The Bredon–Löffler Conjecture

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CONTENTS

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The large commutative diagrams in this article were typeset using Paul Taylor's diagram package.

We give a brief exposition of results of Bredon and others on passage to fixed points from stable C_2 equivariant homotopy (where C_2 is the group of order two) and its relation to Mahowald's root invariant. In particular we give Bredon's easy equivariant proof that the root invariant doubles the stem; the conjecture of the title is equivalent to the Mahowald–Ravenel conjecture that the root invariant never more than triples the stem. Our main result is to verify by computation that the algebraic analogue of this holds in an extensive range: this improves on results of [Mahowald and Shick 1983].

1. INTRODUCTION

Let G be a cyclic group of order two and let ξ be the nontrivial representation of G on $\mathbb{R}$. As usual we let $S^{k\xi+n}$ denote the one point compactification of the representation $k\xi \oplus n$, so that in particular $(S^{k\xi+n})^G = S^n$. We may consider the group $[S^{k\xi}, S^0]^G_n$ of stable equivariant maps $S^{k\xi+n} \to S^0$ [Adams 1984]. The Bredon–Löffler conjecture concerns the fixed-point homomorphism

$$\varphi_k : [S^{k\xi}, S^0]^G_n \to [S^0, S^0]_n,$$

which takes a stable map $f : S^{k\xi+n} \to S^0$ to the fixed-point map $f^G : S^n \to S^0$. Let

$$F_k = F_k\pi_n(S^0) = \operatorname{im}\varphi_k.$$

Because of the natural inclusion $S^{k\xi} \to S^{(k+1)\xi}$, it is clear that $F_k \supseteq F_{k+1}$. By using the fact that the Burnside ring splits when localized away from 2, it is easy to see that $\varphi_k[\frac{1}{2}]$ is projection onto a direct summand for every k. Accordingly we henceforth complete at 2 without modifying the notation.

This was first considered extensively in [Bredon 1967a; 1967b]. A case of particular interest is when $n = 0$, when the codomain is $\mathbb{Z}$; Bredon made calculations that led him to conjecture the exact

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image. The conjecture was proved in [Landweber 1969] using equivariant K-theory.

Theorem 1.1 (Landweber). *Let $k > 0$, and define $a(k)$ by the relation*

$$\log_2 a(k) = \bigl|\{i : 0 < i < k \text{ and } k \equiv 0, 1, 2, 4 \bmod 8\}\bigr|.$$

Then $F_k\pi_0(S^0)$ is the set of multiples of $b(k)$, where $b(k) = 2a(k)$ if $k \not\equiv 0 \bmod 4$ and $b(k) = 4a(k)$ if $k \equiv 0 \bmod 4$.

The analogous theorem for G of odd prime order is proved in [Iriye 1989].

In several ways $n = 0$ is exceptional, and we now consider the case $n > 0$, where the codomain is a finite 2-group. Bredon made the elementary observation that φ_k is obviously surjective if $k \leq n$ since, for any map $g : S^n \to S^0$, the map $f = g \wedge g$ is equivariant for the exchange of factors and hence defines a map $f : S^{n\xi+n} \to S^0$ with $\varphi_n(f) = g$. Thus:

Lemma 1.2 (Bredon). $\pi_n(S^0) = F_0 = F_1 = \cdots = F_n$. $\square$

Bredon did further calculations suggesting that the general effect of increasing k is to impose the requirement that F_k lies in $2^{m(k)}\pi_n(S^0)$, where $m(k)$ tends to infinity with k. His calculations led him to make the conjecture that $F_k = 0$ for $k > 2n$.

Conjecture 1.3 (Bredon–Löffler). *If $k > 2n > 0$, the image of*

$$\varphi_k : [S^{k\xi}, S^0]_n^G \to [S^0, S^0]_n$$

is zero.

Remark. The conjecture is misstated in Problem 5.16 of the problem list from the 1983 Boulder conference [Schultz 1985]. The misstatement refers to the forgetful map $U_k : [S^{k\xi}, S^0]_n^G \to [S^0, S^0]_{k+n}$, which is generally much harder to understand. The forgetful map is discussed in [Greenlees 1992], using calculations of Mahowald.

