The Primitive Distance-Transitive Representations of the Fischer Groups

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CONTENTS

We classify the primitive distance-transitive representations of the Fischer sporadic simple groups and their automorphism groups. It turns out that the only primitive distance-transitive representations of these groups are their rank 3 representations. In the process of our work, we also classify and study the primitive multiplicity-free permutation representations of these Fischer groups. Our methods, which we describe in some detail, demonstrate the use of computational and randomized techniques in the classification of distance-transitive graphs and the study of very large permutation representations.

1. INTRODUCTION

Let G be a permutation group on a finite set V, and Γ an undirected, loopless, connected graph with vertex-set V. Now G has a natural action on $V \times V$, defined by $(v, w)^g = (v^g, w^g)$, and we say that G *acts distance-transitively* on Γ if the G-orbits of this action are precisely the sets $\{(v, w) \mid d_\Gamma(v, w) = i\}$, where $i = 0, 1, \ldots, \operatorname{diam}\Gamma$. (Note that if G acts distance-transitively on Γ, it is necessarily a vertex-transitive and ordered-edge-transitive group of automorphisms of Γ.) The graph Γ is called *distance-transitive* if $\operatorname{Aut}\Gamma$ acts distance-transitively on Γ. The permutation representation of G on V is a *distance-transitive representation* (DTR) if G acts distance-transitively on some (undirected, loopless, connected) graph with vertex-set V. A good general reference for the theory of distance-transitive graphs is [Brouwer et al. 1989].

For our purposes, a *Fischer group* is one of the sporadic groups Fi_{22}, $\mathrm{Fi}_{22}{:}2 = \operatorname{Aut}\mathrm{Fi}_{22}$, $\mathrm{Fi}_{23} = \operatorname{Aut}\mathrm{Fi}_{23}$, Fi_{24}', and $\mathrm{Fi}_{24} = \mathrm{Fi}_{24}'{:}2 = \operatorname{Aut}\mathrm{Fi}_{24}'$. The main purpose of this paper is to classify the graphs

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on which a Fischer group acts primitively and distance-transitively. In the process we also classify the primitive multiplicity-free permutation representations of these groups, and determine the corresponding permutation characters. These results, and the techniques described in this paper, are used in the complete classification [Ivanov et al. 1995] of the primitive multiplicity-free permutation representations of the sporadic simple groups and their automorphism groups, and the graphs on which such a group acts primitively and distance-transitively.

Our classification uses several tools of computational group theory and graph theory, such as character theory algorithms, single and double coset enumeration, permutation group algorithms, and graph theory algorithms. We also illustrate some randomized techniques that we use to study extremely large permutation representations.

We make extensive use of the group theory system GAP [Schönert et al. 1994] and its share library package GRAPE [Soicher 1993] (for computing with graphs with groups acting on them), which includes the *nauty* package [McKay 1990] (for computing automorphism groups of graphs and testing for graph isomorphism). We usually give more information about a graph than is strictly necessary to determine if a given group acts on it distance-transitively.

The groups Fi_{22}, Fi_{23}, and Fi_{24} were constructed by B. Fischer [1969] as 3-transposition groups. A group $G = (G, D)$ is a 3-*transposition group* if it is generated by a conjugacy class D of 3-transpositions (this means the elements of D are involutions whose pairwise products have order 1, 2, or 3).

Our main result is stated at the end of the next section. We use Atlas notation [Conway et al. 1985] throughout this paper for group structures, conjugacy classes, and characters. For example, $429b$ denotes the second character of degree 429, and $429ab$ denotes the sum $429a + 429b$. The ordering of characters we use is that of the GAP version of the Atlas character tables, which agrees with the Atlas ordering in the case of simple groups.

2. ORBITAL GRAPHS, DISTANCE-TRANSITIVE REPRESENTATIONS, AND THE MAIN THEOREM

Throughout this section G is a transitive permutation group on a finite set V.

The orbits of G (acting naturally) on $V \times V$ are called *orbitals*, and the number of these orbitals is called the (permutation) *rank* of G. A directed graph with vertex-set V and edge-set an orbital E is called an *orbital digraph*. If E is an orbital such that $(v, w) \in E$ whenever $(w, v) \in E$, then we call E *self-paired*, and consider the orbital digraph (V, E) to be an undirected (orbital) graph by identifying $(v, w) \in E$ with (w, v). The orbitals for G are in one-to-one correspondence with the orbits on V of the stabilizer G_v of a point $v \in V$: this correspondence maps an orbital E to the set of points $\{w \mid (v, w) \in E\}$. The orbits of G_v on V are called *suborbits* of G, and their lengths are called the *subdegrees* of G.

