All Geometries of the Mathieu Group M_{11} Based on Maximal Subgroups

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CONTENTS

1. Introduction
2. Definitions and Notation
3. The List of Geometries
Acknowledgement
Electronic Availability
References

Using a Cayley program, we get all firm, residually connected geometries whose rank-two residues satisfy the intersection property, on which M_{11} acts flag-transitively, and in which the stabilizer of each element is a maximal subgroup of M_{11}.

1. INTRODUCTION

This paper can be seen as a sequel to [Dehon 1994], where a set of Cayley programs is presented in order to classify all firm, residually connected and flag-transitive geometries for a given group G, with the additional restriction that the stabilizer of any element is a maximal subgroup of G. In [Dehon 1994] these programs are applied to the automorphism group of $\mathrm{PSU}(4,2)$, which has order 51840, and a fairly wild list is obtained.

Here, the programs are applied to the case where G is the Mathieu group M_{11}. The number of nonisomorphic geometries of ranks 1, 2, 3, and 4 satisfying these conditions is equal to 5, 37, 78, and 30, and there are none of rank higher than four. On the basis of this experience and that in [Dehon 1994], we found it interesting to test one further condition, namely the intersection property for all rank-two residues. This gives us 5, 8, 10, and 10 geometries of those ranks. We list these geometries explicitly by their diagram, and determine further properties of them.

The present work is also a step of the systematic search advocated in [Buekenhout 1986]. Our result can be considered as a theorem about the sporadic group M_{11}, obtained by computer research. It confirms earlier observations of geometries made for this group (see for instance [Buekenhout 1995,

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Chapter 22]) and it gives them an "objective" status in view of the concise character of our lists. We observe in particular the frequent occurence of complete graphs and of the Petersen graph among rank-two residues. Also, some of the geometries collected are likely to occur as residues in many interesting geometries for larger sporadic groups. Finally, we observe that our conditions are realized by pairs consisting of a building geometry and a group of Lie type acting on it. Convenient generalizations of buildings applicable to M_{11} may arise from this and similar work.

2. DEFINITIONS AND NOTATION

Most of the following ideas arise from [Tits 1962]; see also [Buekenhout 1995, Chapter 3].

Let G be a group, together with a finite family of subgroups $(G_i)_{i\in I}$. We define the pre-geometry $\Gamma = \Gamma(G, (G_i)_{i\in I})$ as follows.

The set X of elements of Γ consists of all cosets gG_i, for $g \in G$ and $i \in I$. We define an incidence relation $*$ on X by

$$g_1G_i * g_2G_j \iff g_1G_i \cap g_2G_j \neq \varnothing.$$

The *type function* t on Γ is defined by $t(gG_i) = i$. The *type* of a subset Y of X is the set $t(Y)$; its *rank* is the cardinality of $t(Y)$, and we call $|I|$ the *rank* of Γ. A *flag* is a set of pairwise incident elements of X, and a *chamber* of Γ is a flag of type I. An element of type i is also called an *i-element.*

The group G acts on Γ as an automorphism group, by left translation, preserving the type of each element. Indeed, $g \in G$ maps g_1G_i onto gg_1G_i, and $gg_1G_i * gg_2G_j$ amounts to $g_1G_i * g_2G_j$, so the incidence relation is preserved. The action involves a kernel K, which is the largest normal subgroup of G contained in every G_i, for $i \in I$. In this action, the subgroup G_i is the stabilizer of the element of Γ identified with G_i in the construction of Γ.

As in [Dehon 1994], we call Γ a *geometry* if every flag on Γ is contained in some chamber, and we call Γ *flag-transitive* if G acts transitively on all chambers of Γ, hence also on all flags of any given type J, where J is a subset of I.

Assuming that Γ is a flag-transitive geometry and that F is a flag of Γ, the *residue* of F is the pre-geometry

$$\Gamma_F = \Gamma\biggl(\bigcap_{j\in t(F)} G_j,\ \Bigl(G_i \cap \Bigl(\bigcap_{j\in t(F)} G_j\Bigr)\Bigr)_{i\in I\setminus t(F)}\biggr),$$

and we readily see that Γ_F is a flag-transitive geometry.