2. CONNECTION WITH THE ROOT INVARIANT

Mahowald was also led to define a filtration of the 2-completed stable stem $\pi_n(S^0)$. This uses Thom spectra on infinite projective space $\mathbb{RP}^\infty$; if ξ denotes the tautological line bundle, we let $P_k := (\mathbb{RP}^\infty)^{k\xi}$. The use of the same letter ξ is reasonable since $(EG_+ \wedge S^V)/G \simeq BG^V$ for any finite group G and virtual representation V [Lewis et al. 1986, X.6.3]. By a theorem of Lin [1980] we have $S^0 = \underleftarrow{\mathrm{holim}}_k\, \Sigma P_{-k}$, so that for each k we have a map $S^0 \to \Sigma P_{-k}$. Now define the Mahowald filtration by

$$M_k = M_k\pi_n(S^0) = \ker\bigl(\pi_n S^0 \to \pi_n(\Sigma P_{-k})\bigr).$$

Observe that an $x \in M_k \setminus M_{k+1}$ determines a coset

$$R(x) \subseteq [S^n,\, P_{-k}/P_{-k-1}] = \pi_{n+k}(S^0),$$

called the *root invariant* of x. We denote by $|R(x)|$ the degree $n + k$ of this coset. Similarly, if $x \in F_k \setminus F_{k+1}$, the cofibre sequence

$$S^k \wedge G_+ \to S^{k\xi} \to S^{(k+1)\xi}$$

and the adjunction $[S^k \wedge G_+,\, S^0]_n^G \cong [S^k, S^0]_n$ allow us to define a coset

$$B(x) \subseteq [S^k, S^0]_n = \pi_{n+k}(S^0),$$

which we call the *Bredon root invariant* of x.

We begin by showing that Mahowald's filtration is the same as Bredon's and that the two versions of the root invariant agree. The proof that the filtrations agree is extremely short, and for many purposes (such as Corollary 2.3) this is all that is required.

Proposition 2.1. *For any $k \geq 0$ we have $M_k = F_k$. Further, $B(x) = R(x)$ for any $x \in \pi_n(S^0)$.*

Proof. By obstruction theory, since $S^{\infty\xi}$ is nonequivariantly contractible, we see that

$$[X,\, Y \wedge S^{\infty\xi}]_*^G = [X^G, Y^G]_*,$$

and in particular this applies when $X = S^{k\xi}$ and $Y = S^0$. Thus φ_k is induced by the inclusion

$S^0 \to S^{\infty\xi}$, and because of the cofibre sequence $EG_+ \to S^0 \to S^{\infty\xi}$ we have a long exact sequence

$$\cdots \to [S^{k\xi}, EG_+]^G_n \to [S^{k\xi}, S^0]^G_n \xrightarrow{\varphi_k} \\ \xrightarrow{\varphi_k} [S^{k\xi}, S^{\infty\xi}]^G_n \to [S^{k\xi}, \Sigma EG_+]^G_n \to \cdots.$$

Hence

$$\begin{aligned} F_k &= \operatorname{im}\varphi_k \\ &= \ker\big([S^{k\xi}, S^{\infty\xi}]^G_n \to [S^{k\xi}, \Sigma EG_+]^G_n\big) = M_k, \end{aligned}$$

where the last equality uses the fact that

$$\begin{aligned} [S^{k\xi}, \Sigma EG_+]^G_n &= [S^0, \Sigma EG_+ \wedge S^{-k\xi}]^G_n \\ &\cong [S^0, (\Sigma EG_+ \wedge S^{-k\xi})/G]_n \\ &= [S^0, \Sigma P_{-k}]_n \end{aligned}$$

by Adams' isomorphism [Adams 1984, 5.4].

For the second part of the proposition, we need a lemma.

Lemma 2.2. *Let $A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{\partial} \Sigma A$ be a cofibre sequence, and $\Omega Y \xrightarrow{\delta} F \xrightarrow{\tau} X \xrightarrow{\pi} Y$ a fibre sequence. Consider the diagram*

