Conditions on Periodicity for Sum-Free Sets

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CONTENTS

Cameron has introduced a natural one-to-one correspondence between infinite binary sequences and sets of positive integers with the property that no two elements add up to a third. He observed that, if a sum-free set is ultimately periodic, so is the corresponding binary sequence, and asked if the converse also holds. We present here necessary and sufficient conditions for a sum-free set to be ultimately periodic, and show how these conditions can be used to test specific sets. These tests produce the first evidence of a positive nature that certain sets are, in fact, not ultimately periodic.

1. INTRODUCTION

Infinite binary sequences are in natural one-to-one correspondence with *sum-free* sets of positive integers, that is, sets of which no element is the sum of two elements (Section 3). Cameron [1987] observed that, if a sum-free set is ultimately periodic, the corresponding binary sequence is ultimately periodic, and asked whether the converse is also true. This question is still open, but there is some indication that the answer is no: Some apparently aperiodic sets correspond to ultimately periodic binary sequences. Although some of these sets are relatively simple, a proof of their aperiodicity has been elusive, because no method is known that will show that a sum-free set is not ultimately periodic from a consideration of only finitely many elements.

In this work (Section 4) we introduce two new functions g and $\bar{g}$, defined on the positive integers, and we show that the behavior of these functions determines whether a set is ultimately periodic or not. More precisely, we prove that, if its corresponding binary sequence is ultimately periodic, a sum-free set is ultimately periodic if and only if g is bounded, and that if g is not bounded, $\bar{g}(n)$ grows at least as fast as $\log n$.

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In Section 5 we summarize the results of our systematic tests of periodicity over large classes of sum-free sets.

2. DEFINITIONS

Let S be a *sum-free set.* This means that S is a subset of $\mathbb{N}$ (the set of positive integers) and that there are no $x, y, z \in S$ with $x + y = z$. We do not require x and y to be distinct. We denote the set of sum-free sets of positive integers by $\mathcal{S}$.

S is *ultimately complete* if, for all sufficiently large n, either $n \in S$ or there exist $x, y \in S$ such that $x + y = n$.

S is *periodic* if there exists a positive integer m such that, for all $n \geq 1$, we have $n \in S$ if and only if $n + m \in S$.

S is *ultimately periodic* if there exist positive integers m, n_0 such that, for all $n \geq n_0$, we have $n \in S$ if and only if $n + m \in S$. In this case we call n_0 a *preperiod* and m a *period* of S. There is a unique minimum period, since the greatest common divisor of two periods is also a period, but we usually won't insist that either m or n_0 be minimal. For fixed m and n_0 we set

$$\begin{aligned} S_0 &= S \cap \{1, \ldots, n_0 - 1\}, \\ S_{\text{per}} &= S \cap \{n_0, n_0 + 1, \ldots\}, \\ \bar{S} &= S_{\text{per}} \bmod m. \end{aligned}$$

Then $S = S_0 \cup S_{\text{per}}$ and

$$S_{\text{per}} = \{n \in \mathbb{N} : n \geq n_0 \text{ and } n \bmod m \in \bar{S}\}. \qquad (2.1)$$

For example, if S is the set of odd positive integers, which is periodic, we have $\bar{S} = \{1\} \bmod 2$. Removing a finite number of elements from S makes it only ultimately periodic, but does not change $\bar{S}$. For the ultimately periodic set $S = \{1\} \cup \{3n + 2 : n \in \mathbb{N}\}$ we have $\bar{S} = \{2\} \bmod 3$, and so on.

3. THE BIJECTION BETWEEN BINARY SEQUENCES AND SUM-FREE SETS

Let σ be an element of $2^{\mathbb{N}}$, say $\sigma_1\sigma_2\sigma_3\ldots$, where $\sigma_i \in \{0, 1\}$ for every i. We construct the *sum-free set S associated to σ* by testing one integer n at a time, in increasing order, for the possibility of inclusion in S. If n is a sum of integers already in S, it obviously should not be included. Otherwise, the next element of σ dictates whether or not n is included in S.

