}$ intersects (and joins with) any of those in the figure, for values of $\sigma^i(n)<10^{200}$. Three parameters determine the numbers in this figure: we call it a (π_1,π_2,π_3)-*tree*, with π_1 the smallest number in the tree, and π_2, π_3 such that all sequences $\{\sigma^i(n)\}$ with $\pi_1\le n\le\pi_2$ and $\sigma^i(n)<\pi_3$ have nonempty intersection with $\{\sigma^i(\pi_1)\}$. If we first specify π_2 and π_3 (200 and 10^{200}, here) then we may determine successive (π_1,π_2,π_3)-trees for all $\pi_1\le\pi_2$. There are 21 $(\pi_1,200,10^{200})$-trees, having the following values of π_1:

$$\begin{array}{c}2,5,16,19,27,29,33,49,50,52,66,81,\\85,105,146,147,163,170,189,197,199.\end{array} \tag{4.2}$$

The approach here was as follows. We calculated the sequences $\{\sigma^i(n)\}$ for each n, $2 \le n \le 200$, and determined which sequences were such that the first term exceeding 10^{10} equalled such a term from an earlier sequence. There were 21 $(\pi_1, 200, 10^{10})$-trees obtained this way, and these were tested further for intersection by determining the values of the first terms that exceeded 10^{200}. The trees remained distinct, and we conjecture that this will stay true as $\pi_3 \to \infty$.

We also found 64 $(\pi_1, 1000, 10^{100})$-trees.

Some evidence for statement (iii) in the Introduction is provided by the further computations that extend those for Table 2. The following is the list of those $N < 1000$ for which $\widetilde{m}(n) < \widetilde{m}(N)$ for all $n < N$. (We called such numbers N *megaperfect* in a talk at CANT'95, the Computational Algebra and Number Theory conference held at Macquarie University, Sydney, in April 1995.)

N	1	2	3	5	9	11	23
$\widetilde{m}(N)$	1	2	4	5	7	15	16
N	25	29	59	67	101	131	173
$\widetilde{m}(N)$	17	78	97	101	120	174	214
N	202	239	353	389	401	461	659
$\widetilde{m}(N)$	239	261	263	296	380	557	1287

We set

$$h(n) = \frac{(\sigma^{\widetilde{m}(n)}(n))^{1/\widetilde{m}(n)}}{\log \widetilde{m}(n)}.$$

For the last three values of N above, we have

n	401	461	659
$h(n)$	1.1146	1.1276	1.1658

which suggests that $(\sigma^m(n))^{1/m}$ is at least of the same order as $\log m$, as $m \to \infty$, for any n.

With regard to $\widetilde{m}(659)$, we remark that

$$\begin{aligned}\tilde{k}(659) &= 2^{276}3^{100}5^{44}7^{28}11^{21}13^{14}17^{14}19^{8}\ldots\\ &\approx 1.5 \cdot 10^{1183}.\end{aligned}$$

In the calculation of $\widetilde{m}(659)$, we had to factorise a difficult 104-digit composite factor of $\sigma^{1240}(659)$. This number, which we indicate by C104, arose as follows. We found that $2^{372} \parallel \sigma^{1238}(659)$, so that $\sigma(2^{372}) = (2^{373}-1) \mid \sigma^{1239}(659)$. Now, $2^{373}-1 = 25569151 \cdot \mathrm{P105}$, where P105 is a prime number of 105 decimal digits. Consequently, $\sigma(\mathrm{P105}) = (\mathrm{P105}+1) \mid \sigma^{1240}(659)$ and $\mathrm{P105}+1 = 2\cdot 7\cdot \mathrm{C104}$. We were unable to factorise this C104 with the elliptic curve method or with the quadratic sieve method, and therefore asked Peter Montgomery's help, noticing that

$$\mathrm{C104} = \frac{2^{373}-1+25569151}{2\cdot 7\cdot 25569151}.$$

Peter constructed the two polynomials

$$p_1(x) = 5x - 2^{74}, \quad p_2(x) = 500x^5 + \frac{25569151-1}{50},$$

which have the property that

$$p_1(m) \equiv p_2(m) \equiv 0 \pmod{\mathrm{C104}} \quad \text{for } m = 2^{74}5^{-1}.$$

This enabled him to apply the Special Number Field Sieve method [Lenstra and Lenstra 1993] and factorise C104 within two days on SGI workstations at CWI Amsterdam and the Cray C90 at SARA Amsterdam, into the product of 38-digit and 67-digit primes:

$$\begin{aligned}\mathrm{C104} = {}& 18223164902649732703974292810329988561\\ &\times 2949308713532555425842465546059346081 10\text{-}\\ &\qquad 4682577291637010561295300423.\end{aligned}$$

We also used the 21 $(n, 200, 10^{200})$-trees, with $n = \pi_1$ in (4.2), to investigate statements (i), (ii) and (iii). The results are summarised in Table 4. We remark that if statement (iii) is true and the sequence $\{(\sigma^i(n))^{1/i}\}$ is eventually monotone, then (ii) is true, since $(\sigma^{i+1}(n))^{1/(1+i)} > (\sigma^i(n))^{1/i}$ implies

$$\frac{\sigma^{i+1}(n)}{\sigma^i(n)} > (\sigma^i(n))^{1/i}.$$

Our computations strongly suggest that indeed the sequence $\{(\sigma^i(n))^{1/i}\}$ is eventually monotone, for every n.