We call a geometry Γ *firm* if every flag of rank $|I|-1$ is contained in at least two chambers. We call Γ *residually connected* if the incidence graph of each residue of rank at least two is a connected graph. We call Γ *primitive* if G acts primitively on the set of i-elements of Γ, for each $i \in I$. We call Γ *residually primitive* if each residue Γ_F of a flag F is primitive for the group induced on Γ_F by the stabilizer G_F of F.

We also consider a variation of the latter concept. We call Γ *weakly primitive* if there exists some $i \in I$ such that G acts primitively on the set of i-elements of Γ, and we call Γ *residually weakly primitive* if each residue Γ_F of a flag F is weakly primitive for the group induced on Γ_F by the stabilizer G_F.

If Γ is a geometry of rank two with $I = \{0,1\}$ and such that each of its 0-elements is incident with each of its 1-elements, we call Γ a *generalized digon.*

We say that Γ satisfies the *rank-two intersection property* if, in every rank-two residue of Γ other than a generalized digon, any two elements of the same type are incident with at most one element of the other type.

We call Γ *locally 2-transitive* if the stabilizer G_F of any flag F of rank $|I|-1$ acts 2-transitively on the residue Γ_F.

The *diagram* of a firm, residually connected, flag-transitive geometry Γ is the graph whose vertices are the elements of I, plus additional structure as follows. To each vertex $i \in I$ we attach the *order* s_i, which is $|\Gamma_F|-1$, where F is any flag of type $I \setminus \{i\}$. We also attach to i the number n_i of varieties of type i, which is the index of G_i in G, and the subgroup G_i. Two vertices $i, j \in I$ are *not* joined by an edge of the diagram if a residue Γ_F of type $\{i, j\}$ is a generalized digon. Otherwise, i and j are joined by an edge endowed with three positive integers d_{ij}, g_{ij}, and d_{ji}. The *gonality* g_{ij} is equal to half the girth of the incidence graph of a residue Γ_F of type $\{i, j\}$. The *i-diameter* d_{ij} is the greatest distance from some fixed i-element to any other element in Γ_F, and the *j-diameter* d_{ji} is defined analogously.

On a picture of the diagram, this structure will be depicted as follows:

$$s_i/n_i/G_i \; \textcircled{\scriptsize i} \overset{d_{ij}\ g_{ij}\ d_{ij}}{\rule[0.5ex]{6em}{0.4pt}} \textcircled{\scriptsize j} \; s_j/n_j/G_j$$

In view of the rank-two intersection property, we have $g_{ij} \geq 3$. Moreover, $g_{ij} \leq d_{ij}$, $g_{ij} \leq d_{ji}$, and $|d_{ij} - d_{ji}| \leq 1$. If $g_{ij} = d_{ij} = d_{ji}$, we call Γ_F a *generalized g-gon*, and we don't write d_{ij} and d_{ji} on the picture. Observe that generalized digons are characterized by $g_{ij} = d_{ij} = d_{ji} = 2$.

Let J be a nonempty subset of the type set I. The *J-truncation* of Γ is the pre-geometry $J_\Gamma = \Gamma(G, (G_i)_{i \in J})$.

Assuming that G acts with a trivial kernel on the set X of elements of Γ, a *correlation* of the pair (G, Γ) is an automorphism α of the incidence graph $(X, *)$ mapping the permutation group (G, X) onto itself and such that $t(x) = t(y)$ implies $t(\alpha(x)) = t(\alpha(y))$ for any $x, y \in X$. The group of all correlations of (G, Γ) is called $\mathrm{Cor}\,\Gamma$.

As to notation for groups, we follow the Atlas [Conway et al. 1985]. The symbol : stands for a split extension, the "hat" symbol $\hat{\ }$ for a nonsplit extension, and $\times$ for a direct product.

3. THE LIST OF GEOMETRIES

The following tables list the data for the five rank-one geometries (Table 1), eight rank-two geometries (Table 2), ten rank-three geometries (Table 3), and ten rank-four geometries (Table 4). We recall that each geometry is firm, residually connected, flag-transitive, and primitive, and that it satisfies the rank-two intersection property.