Formally, we define sets S_i and U_i inductively, starting with $S_0 = U_0 = \varnothing$. Let n_i be the least element of $\mathbb{N}$ that is not in $S_{i-1} \cup (S_{i-1} + S_{i-1}) \cup U_{i-1}$, where $A + B = \{a + b : a \in A,\ b \in B\}$. Then define

$$S_i = \begin{cases} S_{i-1} \cup \{n_i\} & \text{if } \sigma_i = 1, \\ S_{i-1} & \text{if } \sigma_i = 0; \end{cases}$$

$$U_i = \begin{cases} U_{i-1} & \text{if } \sigma_i = 1, \\ U_{i-1} \cup \{n_i\} & \text{if } \sigma_i = 0. \end{cases}$$

Let $S = \bigcup_i S_i$; then, since each S_i is sum-free, and since $S_i \subset S_{i+1}$, the union S is also sum-free.

We write $S = \theta(\sigma)$, defining a map $\theta : 2^{\mathbb{N}} \to \mathcal{S}$. For example,

$$\begin{aligned} \theta(1111111111\ldots) &= \{1, 3, 5, 7, 9, 11, 13, 15, \ldots\}, \\ \theta(0101010101\ldots) &= \{2, 5, 8, 11, \ldots\}, \\ \theta(1010101010\ldots) &= \{1, 4, 7, 10, \ldots\}, \\ \theta(1010010101\ldots) &= \{1, 4, 8, 11, 14, \ldots\}. \end{aligned}$$

This map is a bijection. Indeed, if S is a sum-free set, we define a ternary sequence τ by setting

$$\tau_n = \begin{cases} 1 & \text{if } n \in S, \\ * & \text{if } n \in S + S, \\ 0 & \text{otherwise}; \end{cases} \qquad (3.1)$$

then convert this to a binary sequence σ by deleting all $*$'s. It is an easy exercise to check that this correspondence is inverse to θ. We formalize the $*$-erasing procedure since the notation will be useful later: Let ν be the unique increasing bijection from $\mathbb{N}$ onto the set $\mathbb{N} \setminus (S + S) = \tau^{-1}(\{0, 1\})$; then $\sigma_n = \tau_{\nu(n)}$.

The bijection $\theta : 2^{\mathbb{N}} \to \mathcal{S}$ is actually a homeomorphism if $2^{\mathbb{N}}$ is given the dyadic metric (two sequences are at distance 2^{-k} if they differ for the first time at the $(k+1)$-st place) and $\mathcal{S}$ is given the analogous metric (the distance between S_1 and S_2

is 2^{-k} if $k+1$ is the least element in the symmetric difference $S_1 \triangle S_2$).

We will see in Lemma 4.3 that, *if S is ultimately periodic, so is $\theta^{-1}(S)$.*

Proposition 3.1. *S is ultimately complete if and only if $\theta^{-1}(S)$ contains only finitely many zeros.*

Proof. In the construction of $\theta(\sigma)$, an element is not included if and only if either it is a sum of smaller elements already in the set, or the corresponding term in the binary sequence is zero. Thus, if S is ultimately complete, we can only have finitely many elements excluded because of zeros in $\theta^{-1}(S)$. □

This implies that the set of ultimately complete sum-free sets is countable. By contrast:

Proposition 3.2. *There are uncountably many maximal sum-free sets.*

Naturally, a sum-free set is *maximal* if it cannot be enlarged without destroying the sum-free property.