$$\begin{tikzcd}
{[A,\Omega Y]} \arrow[r, "\delta_*"] & {[A,F]} \arrow[r, "\tau_*"] & {[A,X]} \arrow[r, "\pi_*"] & {[A,Y]} \\
{[B,\Omega Y]} \arrow[r, "\delta_*"] \arrow[u, "i^*"] & {[B,F]} \arrow[r, "\tau_*"] \arrow[u, "i^*"] & {[B,X]} \arrow[r, "\pi_*"] \arrow[u, "i^*"] & {[B,Y]} \arrow[u, "i^*"] \\
{[C,\Omega Y]} \arrow[r, "\delta_*"] \arrow[u, "j^*"] & {[C,F]} \arrow[r, "\tau_*"] \arrow[u, "j^*"] & {[C,X]} \arrow[r, "\pi_*"] \arrow[u, "j^*"] & {[C,Y]} \arrow[u, "j^*"] \\
{[\Sigma A,\Omega Y]} \arrow[r, "\delta_*"] \arrow[u, "\partial^*"] & {[\Sigma A,F]} \arrow[r, "\tau_*"] \arrow[u, "\partial^*"] & {[\Sigma A,X]} \arrow[r, "\pi_*"] \arrow[u, "\partial^*"] & {[\Sigma A,Y]}. \arrow[u, "\partial^*"]
\end{tikzcd}$$

If $f \in [C,X]$ satisfies $\pi f j = 0$, then

$$-(\partial^*)^{-1}(\pi_* f) \subseteq (\delta_*)^{-1} i^* (\tau_*)^{-1}(j^* f).$$

Remark. We would have equality except that the indeterminacy on the left is $(\Sigma i)^*[\Sigma B, Y]$, while that on the right is $(\Sigma i)^*[\Sigma B, Y] + \pi_*[\Sigma A, X]$.

Proof. This is a simple exercise in the manipulation of cone coordinates and the standard equivalences Cofibre $j \simeq \Sigma A$ and Fibre$(\tau) \simeq \Omega Y$ used to establish the Barratt–Puppe sequences. Any element of

$$(\partial^*)^{-1}(\pi_* f) \subseteq [\text{Cofibre}\, j,\, Y]$$

provides a null homotopy $H : B \wedge I \to Y$ of $\pi f j$. The adjoint $\tilde{H} : B \to F(I, Y)$ together with fj defines an element of $(\tau_*)^{-1}(j^* f)$. The given map f provides a null homotopy $A \wedge I \to X$ whose adjoint $A \to F(I, X)$ can be combined with the map in $(\tau_*)^{-1}(j^* f)$ to produce an element of

$$(\delta_*)^{-1} i^* (\tau_*)^{-1}(j^* f) \subseteq [A, \text{Fibre}\,\tau].$$

The equivalences Cofibre $j \simeq \Sigma A$ and Fibre $\tau \simeq \Omega Y$ then convert these maps into maps $\Sigma A \to Y$ and $A \to \Omega Y$ whose adjoints are negatives of one another. $\square$

Returning to the proof of Proposition 2.1, we take as our cofibre sequence

$$S^k \wedge G_+ \to S^{k\xi} \to S^{(k+1)\xi} \to S^{k+1} \wedge G_+$$

and as our fibre sequence the negative of the cofibre sequence

$$EG_+ \to S^0 \to S^{\infty\xi} \to \Sigma EG_+$$

(recall that the negative of a fibre sequence is a cofibre sequence and vice versa). Omitting the three minus signs in each row, we have the diagram in Figure 1.

If $\alpha \in [S^0, S^0]_n = [S^{(k+1)\xi}, S^{\infty\xi}]^G_n$ is in $F_k \setminus F_{k+1}$, then $R(\alpha)$ is the lifting to the lower right corner while $B(\alpha)$ is the lifting to the upper left. Putting the minus signs back in so that we may apply the lemma, we see that $R(\alpha)$ changes sign while $B(\alpha)$ does not. Hence $R(\alpha) \subseteq B(\alpha)$. Finally, the indeterminacies are the same in this case, since $[S^k, S^{\infty\xi}] = 0$. $\square$