Γ	structure of G_0	$\#\Gamma$
1–1	M_{10}	11
1–2	$L(2, 11)$	12
1–3	$M_9 : 2$	55
1–4	$M_8 : S_3$	165
1–5	S_5	66

TABLE 1. Summary of rank-one geometries, that is, maximal subgroups up to conjugacy. This is Atlas information [Conway et al. 1985]. The last column gives the number of elements of Γ. We note that $M_8 : S_3 = 2\hat{\ }S_4$.

We report on residual primitivity, residual weak primitivity, and local two-transitivity ($2\mathrm{T}_1$). We also give the structure of the Boolean lattice whose members are intersections of some G_i; in the lattices, a bold line means maximal inclusion.

Whenever the diagram of the geometry has a nontrivial automorphism, we give in the table caption the group of correlations $\mathrm{Cor}\,\Gamma$.

If the subgroup G_i acts with a nontrivial kernel K_i on the residue of the element G_i of Γ, we describe G_i as $\boxed{K_i}.G_i/K_i$, where the symbol . stands for :, $\hat{\ }$, or $\times$.

A picture of the subgroup lattice of M_{11} can be found in [Buekenhout 1986].

Table 5 shows how the geometries of different rank are related by truncation.

ACKNOWLEDGEMENT

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ELECTRONIC AVAILABILITY

The computations leading to the results described in this work were originally performed in Cayley [Cannon 1984]. The programs have since been rewritten in C++ and Magma, and can be obtained from http://cso.ulb.ac.be/~dleemans. For information on Magma see http://www.maths.usyd.edu.au:8000/comp/magma/Overview.html.

Γ	structure of Γ	$\bigcap_{i\in I} G_i$	RWP	RP	$2T_1$
2–1	$10/11/M_{10}\ \bigcirc \qquad \bigcirc\ 11/12/L(2,11)$	A_5	no	no	no
2–2	$54/55/M_9{:}2\ \bigcirc \qquad \bigcirc\ 11/12/L(2,11)$	D_{12}	no	no	no
2–3	$1/11/M_{10}\ \bigcirc\overset{3\;3\;4}{\text{———}}\bigcirc\ 9/55/\boxed{M_9}{:}2$	M_9	yes	yes	yes
2–4	$1/12/L(2,11)\ \bigcirc\overset{3\;3\;4}{\text{———}}\bigcirc\ 10/66/\boxed{A_5}{:}2$	A_5	yes	yes	yes
2–5	$3/55/M_9{:}2\ \bigcirc\overset{5\;3\;6}{\text{———}}\bigcirc\ 8/165/\boxed{M_8}{:}S_3$	$M_8{:}2$	yes	yes	yes
2–6	$3/55/M_9{:}2\ \bigcirc\overset{4\;3\;4}{\text{———}}\bigcirc\ 11/165/M_8{:}S_3$	D_{12}	no	no	no
2–7	$3/66/S_5\ \bigcirc\overset{5\;3\;5}{\text{———}}\bigcirc\ 9/165/M_8{:}S_3$	D_{12}	yes	yes	no
2–8	$7/165/M_8{:}S_3\ \bigcirc\overset{5\;3\;5}{\text{———}}\bigcirc\ 7/165/M_8{:}S_3$	S_3	no	no	no

TABLE 2. Summary of rank-two geometries. The last three columns say whether the geometry is residually weakly primitive (RWP), residually primitive (RP), and locally two-transitive ($2T_1$). See page 103 for other notations. *Remarks:* Geometries 2–1 and 2–2 are generalized digons and correspond to factorizations $M_{11} = G_0G_1$ of [Liebeck 1990]; 2–3 and 2–4 are complete graphs, while 2–5 is a truncation of 3–1 (see next section). In 2–6 and 2–7, we have $M_8 : S_3 = \boxed{2}\hat{\ }S_4$. Geometry 2–8 is self-dual, and its group of correlations is $M_{11} \times 2$.