Proof. Consider the set $\{9, 11, 14, 16, 19, 21, \ldots\} = \{5k \pm 1 : k = 2, 3, \ldots\}$, which is clearly sum-free. If we add to this set the element 2, we find that the only solutions to the equation $x + y = z$ are of the form $5k + 1 = (5k - 1) + 2$. Now consider an arbitrary partition of $\{2, 3, 4, 5, \ldots\}$ into two parts N_1 and N_2, and define

$$S_{N_1,N_2} = \{2\} \cup \{5k-1 : k \in N_1\} \cup \{5k+1 : k \in N_2\}.$$

This set is sum-free, since by definition $N_1 \cap N_2 = \varnothing$. Then no integer of the form $5k - 1$, for $k \in N_2$, or of the form $5k + 1$, for $k \in N_1$, can be adjoined to S_{N_1,N_2}, since such integers are differences or sums of pairs of elements in S_{N_1,N_2}. Now extend S_{N_1,N_2} to a maximal sum-free set T_{N_1,N_2}, using Zorn's lemma. By the preceding comments, T_{N_1,N_2} and T_{M_1,M_2} are distinct if $N_1 \neq M_1$. Since there are uncountably many partitions of $\{2, 3, 4, \ldots\}$, we have proved the proposition. □

Since the lower asymptotic density of T_{N_1,N_2} is at least $\frac{1}{5}$, we get the following result, which answers a question of Stewart (personal communication).

Corollary 3.3. *There are uncountably many aperiodic maximal sum-free sets of positive lower density.*

4. PERIODICITY OF SUM-FREE SETS

We shall now consider one of the most intriguing questions regarding sum-free sets, namely the relationship between the periodicity of a binary string σ and that of the associated sum-free set $\theta(\sigma)$. Cameron (personal communication) has asked if either of these statements is true:

Conjecture 4.1. *A binary string σ is ultimately periodic if and only if $\theta(\sigma)$ is ultimately periodic.*

Conjecture 4.2. *A binary string σ has only finitely many zeros if and only if $\theta(\sigma)$ is ultimately periodic and ultimately complete* (by Proposition 3.1, this is the same as saying that any ultimately complete sum-free set is ultimately periodic).

Clearly 4.1 implies 4.2, but not necessarily vice versa. Lemma 4.3 below shows that the "if" part of the first conjecture holds, and that of the second follows of course from Proposition 3.1. The converses are still open; however, since the questions were first posed, we have found evidence to suggest that Conjecture 4.1 is false, and Cameron [1987] has found evidence that 4.2 may also be false.

Lemma 4.3 [Cameron 1987]. *If $S = \theta(\sigma)$ is ultimately periodic, so is σ.*

Proof. Let n_0 be a preperiod and m a period of S. Consider the ternary sequence τ associated with S via (3.1). For $n \geq 2n_0$, we have $\tau_n = 1 \iff n \in S \iff n \in S_{\text{per}} \iff n \bmod m \in \bar{S}$ by (2.1), and also

$$\begin{aligned} \tau_n = * &\iff n \in S + S \\ &\iff n \in (S_{\text{per}} + S_{\text{per}}) \cup (S_{\text{per}} + S_0) \\ &\iff n \bmod m \in (\bar{S} + \bar{S}) \cup (\bar{S} + (S_0 \bmod m)) \end{aligned}$$

because n is too large to be in $S_0 + S_0$. Thus τ_n depends solely upon the congruence class of n mod m. This shows that τ is ultimately periodic, and therefore so is σ. A period of σ is given by $\nu^{-1}(n+m) - \nu^{-1}(n)$, where $n \in S$ exceeds $2n_0$ (the map ν is defined after (3.1)). □

Where will we run into difficulties when we try to reverse this proof? The crucial step involves the erasing of the $*$'s in τ: given a periodic sequence σ it is easy to insert $*$'s in such a way that the resulting ternary sequence is aperiodic (for example, insert a $*$ after every p_k-th 1, where p_k is the k-th prime). Of course, it is unlikely that such insertions would leave a sum-free set: Conjecture 4.1 states essentially that only by inserting in a periodic manner is it possible to ensure that S is sum-free.