Now if G on V is a distance-transitive representation, then a corresponding distance-transitive graph must have vertex-set V, and edge-set a self-paired orbital of G. Indeed, if G on V is a DTR, then all its orbitals must be self-paired, which is equivalent to the property that the permutation representation of G on V is the sum of distinct complex irreducible representations, each of which is writable over the reals [Brouwer et al. 1989]. Furthermore, if G acts distance-transitively on the graph (V, E), then the suborbit corresponding to the orbital E is a suborbit of the smallest or the second smallest length greater than 1 [Brouwer et al. 1989].

Now suppose that $V_1 = \{v\}$, V_2, ..., V_r is an ordering of the orbits of G_v, with respective representatives $v_1 = v$, v_2, ..., v_r. Let $\Gamma = (V, E)$ be a (di)graph on which G acts as a vertex-transitive group of automorphisms, and define

$$a_{ij} = |\{(v_i, w) \in E \mid w \in V_j\}|.$$

Note that a_{ij} does not depend on the choice v_i of suborbit representative. The $r \times r$ integer matrix $A = (a_{ij})$ is called the *collapsed adjacency* *matrix* for Γ (with respect to G and the suborbit ordering). Much information about Γ can be read off directly from its collapsed adjacency matrix [Praeger and Soicher]. In particular, G acts distance-transitively on Γ if and only if for some ordering $V_1 = \{v\}, V_2, \dots, V_r$ of the suborbits, the corresponding collapsed adjacency matrix is tridiagonal, with all entries nonzero on the upper and lower diagonals.

We are now in a position to state our main theorem. The representations and graphs described by this theorem are well-known (see, for example, [Brouwer et al. 1989]).

Theorem 2.1. *Suppose that* $G = \mathrm{Fi}_{22}$, $\mathrm{Fi}_{22}{:}2$, Fi_{23}, Fi_{24}', *or* Fi_{24}. *Then the primitive distance-transitive representations of G are precisely its (well-known) rank 3 representations, described below. The corresponding distance-transitive graphs come in complementary pairs, and the list below gives their collapsed adjacency matrices.*

1. If $G = \mathrm{Fi}_{22}$ or $\mathrm{Fi}_{22}{:}2$, then G acts primitively with permutation rank 3 on the conjugacy class of 3-transpositions of $G' = \mathrm{Fi}_{22}$. The subdegrees are 1, 693, 2816, and the collapsed adjacency matrices are:

$$\begin{matrix} 0 & 693 & 0 \\ 1 & 180 & 512 \\ 0 & 126 & 567 \end{matrix} \qquad \begin{matrix} 0 & 0 & 2816 \\ 0 & 512 & 2304 \\ 1 & 567 & 2248 \end{matrix}$$

2. Let $G = \mathrm{Fi}_{22}$. Then G contains exactly two conjugacy classes of maximal subgroups $O_7(3)$, and these classes are interchanged by an outer automorphism of G. The group G acts on each of these classes with permutation rank 3, and these two representations give rise to the same complementary pair of graphs. The subdegrees are 1, 3159, 10920, and the collapsed adjacency matrices are:

$$\begin{matrix} 0 & 3159 & 0 \\ 1 & 918 & 2240 \\ 0 & 648 & 2511 \end{matrix} \qquad \begin{matrix} 0 & 0 & 10920 \\ 0 & 2240 & 8680 \\ 1 & 2511 & 8408 \end{matrix}$$

3. If $G = \mathrm{Fi}_{23}$, then G acts primitively with permutation rank 3 on the conjugacy class of 3-transpositions of G. The subdegrees are 1, 3510, 28160, and the collapsed adjacency matrices are:

$$\begin{matrix} 0 & 3510 & 0 \\ 1 & 693 & 2816 \\ 0 & 351 & 3159 \end{matrix} \qquad \begin{matrix} 0 & 0 & 28160 \\ 0 & 2816 & 25344 \\ 1 & 3159 & 25000 \end{matrix}$$

4. Let $G = \mathrm{Fi}_{23}$. Then G contains exactly one conjugacy class of maximal subgroups $O_8^+(3){:}S_3$, on which G acts with permutation rank 3. The subdegrees are 1, 28431, 109200, and the collapsed adjacency matrices are:

$$\begin{matrix} 0 & 28431 & 0 \\ 1 & 6030 & 22400 \\ 0 & 5832 & 22599 \end{matrix} \qquad \begin{matrix} 0 & 0 & 109200 \\ 0 & 22400 & 86800 \\ 1 & 22599 & 86600 \end{matrix}$$