Γ	structure of Γ	rank-2 residues	lattice	RWP	RP	$2T_1$
3–1	$\bigcirc\overset{3}{\text{———}}\bigcirc\overset{3\;3\;4}{\text{———}}\bigcirc$ $1/11/M_{10}\quad 1/55/M_9{:}2\quad 8/165/\boxed{M_8}{:}S_3$		$M_{10}\quad M_9{:}2\quad M_8{:}S_3$ $M_9\quad M_8{:}2\quad M_8{:}2$ $M_8 = Q_8$	yes	yes	yes
3–2	$\bigcirc\overset{3\;3\;4}{\text{———}}\bigcirc\overset{5\;3\;5}{\text{———}}\bigcirc$ $2/11/M_{10}\quad 3/165/M_8{:}S_3\quad 3/55/M_9{:}2$	$M_{10} = \bigcirc\overset{5\;3\;5}{\text{———}}\bigcirc$ $3/45/M_8{:}2\quad 3/45/M_8{:}2$	$M_{10}\quad M_8{:}S_3\quad M_9{:}2$ $M_8{:}2\quad M_8{:}2\quad D_{12}$ 2^2	no	no	no
3–3	$\bigcirc\overset{5\;3\;5}{\text{———}}\bigcirc\overset{3\;3\;4}{\text{———}}\bigcirc$ $2/55/M_9{:}2\quad 2/165/M_8{:}S_3\quad 3/66/S_5$	$S_5 = \bigcirc\overset{5\;3\;5}{\text{———}}\bigcirc$ $2/10/D_{12}\quad 2/10/D_{12}$	$M_9{:}2\quad M_8{:}S_3\quad S_5$ $M_8{:}2\quad D_{12}\quad D_{12}$ 2^2	no	no	no

TABLE 3. Summary of rank-three geometries. See page 103 for notations. *Remarks:* In 3–2 and 3–3, we have $M_8 : S_3 = \boxed{2}\hat{\ }S_4$. The S_5 in 3–3 is Desargues's configuration [Buekenhout et al. 1995].

Γ	structure of Γ	rank-2 residues	lattice	RWP	RP	$2T_1$
3–4	$2/55/M_9{:}2 \overset{7\,5\,7}{—} 2/165/M_8{:}S_3 \overset{3\,3\,4}{—} 3/12/L(2,11)$	$L(2,11) = 2/55/D_{12} \overset{7\,5\,7}{—} 2/55/D_{12}$	$M_9{:}2$ $M_8{:}S_3$ $L(2,11)$ / $M_8{:}2$ D_{12} D_{12} / 2^2	no	no	no
3–5	$2/11/M_{10} \overset{4\,3\,4}{—} 2/165/M_8{:}S_3 \overset{5\,3\,6}{—} 7/165/M_8{:}S_3$	$M_8{:}S_3 = 2/8/S_3 \overset{4\,3\,4}{—} 2/8/S_3$; $M_{10} = 2/45/M_8{:}2 \overset{5\,3\,6}{—} 7/120/S_3$	M_{10} $M_8{:}S_3$ $M_8{:}S_3$ / $M_8{:}2$ S_3 S_3 / 2	no	no	no
3–6	$11/12/L(2,11)$ $1/11/M_{10} \overset{3\,3\,4}{—} 9/55/M_9{:}2$		$L(2,11)$ M_{10} $M_9{:}2$ / A_5 D_{12} M_9 / S_3	no	no	no
3–7	triangle: $1/11/M_{10}$, $4/66/S_5$, $4/66/S_5$; edges $4\,3\,3$, $3\,3\,4$, $5\,3\,5$	$M_{10} = 4/36/5{:}4 \overset{5\,3\,5}{—} 4/36/5{:}4$	M_{10} S_5 S_5 / $5{:}4$ $5{:}4$ D_8 / 4	yes	no	yes
3–8	triangle: $8/165/\boxed{M_8}{:}S_3$, $1/11/M_{10}$, $1/11/M_{10}$; edges $3\,3\,4$, $4\,3\,3$, 3		M_{10} M_{10} $M_8{:}S_3$ / M_9 $M_8{:}2$ $M_8{:}2$ / M_8	yes	yes	yes
3–9	triangle: $1/12/L(2,11)$, $2/165/M_8{:}S_3$, $2/66/S_5$; edges $6\,5\,5$, $3\,3\,4$, $7\,5\,7$	$L(2,11) = 2/55/D_{12} \overset{7\,5\,7}{—} 2/55/D_{12}$	$M_8{:}S_3$ $L(2,11)$ S_5 / D_{12} D_8 D_{12} / 2^2	yes	no	yes
3–10	triangle: $2/66/S_5$, $2/66/S_5$, $2/66/S_5$; edges $6\,3\,5$, $5\,3\,6$, $4\,3\,4$	$S_5 = 3/15/D_8 \overset{4\,3\,4}{—} 3/15/D_8$; $S_5 = 2/15/D_8 \overset{5\,3\,6}{—} 3/20/S_3$	S_5 S_5 S_5 / D_8 S_3 D_8 / 2	no	no	no

TABLE 3. Summary of rank-three geometries (continued). *Remarks:* In 3–4 and 3–9, we have $M_8 : S_3 = \boxed{2}\hat{\cdot}S_4$. Geometries 3–7, 3–8, and 3–10 are self-dual, and their correlation group is $M_{11} \times 2$.