In trying to prove the "only if" part of the conjectures, one might be helped by a sufficient criterion that ensures that a sum-free set is ultimately periodic. The following lemma is one such criterion: it says that if a sum-free set S is ultimately periodic, this fact can be proved by considering only $\theta^{-1}(S)$ and a finite prefix of S.

Lemma 4.4. *Let S be a sum-free set, with associated binary sequence σ, and let m be an integer. Set*

$$S_k = S \cap \{km+1,\, km+2,\, \dots,\, (k+1)m\}$$

for $k = 1, 2, 3$, and $t_k = \nu^{-1}(\max S_k)$. Suppose that $S_3 = S_2 + m = S_1 + 2m$, and that $\sigma_{t+t_2} = \sigma_{t+t_3}$ for all $t > 0$—in particular, σ is ultimately periodic of period $p = t_3 - t_2$. Then S is ultimately periodic of period m and preperiod m.

Proof. We show by induction that $n \geq m$ is in S if and only if $n+m$ is in S. This is true of $n = m+1, \dots, 3m$ because S_2 and S_3 are translates of S_1. Therefore we can take $n > 3m$, and assume that the claim is true for all lesser values of n ($\geq m$).

In fact we will show that $\tau_{n+m} = \tau_n$. Take first the case $\tau_n = *$, that is, $n \notin S + S$. Then also $n+m \notin S+S$; otherwise, express $n+m$ as $x+y$ with $x, y \in S$ and $x \in \{2m+1, \dots, n-1\}$ (using the fact that $n > 3m$), and apply the induction assumption to write $n = (x-m) + y \in S + S$. Analogously, $n + m \notin S + S$ implies $n \notin S + S$; otherwise, set $n = x+y$ with $x, y \in S$ and $x \in \{m+1, \dots, n-1\}$, and apply the induction assumption. This shows that $\tau_{n+m} = * \iff \tau_n = *$.

On the other hand, if $\tau_n \neq *$ and $\tau_{n+m} \neq *$, we have $\nu^{-1}(n+m) = \nu^{-1}(n) + p$ (by induction; the base case is $n = \max S_2$ and $n + m = \max S_3$, and n is in the image of ν if and only $n + m$ is). But then $\tau_{n+m} = \sigma_{\nu^{-1}(n+m)} = \sigma_{\nu^{-1}(n)+p} = \sigma_{\nu^{-1}(n)} = \tau_n$, where the second-to-last equality comes from the lemma's assumption. □

In order to test Cameron's conjectures, we generated the sum-free sets corresponding to periodic binary inputs with period at most seven. For all inputs with periods of length at most four, the corresponding sum-free set was ultimately periodic, with a small preperiod (usually fewer than 10) and a small period (always less than 25). Of the thirty inputs with periods of length five, all but three ($\dot{0}100\dot{1}$, $\dot{0}101\dot{0}$, $\dot{1}001\dot{0}$) gave sum-free sets that were quickly periodic. The set

$$\theta(\dot{0}100\dot{1}) = \{2,6,9,14,19,26,29,36,39,47,54,64,69,79,84,91,\dots\},$$

certainly appears to be aperiodic: the sequence of differences between consecutive elements up to 10^7 exhibits long strings that are repeated, separated by short "glitches" that show no sign of settling down to be periodic. Other potential counterexamples to Cameron's conjecture will be exhibited in Section 5.

This, of course, is all evidence of a rather flimsy type: "We looked, but we couldn't find anything". We shall now state theorems that lead to more positive evidence that certain sum-free sets, $\theta(\dot{0}100\dot{1})$ among them, are aperiodic.