This means that the results for the fixed-point filtration apply equally well to Mahowald's filtration. It is a historical curiosity that the results about Mahowald's filtration were reproved later by more complicated means. Thus Bredon's easy observation (Lemma 1.2) that φ_k is surjective for $k \leq n$ becomes a theorem in [Jones 1985]:

$$\begin{array}{ccccccc}
[S^k,\,EG_+]_n & \xrightarrow{\cong} & [S^k,\,S^0]_n & \longrightarrow & [S^k,\,S^{\infty\xi}]_n = 0 & \longrightarrow & [S^k,\,\Sigma EG_+]_n \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
[S^{k\xi},\,EG_+]_n^G & \longrightarrow & [S^{k\xi},S^0]_n^G & \xrightarrow{\varphi_k} & [S^{k\xi},\,S^{\infty\xi}]_n^G & \longrightarrow & [S^{k\xi},\,\Sigma EG_+]_n^G \\
\uparrow & & \uparrow & & \uparrow\cong & & \uparrow \\
[S^{(k+1)\xi},\,EG_+]_n^G & \longrightarrow & [S^{(k+1)\xi},\,S^0]_n^G & \xrightarrow{\varphi_{k+1}} & [S^{(k+1)\xi},\,S^{\infty\xi}]_n^G & \longrightarrow & [S^{(k+1)\xi},\,\Sigma EG_+]_n^G \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
[S^{k+1},\,EG_+]_n & \longrightarrow & [S^{k+1},S^0]_n & \longrightarrow & [S^{k+1},\,S^{\infty\xi}]_n = 0 & \longrightarrow & [S^{k+1},\,\Sigma EG_+]_n
\end{array}$$

FIGURE 1. Diagram arising from the cofibre sequence $S^k \wedge G_+ \to S^{k\xi} \to S^{(k+1)\xi} \to S^{k+1} \wedge G_+$ and the fibre sequence equal to the negative of the cofibre sequence $EG_+ \to S^0 \to S^{\infty\xi} \to \Sigma EG_+$. See preceding page.

Corollary 2.3 (Root invariant doubles the stem). *If $x \in \pi_n(S^0)$ then $|R(x)| \geq 2n$.* □

This applies also to odd primes for H. R. Miller's analogue, which says that if $|x| = 2k-\varepsilon$, with $\varepsilon = 0$ or 1, then $|R(s)| \geq 2(pk - \varepsilon)$ (that is, essentially, that the root invariant must multiply the stem by at least p), although the equivariant proof may lose one stem in the estimate. Bredon's idea was used in [Greenlees and May 1995] to give a result for arbitrary finite groups.

The Bredon–Löffler conjecture, combined with Proposition 2.1, becomes the Mahowald–Ravenel conjecture bounding the degree of the root invariant. Indeed if the Bredon–Löffler conjecture holds then $F_{2n+1} = 0$ so that the last possible nontrivial subquotient is F_{2n}/F_{2n+1}.

Conjecture 2.4 (Mahowald–Ravenel). *If $x \in \pi_n(S^0)$, then $|R(x)| \leq 3n$.*

Mahowald and Ravenel also observe [1993, p. 871] that $|R(x)| \leq (p+1)|x|$ at odd primes in all known cases, although they do not make the formal conjecture that this is generally true.

Although we will not use it here, Proposition 2.1 provides us with an interesting description of the root invariant that is probably well known to the experts, but does not seem to be in the literature.

Proposition 2.5. *For $x \in \pi_n(S^0)$ we may calculate the root invariant of x by finding the largest k for which $x \in \operatorname{im}\varphi_k$ and then setting $R(x) = U_k\varphi_k^{-1}(x)$.*

Proof. The Bredon root invariant is the composite of φ_k^{-1}, for k maximal, with the homomorphism induced by the attaching map $S^k \wedge G_+ \to S^{k\xi}$ and the adjunction isomorphism:

$$[S^{k\xi},\,S^0]_n^G \to [S^k \wedge G_+,\,S^0]_n^G = [S^k,\,S^0]_n.$$

Since the attaching map is the adjoint of the (non-equivariant) identity map $S^k \to S^{k\xi}$, the result follows. □

3. THE ALGEBRAIC CONJECTURE

The groups involved in the conjecture can be calculated by Adams spectral sequences in various ways. The algebraic version of the conjecture arises as the E_2 analogue of the geometric conjecture. In a particular case the algebraic conjecture may or may not prove the geometric one; this is discussed in detail in [Mahowald and Ravenel 1993, 2.9]. See [Mahowald and Shick 1983; Shick 1987] for further discussion of the algebraic root invariant.