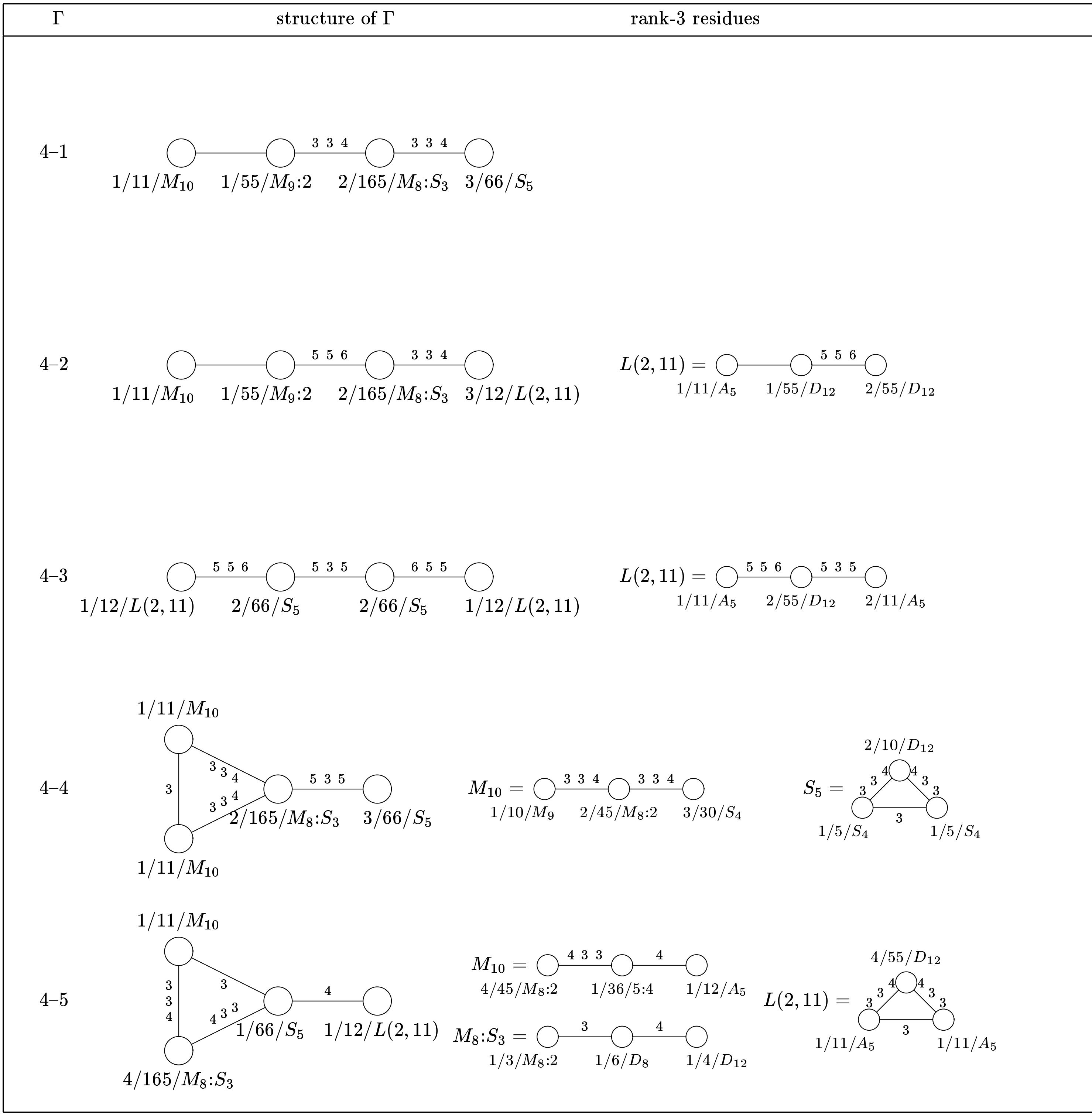
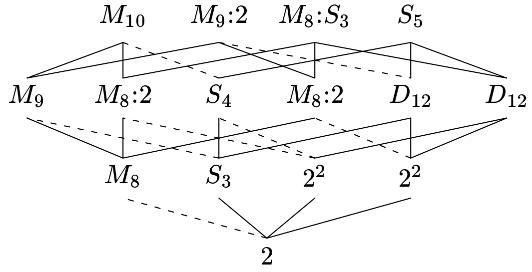
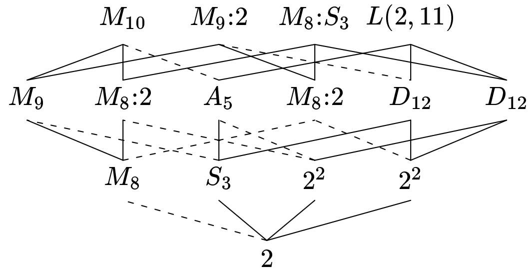
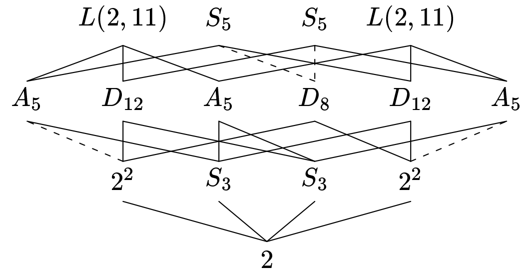
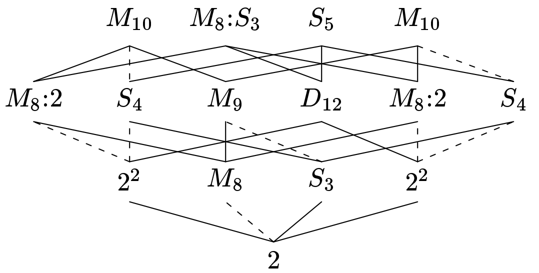
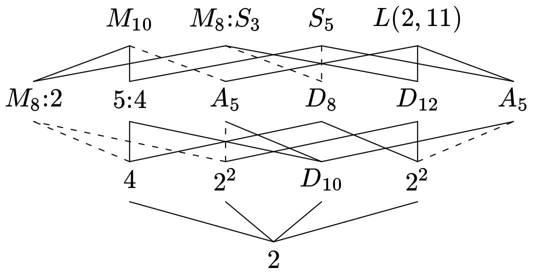