Define functions g and $\bar{g}$ on $\mathbb{N}$ by

$$g(n) = \begin{cases} 0 & \text{if } n \notin S+S, \\ \min\{x \in S : x+y = n \text{ for some } y \in S\} & \\ \qquad \text{otherwise;} & \end{cases}$$

$$\bar{g}(n) = \max_{k \leq n} g(k).$$

Theorem 4.5. *S is ultimately periodic if and only if σ is ultimately periodic and $\bar{g}$ is ultimately constant (that is, g is bounded).*

Proof. For S ultimately periodic, with preperiod n_0 and period m, we get $g(n) < n_0 + m$ using the equivalence $\tau_n = * \iff n \in (S_{\text{per}} + S_{\text{per}}) \cup (S_{\text{per}} + S_0)$ from the proof of Lemma 4.3 (for $n \in S_{\text{per}} + S_{\text{per}}$, one of the summands can be taken less than $n_0 + m$, by periodicity).

Conversely, suppose $\sigma = \theta^{-1}(S)$ is ultimately periodic of period p, and take r large enough that $\nu^{-1}(r)$ is in the periodic part of σ. Suppose also that $g(n) \le k$ for all n. Define $S_n = S \cap \{n+1, n+2, \dots, n+k\}$ for $n \ge 0$, so that $S + S = S_0 + S$. Then, for $n > r$, the question whether $n + k + 1$ belongs to S, and thus to S_{n+1}, depends only on S_0, S_n, and $\sigma(j_n)$, where $j_n = \nu^{-1}(\max S_n)$; also $\sigma(j_n)$ depends only on $j_n \bmod p$. Setting $T_n = S_n - n$ and $i_n = j_n \bmod p$, it follows that T_{n+1} and i_{n+1} are determined by T_n and i_n (the dependence being controlled by S_0 and σ, which are fixed). T_n has at most 2^k values as n varies, and i_n has at most p values, so there exist n_0 and $n_0 + m$, both in the interval $\{r, r+1, \dots, r + 2^k p\}$, such that $T_{n_0} = T_{n_0+m}$ and $i_{n_0} = i_{n_0+m}$. Thereafter, T_n and T_{n+m} coincide. This proves that S is ultimately periodic with preperiod n_0 and period m. □

In fact, Theorem 4.5 can be strengthened: for a sum-free set that is not ultimately periodic, $\bar{g}$ must grow at least logarithmically, as we now show.

Theorem 4.6. *Suppose $\sigma = \theta^{-1}(S)$ is ultimately periodic, with period p, and take $r > 2 \min S$ large enough that $\nu^{-1}(r)$ is in the periodic part of σ. If there exists $N > 4r$ such that*

$$\bar{g}(N) < \log_2 \frac{N - 4r}{4p}, \tag{4.1}$$

then S is ultimately periodic.

Proof. Let $N > 4r$ satisfy (4.1), and set $k = \bar{g}(N)$: thus $N \ge 4(r + 2^k p)$. Then, copying the notation and reasoning from the proof of Theorem 4.5, we can find n_0 and $n_0 + m$, both in the interval $\{r, r+1, \dots, r + 2^k p\}$, such that $T_{n_0} = T_{n_0+m}$ and $i_{n_0} = i_{n_0+m}$. Thereafter, T_n and T_{n+m} coincide at least until $n + m = N$. Replace m by the least multiple of m greater than n_0; this number is still bounded by $r + 2^k p \le \frac{1}{4}N$. But now m satisfies the conditions of Lemma 4.4, proving that S is ultimately periodic. □

Computing the values of $\bar{g}(n)$ for the set $\theta(\dot{0}100\dot{1})$, for all $n \le 200\,000$, we find that $\bar{g}$ appears to be very far from bounded: in fact it seems to increase in a roughly linear fashion, throughout the whole range $n < 10^7$. See Table 1.