Now consider the diagram in Figure 2. The conjecture is that for $k > 2n$ the map φ_k is zero, or equivalently, that Mahowald's map is a monomorphism. The diagram shows that this is equivalent to α being a monomorphism.

$$
\begin{tikzcd}
\cdots \arrow[r] & {[S^{k\xi},\, S^0]^G_n} \arrow[r, "\varphi_k"] & {[S^{k\xi},\, S^{\infty\xi}]^G_n} \arrow[r, "\text{Mahowald}"] & {[S^{k\xi},\, \Sigma EG_+]^G_n} \arrow[r] & \cdots \\
& & {[S^{\infty\xi},\, S^{\infty\xi}]^G_n} \arrow[u, "\cong"'] \arrow[r, "\cong"] & {[S^{\infty\xi},\, \Sigma EG_+]^G_n,} \arrow[u, "\alpha"'] &
\end{tikzcd}
$$

FIGURE 2. The algebraic version of the Bredon–Löffler conjecture says that, for $k > 2n$, Mahowald's map is a monomorphism. Here the horizontal isomorphism is Lin's Theorem [Lin 1980], and the vertical isomorphism is elementary obstruction theory.

The advantage of considering α is that it relates groups of maps into free G-spaces, for which there is a uniform method of calculation [Greenlees 1988]. To explain this we summarise the properties of the relevant cohomology theory $f_G^*(X)$. We have used the letter f (for 'free') to ensure consistency with [Greenlees and May 1995], although the letter c was used in [Greenlees 1988]. The theory is so named because of its representing spectrum, and the definition of its homology theory:

$$f_*^G(X) = H_*(EG_+ \wedge_G X;\, \mathbb{F}_2).$$

In general it is not so easy to describe the cohomology theory, but for our purposes it suffices to know two facts:

(i) if X is G-free, $f_G^*(X) = H^*(EG_+ \wedge_G X;\, \mathbb{F}_2)$, and
(ii) f admits Thom isomorphisms

$$f_G^*(S^\xi \wedge X) \cong f_G^*(S^1 \wedge X)$$

for any X.

These facts allow one to calculate the ring of operations

$$f_G^* f = H^*(BG_+) \otimes A,$$

where A is the mod 2 Steenrod algebra, and the modules $f_G^*(S^{k\xi})$ over it, as in [Greenlees 1988]. To state the result we consider the module $L = H^*(BG_+)[x^{-1}] = \mathbb{F}_2[x,\, x^{-1}]$ as a module over A. The submodule nonzero in degrees $\geq a$ is denoted L_a. The quotient nonzero in degrees $\leq b$ is denoted L^b, and the subquotient that is nonzero only in degrees from $[a, b]$ is denoted L_a^b.

Lemma 3.1. (a) $f_G^*(S^{\infty\xi}) = \Sigma L$.
(b) $f_G^*(S^{k\xi}) = \Sigma L^{k-1}$.
(c) *For* $0 \leq k \leq l$, *the inclusion maps*

$$S^{k\xi} \to S^{l\xi} \to S^{\infty\xi}$$

induce the projections

$$\Sigma L^{k-1} \leftarrow \Sigma L^{l-1} \leftarrow \Sigma L. \qquad \square$$

The main theorem of [Greenlees 1988] states in this case that for any G-free, 2-complete, bounded below spectrum Y of finite type there is an Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{H^*(BG_+)\otimes A}(f_G^*(Y),\, f_G^*(X)) \Longrightarrow [X, Y]^G_*.$$

In particular this applies to calculate maps into EG_+ itself. In this case, since

$$f_G^*(EG_+) = H^*(BG_+),$$

by change of rings the spectral sequence becomes

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_A(\mathbb{F}_2,\, f_G^*(X)) \Longrightarrow [X,\, EG_+]^G_*.$$

Therefore, the E_2 analogue of Conjecture 1.3 is as follows (where the suspension from ΣEG_+ has been cancelled with that from Lemma 3.1).

Conjecture 3.2. *The natural quotient map* $L \to L^{k-1}$ *induces a monomorphism of* $\mathrm{Ext}^{s,t}_A(\mathbb{F}_2,\, \cdot\,)$ *for* $k > 2(t-s) > 0$.