TABLE 4. Summary of rank-four geometries. See page 103 for notations. *Remarks:* Geometry 4–1 is the Steiner system $S(4,5,11)$. In 4–2, each $L(2,11)$-element is incident with each M_{10}-element and with each

lattice	RWP	RP	$2T_1$
	no	no	no
	no	no	no
	yes	no	yes
	no	no	no
	no	no	no

($M_9{:}2$)-element. Geometries 4–3 and 4–4 are self-dual, and their group of correlations is $M_{11} \times 2$; for the S_5 in 4–4, see [Buekenhout et al. 1995]. In 4–1, 4–2, 4–4, and 4–5, we have $M_8{:}S_3 = \boxed{2}\hat{\cdot}(2^2{:}S_3)$.

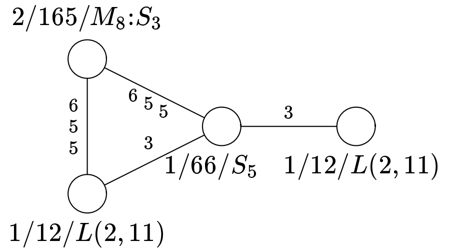
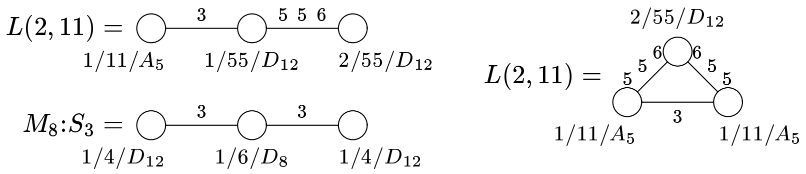