If it could be shown for such a set S that such behavior continues, namely that there exist an infinite number of n such that $g(n)/n$ is close to $\frac{1}{2}$,

n	$\bar{g}(n)$	n	$\bar{g}(n)$	n	$\bar{g}(n)$	n	$\bar{g}(n)$	n	$\bar{g}(n)$	n	$\bar{g}(n)$	n	$\bar{g}(n)$
4	2	242	121	1820	597	4632	2068	14779	7104	47437	23304	121318	57969
12	6	274	137	1850	627	4945	2381	16129	7675	49313	24133	126698	63349
18	9	322	161	2028	805	5128	2564	19678	9839	50678	25180	137806	65796
33	14	348	174	2058	835	6053	2676	22914	11457	50996	25498	142928	71464
52	26	362	181	2103	880	6411	3034	24624	12312	65250	28709	171101	81091
72	36	637	237	2356	1133	6674	3297	27324	13394	68410	30974	188656	82178
94	47	647	247	2371	1148	6709	3332	30140	14127	75499	37613	199466	99733
133	54	690	345	2401	1178	6754	3377	40677	15179	82800	38422	....	....
182	91	885	430	2446	1223	10360	4014	43908	16281	88756	44378	605846	1211692
192	96	1288	445	3650	1522	11144	4798	43948	21974	111332	54455	....	....
227	106	1457	577	4394	1795	12692	6346	46355	22222	112419	55542	4621889	9662060

TABLE 1. For $S = \theta(\dot{0}100\dot{1})$, the table gives the points $n \le 200\,000$ at which $\bar{g}(n)$ has just increased, and the corresponding values $\bar{g}(n)$. Also given are the largest $n < 10^7$ for which $g(n) = \frac{1}{2}n$ (penultimate entry), and the largest $n < 10^7$ for which $\bar{g}$ increases (last entry).

say, it would follow immediately from Theorem 4.5 that S is aperiodic; it does not, however, appear that it is a simple matter to prove this.

5. COMPUTATIONAL EVIDENCE

If we could prove $\theta(\dot{0}100\dot{1})$ is aperiodic, there would be no need to list further potential counterexamples to Cameron's conjecture. Since we couldn't, we found it to be of some value to test periodicity over large classes of sum-free sets, in the hope that a recognizable pattern to the counterexamples might eventually emerge. Table 2 summarizes the possible counterexamples we have found among all periodic binary sequences σ of periods 5, 6 or 7. This includes the three potential counterexamples mentioned earlier.

$\dot{0}100\dot{1}$	$\dot{0}1000\dot{1}$	$\dot{0}01000\dot{1}$	$\dot{0}10101\dot{1}$
$\dot{0}101\dot{0}$	$\dot{0}1100\dot{1}$	$\dot{0}01001\dot{0}$	$\dot{0}10110\dot{1}$
$\dot{1}001\dot{0}$	$\dot{0}1110\dot{0}$	$\dot{0}10000\dot{1}$	$\dot{0}11000\dot{1}$
	$\dot{1}0001\dot{0}$	$\dot{0}10001\dot{0}$	$\dot{1}00001\dot{0}$
	$\dot{1}0100\dot{1}$	$\dot{0}10010\dot{0}$	$\dot{1}00010\dot{0}$
	$\dot{1}0101\dot{1}$	$\dot{0}10010\dot{1}$	$\dot{1}00011\dot{0}$
		$\dot{0}10101\dot{0}$	$\dot{1}01010\dot{0}$

TABLE 2. Periodic binary sequences whose associated sum-free sets are incomplete and appear to be aperiodic (aperiodicity checked up to 10^7).

We note that periodicity in sum-free sets need not arrive quickly. $S = \theta(\dot{0}11001\dot{1})$ has minimal period $m = 10\,710$, after a transient phase of approximately 89 000 terms. Moreover, the largest integer $n \in S$ for which $n + m \notin S$ is $n = 489\,115$, and the largest integer $n \notin S$ for which $n + m \in S$ is $n = 489\,108$.

In addition to periodic binary sequences of periods up to 7, we studied those having period 3 and preperiod 2 (that is, of the form $uv\dot{x}y\dot{z}$), and those having period 2 and preperiod 5. The potentially aperiodic sum-free sets among them (also checked up to 10^7) are $\theta(00\dot{0}0\dot{1})$, $\theta(00001\dot{1}\dot{0})$, $\theta(11000\dot{0}\dot{1})$, and $\theta(00110\dot{0}\dot{1})$. These are the simplest such cases, that is, the binary inputs simultaneously have minimal preperiod and minimal period.