For purposes of calculation, it is more convenient to dualize so that we are dealing with modules that are bounded below. Suppose M, N and R are A-modules. With the diagonal action on $R \otimes M$ and the conjugation action, $(af)(m) = \Sigma\, a' f(\chi(a'')m)$, on $\mathrm{Hom}(M, N)$, we have a natural isomorphism

$$\mathrm{Ext}_A(R,\, \mathrm{Hom}(M, N)) \cong \mathrm{Ext}_A(R \otimes M,\, N),$$

$$
\begin{tikzcd}
\mathrm{Ext}_A(\mathbb{F}_2,\, L) \arrow[r] \arrow[d, equal] & \mathrm{Ext}_A(\mathbb{F}_2,\, L^{k-1}) \arrow[d, equal] \\
\mathrm{Ext}_A(\mathbb{F}_2,\, (\Sigma L)^*) \arrow[r] \arrow[d, equal] & \mathrm{Ext}_A(\mathbb{F}_2,\, (\Sigma L_{-k})^*) \arrow[d, equal] \\
\mathrm{Ext}_A(\Sigma L,\, \mathbb{F}_2) \arrow[r] \arrow[d, equal] & \mathrm{Ext}_A(\Sigma L_{-k},\, \mathbb{F}_2) \arrow[d, equal] \\
\mathrm{Ext}_A(\mathbb{F}_2,\, \mathbb{F}_2) \arrow[r] & \mathrm{Ext}_A(\Sigma L_{-k},\, \mathbb{F}_2)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
L \arrow[r] & L^{k-1} \\
\Sigma L \arrow[d, equal] & \Sigma L_{-k} \arrow[l] \arrow[d, equal] \\
\Sigma L \arrow[d] & \Sigma L_{-k} \arrow[l] \arrow[d, equal] \\
\mathbb{F}_2 & \Sigma L_{-k} \arrow[l]
\end{tikzcd}
$$

FIGURE 3. Algebraic dualities and homomorphisms, and the induced isomorphisms and homomorphisms of Ext groups.

which specializes to $\mathrm{Ext}_A(\mathbb{F}_2,\, M^*) \cong \mathrm{Ext}_A(M,\, \mathbb{F}_2)$. Using the identification $L^* \cong \Sigma L$, and, more generally, $(L_a^b)^* \cong \Sigma L_{-b-1}^{-a-1}$, together with the isomorphism

$$\mathrm{Ext}_A^{s,t}(\Sigma L,\, \mathbb{F}_2) \cong \mathrm{Ext}_A^{s,t}(\mathbb{F}_2,\, \mathbb{F}_2),$$

which is the main result of [Lin et al. 1980], we reach the following convenient algebraic statement of 3.2. (It may help to look at Figure 3, which displays the dualities.)

Conjecture 3.3. *The natural quotient map* $\Sigma L_{-k} \to \mathbb{F}_2$ *induces a monomorphism of* $\mathrm{Ext}_A^{s,t}(\,\cdot\,, \mathbb{F}_2)$ *for* $k > 2(t-s) > 0$.

This is the version of the conjecture that we have used for calculations.

4. THE CALCULATIONS

If we let $P(k,n)$ be the statement that

$$\psi_k : \mathrm{Ext}_A^{s,t}(\mathbb{F}_2,\, \mathbb{F}_2) \to \mathrm{Ext}_A^{s,t}(\Sigma L_{-k},\, \mathbb{F}_2)$$

is a monomorphism whenever $t-s=n$, Conjecture 3.3 is the claim that $P(k,n)$ holds if $0 < 2n < k$. The factorization $\Sigma L_{-(k+1)} \to \Sigma L_{-k} \to \mathbb{F}_2$ shows that $P(k,n) \Rightarrow P(k+1,\, n)$. Thus, to verify the conjecture experimentally for small values of k and n, we may confine attention to odd k.

Using the programs described in [Bruner 1992], together with more recent extensions that compute induced chain maps, we have computed ψ_k for odd $k \le 55$, for $s < 20$, and a range of values of t. The range of t was chosen so as to determine, for as many elements x of $\mathrm{Ext}_A(\mathbb{F}_2,\, \mathbb{F}_2)$ as possible in a reasonable amount of computer time, the minimum odd k for which $\psi_k(x)$ is nonzero in $\mathrm{Ext}_A(\Sigma L_{-k},\, \mathbb{F}_2)$. We refer to the number k as the *odd cell filtration* of x. The Bredon–Mahowald filtration of x will then be either $k-1$ or $k-2$.

Table 1 displays the results obtained.