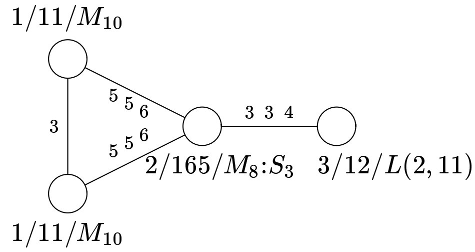
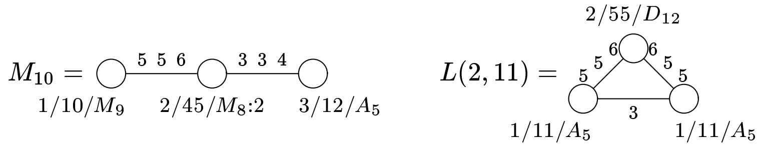
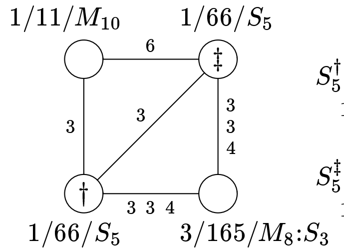
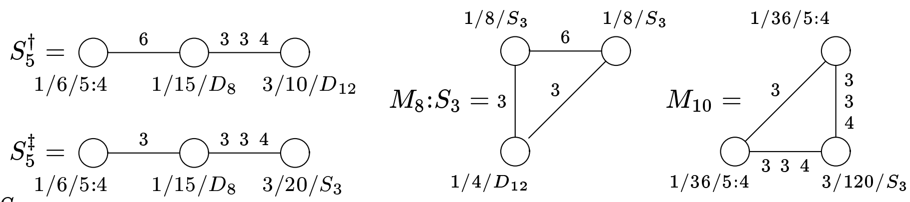
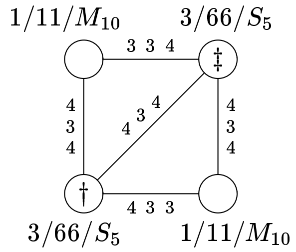
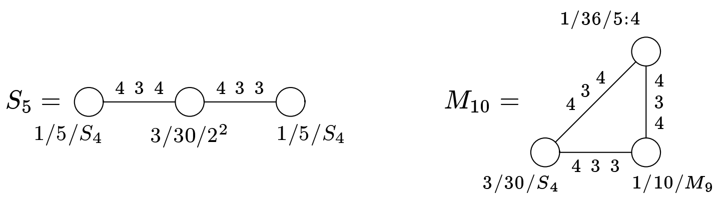
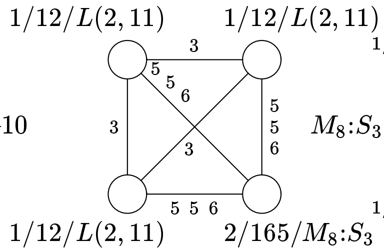
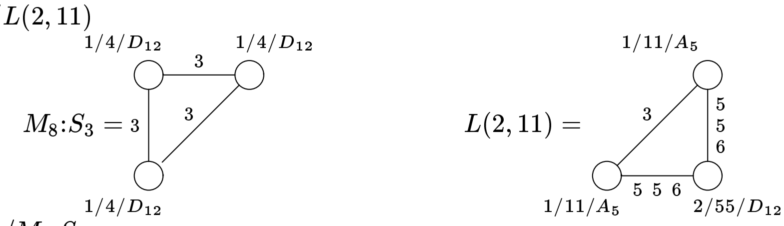
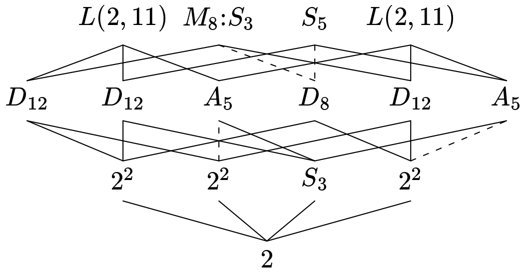
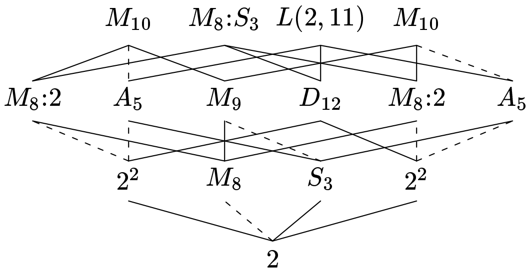
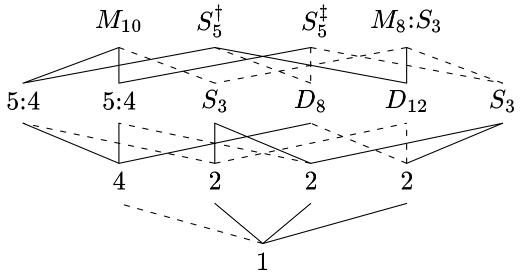
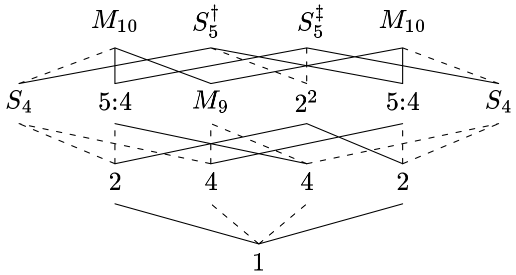
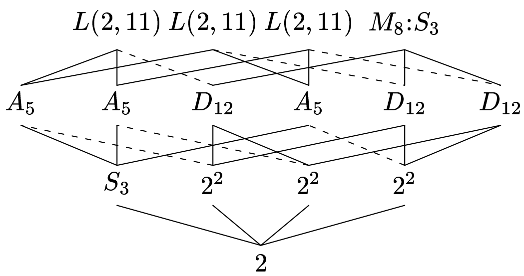
Γ	structure of Γ	rank-3 residues
4–6		
4–7		
4–8		
4–9		
4–10		