Cameron [1987] found the first potentially aperiodic complete sum-free set; it is entry 1 in Table 4 (the notation will be explained shortly). The existence of such a set suggests that Dickson's problem [Dickson 1934; Guy 1980, Problem E32] may have a negative solution. Queneau [1972] and Finch [1992] have studied a variation of this problem involving what are known as 0-additive sequences; an update on this direction of research appears in [Guy 1993].

By the *base* of an ultimately complete sum-free set $S = \{s_1 < s_2 < \cdots < s_n < \cdots\}$ we mean the minimal set of S-elements $B = \{s_1, s_2, \ldots, s_n\}$ such that recursive application of the greedy algorithm, starting with B, gives the sum-free set S.

By the phrase "all sum-free bases up to p" we mean the collection of all sets B that are bases of ultimately complete sum-free sets S and whose largest element is at most p. For example, the sum-free bases up to 7 are

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\},$$
$$\{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{2,5\}, \{2,6\}, \{2,7\},$$
$$\{3,5\}, \{3,7\}, \{4,6\}, \{4,7\}, \{5,7\},$$
$$\{1,3,7\}, \{1,4,7\}, \{4,5,7\}.$$

We examined each of the 76 080 sum-free bases up to 27 and determined whether each of the corresponding complete sum-free sets were periodic (checked up to 10^7). We did the same for all sum-free bases up to 35 with three or fewer elements. All apparently aperiodic cases (for which g appears to be unbounded and no pattern is seen) are listed in Table 3. Table 4, by contrast, lists those cases that we classify as tentatively periodic. Entry 1 in this table is Cameron's example. Entry 6 is the same, minus one term, as $\{15, 16, 18, 21, 22, 24, 27\}$, which is not listed to avoid duplication. Entry 7 is unexpected: the maximum g-value is quite small, but no clear signs of periodicity are apparent.

We stress that periodicity need not arrive quickly. For example, the periodic complete sum-free set S based on $\{10, 14, 15, 17, 22\}$ has minimal period $m = 2\,875\,722$ after a transient phase of approximately 584 000 terms. The largest integer $n \in S$ for

$\{8,18,30\}$	$\{1,3,8,20,26\}$
$\{8,27,32\}$	$\{2,15,16,23,27\}$
$\{9,16,29\}$	$\{5,6,14,23,27\}$
$\{9,26,32\}$	$\{3,12,17,19,21,27\}$
$\{9,28,35\}$	$\{10,13,15,16,17,24\}$
$\{10,18,34\}$	$\{10,15,16,18,22,27\}$
$\{11,26,35\}$	$\{12,15,17,18,19,25\}$
$\{12,21,35\}$	$\{14,16,17,18,21,27\}$
$\{9,21,24,27\}$	$\{6,14,17,18,22,25,27\}$
$\{11,16,17,26\}$	

TABLE 3. Apparently aperiodic complete sum-free sets listed by base (checked up to 10^7).

#	base	$\bar{g}(10^7)$	est. per.
1	$\{3,4,13,18,24\}$	2937317	3274006
2	$\{8,14,15,17,26\}$	2898098	?
3	$\{14,15,16,18,21,26\}$	1349528	?
4	$\{14,15,18,20,22,24,26\}$	1424518	1291498
5	$\{4,17,18,19,24,27\}$	3132839	1022104
6	$\{15,16,18,22,24,27\}$	2330099	2673770
7	$\{4,21,32\}$	770538	?

TABLE 4. Tentatively periodic complete sum-free sets listed by base ($\bar{g}(10^7) \le 3.2 \cdot 10^6$). The last column gives our best estimate of the period.

which $n+m \notin S$ is $n = 4\,562\,648$, and the largest integer $n \notin S$ for which $n+m \in S$ is $n = 4\,453\,256$.

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