5. CONCLUSIONS

The calculations show that the conjecture holds for all $t-s < 30$, and that $P(65, 30)$ holds, just short of the conjectured $P(61, 30)$. Previously, the conjecture was known to hold for $t-s \le 16$ from the calculations of Ext root invariants in [Mahowald and Shick 1983], so our calculations nearly double the stem through which the conjecture is known.

Note that the Adams–Barratt elements $P^i h_1^j$, in bidegrees $(s,\, t-s) = (j+4i,\, j+8i)$, have maximal filtration for $j = 2$ or 3, showing that the conjecture is sharp for these stems. The elements $P^i h_1$ have less than maximal filtration, by an amount that suggests some influence of divisibilty by 2, while the remaining Adams–Barratt elements (that is, elements of least positive stem in each Adams filtration) appear to have odd cell filtration 2 less than maximal. Since this occurs near the vanishing line, it may be amenable to proof by the techniques that establish the vanishing line and the range of

$t-s$	$s=0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	3	3	5	9	11	11	13	17	19	19	21	25	27	27	29	33	35	35	37	41
1		3																			
2			5																		
3		5	7	7																	
4																					
5																					
6			9																		
7		9	11	11	13																
8			11	13																	
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14			17	19	19	21	25														
15		17	19	19	21	21, 25	27	27	29												
16			19				23	25													
17				21	23	23	25	31	31	33											
18			21	23	23, 23	25					37										
19				23						35	37	39									
20					25	27	29														
21				25		27															
22					27				31	37	41										
23					29	29	31	31	37	39, 41	43	43	45								
24						31					43	45									
25									39	39	41	45	47	47							
26							41	37	39	41					53						
27														51	53	55					
28									41	43	45										
29								37	41	43											
30			33	35	35	37	37	39	41	45	49	51	51	53	65						
31		33	35	35, 35	37	37, 41	43	43	41, 45	45, 49	47, 51	51	53	53, 65	?	?	?				
32			35		37		39	41	43	45					?	?					
33				37	39	39		45					?	?	?	?	?	?			
34			37	39	39		41		43	45	65	49	?	?					?		
35						41		43	45	47								?	?	?	
36							43						?	?, ?	?						
37				41		43	45	45, 47	47, 49	?	?	?	?	?							
38			41	43	43, 43	45	45, 45	47	49	?							?	?	?		
39				43	45	45	47, 47	49		?			?	?	?	?	?	?	?, ?	?	?

TABLE 1. At column s, row $t-s$, we show the odd cell filtration for a basis of $\mathrm{Ext}_A^{s,t}(\mathbb{F}_2,\mathbb{F}_2)$, when the entry is 55 or less. (For typographical reasons, we have exchanged rows and columns from the usual Adams spectral sequence.) Thus, an entry k means that the algebraic root invariant increases the stem by $k-1$ or $k-2$, and the conjecture for the n-th stem is that all entries in the row $t-s=n$ are at most $2n+1$. If the entry is 65, this means that the element maps nontrivially to L_{-65}. (We computed the map for $k=60$ and 65 as part of our initial exploration of the problem.) If the entry is a question mark, we have no information about this bidegree. The values 55 and less are minimal; that is, if the entry at column s, row $t-s$ is $k \le 55$, we have computed the minimal resolution of $L_{-(k-2)}$ through at least internal degree t to verify that the entry should not be $k-2$.

Adams periodicity. Equally, one might hope to prove the geometric conjecture on the image of J by using an e-invariant based on equivariant K-theory.

Inspecting Table 1, we see an interesting pattern emerge. The line below which the homomorphism $\mathrm{Ext}_A^{s,t}(\mathbb{F}_2,\mathbb{F}_2) \to \mathrm{Ext}_A^{s,t}(\Sigma L_{-k},\mathbb{F}_2)$ is monic appears to have slope $-\frac{1}{2}$ and $(t-s)$-intercept k (in the