TABLE 4. Summary of rank-four geometries (continued). *Remarks:* In 4–6 and 4–7, we have $M_8{:}S_3 = \boxed{2}\hat{\ }(2^2{:}S_3)$. Geometries 4–7 and 4–9 are self-dual, with $\operatorname{Cor}\Gamma = M_{11} \times 2$, while 4–10 has $\operatorname{Cor}\Gamma = M_{11} \times S_3$. The symbols

lattice	RWP	RP	$2T_1$
	yes	no	yes
	no	no	no
	no	no	no
	no	no	no
	yes	no	yes

† and ‡ in the nodes of the diagram in 4–8 and 4–9 serve to distinguish between the respective entries on the top row of the intersections lattice and (in 4–8) between rank-three residues.

Γ_1	Γ_2	subgroups removed
4–1	3–1	S_5
4–1	3–3	M_{10}
4–2	3–1	$L(2,11)$
4–2	3–4	M_{10}
4–2	3–6	$M_8 : S_3$
4–3	2–4	$L(2,11)$, S_5 (any (5,5,6)-edge)
4–4	3–8	S_5
4–4	2–7	M_{10}, M_{10}
4–5	2–1	S_5, $M_8 : S_3$
4–5	2–4	M_{10}, $M_8 : S_3$
4–6	3–9	$L(2,11)$ (the vertex of degree 1)
4–6	2–4	$L(2,11)$, $M_8 : S_3$ (the (5,5,6)-edge)
4–7	3–8	$L(2,11)$
4–7	2–1	M_{10}, $M_8 : S_3$ (any (5,5,6)-edge)
4–8	3–7	$M_8 : S_3$
4–8	2–7	M_{10}, S_5 (the (6)-edge)
3–1	2–3	$M_8 : S_3$
3–1	2–5	M_{10}
3–2	2–6	M_{10}
3–3	2–5	S_5
3–3	2–7	$M_9 : 2$
3–4	2–2	$M_8 : S_3$
3–4	2–5	$L(2,11)$
3–5	2–8	M_{10}
3–6	2–1	$M_9 : 2$
3–6	2–2	M_{10}
3–6	2–3	$L(2,11)$

TABLE 5. For each row of the table, the truncation obtained from Γ_1 by deleting the elements corresponding to the subgroup(s) mentioned is isomorphic to Γ_2.

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