We turn finally to statement (v). As evidence in favour of this statement, we showed that every number up to 400 occurs as a divisor in the sequence $\{\sigma^i(n)\}$, for each of the 21 values of n in (4.2). The results are summarised in Table 5.

n	j_1	α_1	β_1	$\beta_1/\log j_1$	j_2	α_2	β_2	$\beta_2/\log j_2$
2	146	6.2437	4.8927	0.98176	263	7.8129	5.7938	1.03978
5	144	6.8248	4.9610	0.99822	262	7.3602	5.8341	1.04773
16	143	6.3581	5.0681	1.02120	260	7.2318	5.9191	1.06445
19	140	6.2237	5.2215	1.05663	257	7.4125	6.0250	1.08576
27	138	6.6011	5.3063	1.07692	256	7.4307	6.0797	1.09640
29	143	6.9807	5.0686	1.02131	260	7.3834	5.9227	1.06511
33	142	6.3337	5.1231	1.03375	259	7.6907	5.9330	1.06770
49	142	6.8223	5.0856	1.02619	260	7.3791	5.9128	1.06332
50	141	7.1219	5.1384	1.03831	258	7.7576	5.9640	1.07403
52	140	6.3248	5.2049	1.05328	257	8.3219	6.0347	1.08752
66	139	6.4359	5.2554	1.06504	255	7.4043	6.0885	1.09876
81	140	6.9101	5.1895	1.05016	257	8.1663	6.0044	1.08205
85	143	6.7800	5.0216	1.01183	260	7.8790	5.8813	1.05765
105	141	7.0380	5.1771	1.04614	258	7.9647	5.9891	1.07854
146	138	6.0071	5.3216	1.08003	255	7.1539	6.1125	1.10309
147	139	6.6003	5.2756	1.06914	256	8.1533	6.0440	1.08996
163	139	7.1172	5.2817	1.07037	256	7.4892	6.0688	1.09443
170	138	6.8193	5.3547	1.08675	255	7.7101	6.1182	1.10411
189	138	6.9452	5.3358	1.08291	256	8.1988	6.0853	1.09741
197	139	6.6808	5.2831	1.07065	256	8.0667	6.0462	1.09036
199	139	5.9943	5.2720	1.06841	256	7.5814	6.0618	1.09317

TABLE 4. For each n, j_1 is the smallest value of i such that $\sigma^i(n) > 10^{100}$, j_2 is the smallest value of i such that $\sigma^i(n) > 10^{200}$, and $\alpha_u = \sigma^{j_u+1}(n)/\sigma^{j_u}(n)$, $\beta_u = (\sigma^{j_u}(n))^{1/j_u}$, for $u = 1, 2$.

		"hard" divisors d							
n	j_2	239	283	293	347	353	359	383	389
2	263	290							370*
5	262	290				275			293*
16	260	346*				274	265		296
19	257	307*							295
27	256	295*		287					
29	260	287	271				262		301*
33	259	322*							
49	260			295					299*
50	258		261			268*			265
52	257	323*							297
66	255	263							285*
81	257	352*				307		271	278
85	260					264*			262
105	258	281*							
146	255	281			316*				263
147	256	300				285		293	328*
163	256								266*
170	255			283*					
189	256	303						282	305*
197	256			285		257			289*
199	256	292*							276

TABLE 5. Smallest value of i for which $d \mid \sigma^i(n)$, for given d (see top of next page for full explanation).

We give in that table the "hard" divisors, those that did not divide any term of $\{\sigma^i(n)\}$ for some n in (4.2) and $i \le j_2$, with j_2 as in Table 4; and, for each such divisor d, we give the first index $i > j_2$ for which $d \mid \sigma^i(n)$. The largest such index for each n is marked by *, so every number up to 400 divides a term of this sequence for some value of i up to the marked value.

For example, all positive integers less than or equal to 400, except 239 and 389, divide a term of the sequence $\{\sigma^i(2)\}$ for some value of i with $0 \le i \le j_2$, where $j_2 = 263$ is the index of the first term in this sequence that exceeds 10^{200}; furthermore, $239 \mid \sigma^{290}(2)$ and $389 \mid \sigma^{370}(2)$.

Not surprisingly, the larger megaperfect numbers less than 400 are in the list of hard divisors.

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