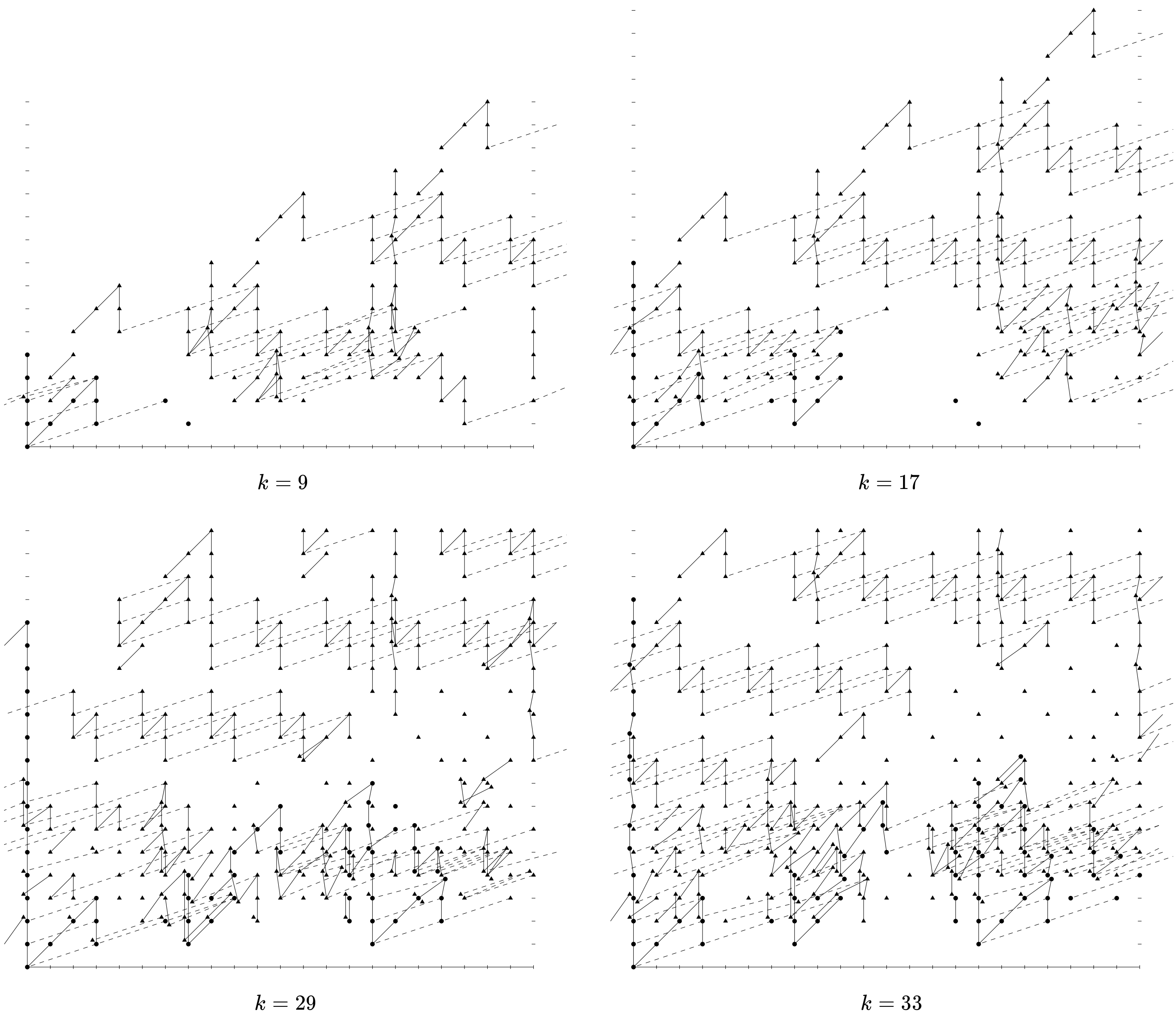

FIGURE 4. $\mathrm{Ext}_A^{s,t}(\Sigma L_{-k}, \mathbb{F}_2)$, for representative values of k. Elements in the image of $\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ are indicated by $\bullet$, and all others by $\blacktriangle$. The orientation is conventional, with $t-s$ horizontal; the leftmost column of each chart represents $t-s=0$. Multiplication by h_0, h_1, and h_2 is indicated by solid vertical, solid diagonal, and dotted lines, respectively. The number of dots at each position $(s,\, t-s)$ is the $\mathbb{F}_2$-rank of $\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$.

conventional display with s vertical and $t-s$ horizontal), which forms a kind of k-dual to the Adams vanishing line. If this is true, then Conjecture 3.3 would follow from the intersection of this line with the Adams vanishing line in $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$.

The results of some representative Ext calculations are displayed in Figure 4.

There now seems to be considerable evidence that the Bredon–Löffler conjecture holds at the level of Ext groups.

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