5. If $G = \mathrm{Fi}_{24}'$ or Fi_{24}, then G acts primitively with permutation rank 3 on the conjugacy class of 3-transpositions of Fi_{24}. The subdegrees are 1, 31671, 275264, and the collapsed adjacency matrices are:

$$\begin{matrix} 0 & 31671 & 0 \\ 1 & 3510 & 28160 \\ 0 & 3240 & 28431 \end{matrix} \qquad \begin{matrix} 0 & 0 & 275264 \\ 0 & 28160 & 247104 \\ 1 & 28431 & 246832 \end{matrix}$$

We shall prove this theorem by showing that there are no other primitive DTRs for the Fischer groups.

3. THE GENERAL APPROACH

We discuss here our general approach to classifying the primitive DTRs of a given finite group G.

First, a permutation representation of G is primitive if and only if it is equivalent to a representation of G acting on the (right) cosets of a maximal subgroup. The maximal subgroups of Fi_{22} and $\mathrm{Fi}_{22}{:}2$ are determined in [Wilson 1984; Kleidman and Wilson 1987], those of Fi_{23} in [Kleidman et al. 1989], and those of Fi_{24}' and Fi_{24} in [Linton and Wilson 1991].

Next, for a permutation representation ρ to be a DTR, it is necessary that ρ be *multiplicity-free*, that is, the sum of distinct complex irreducible representations. Furthermore, if ρ is a DTR then each of these distinct irreducible representations must be writable over the reals, or equivalently, have a character with Frobenius–Schur indicator $+1$. The next section contains a general discussion on practical computational methods to determine if a given permutation representation is multiplicity-free, and in Section 5 the multiplicity-free primitive representations of the Fischer groups are classified, and their characters determined.

The problem then boils down to that of determining if a given (multiplicity-free) primitive representation of G on V is a DTR.

We explicitly construct some such representations using single or double coset enumeration, and calculate collapsed adjacency matrices for the orbital graphs corresponding to the two smallest subdegrees greater than 1 (using the method described in [Praeger and Soicher]). Then a trivial examination of these collapsed adjacency matrices determines if the representation is a DTR.

However, the primitive representation of G on V may be too large or difficult to construct directly, but if we can construct another representation of G as a vertex-transitive group of automorphisms of some graph $\Gamma = (W, F)$, such that the stabilizer H of $v \in V$ acts intransitively on W, then we can try to show that G on V is not a DTR as follows.

Let Δ be the (proper) subgraph of Γ induced on some orbit of H on W. Then the action of G on V is equivalent to the action of G on the orbit Δ^G of subgraphs of Γ (as H is maximal in G, it must be the full G-stabilizer of Δ). We may then use various computational tricks (often involving randomized techniques) to determine a set of representatives of the H-orbits on Δ^G. We sometimes distinguish the H-orbit containing Δ^{g_1} from that containing Δ^{g_2} by showing that $\Delta \cap \Delta^{g_1}$ is not isomorphic to $\Delta \cap \Delta^{g_2}$, and here the *nauty* package [McKay 1990] is useful.

Now, given H-orbit representatives $\Delta_1 = \Delta$, $\Delta_2, \ldots, \Delta_r$, we determine $H_i = \mathrm{stab}_H(\Delta_i)$ for $i = 1, \ldots, r$ (using GAP, say), and then obtain the subdegrees $d_i = |H|/|H_i|$.

Now define Σ_i to be the graph with vertex-set Δ^G, and edge-set the orbit $\{\Delta, \Delta_i\}^G$ (where $i > 1$). We can usually show that G does not act distance-transitively on a given Σ_i as follows. (In general, our aim is to find a G-invariant relation $\sim$ on Δ^G, and $X, Y \in \Delta^G$, such that $d(\Delta, X) = d(\Delta, Y)$ in Σ_i, $\Delta \sim X$, but $\Delta \not\sim Y$.) First, we calculate an element $g_i \in G$ such that $\Delta^{g_i} = \Delta_i$. We then determine various subgraphs of Γ of the form $\Delta_i^{hg_i}$, for random $h \in H$. Such subgraphs are joined to Δ_i in Σ_i, and we can usually find two such subgraphs X, Y such that $\Delta \cap X \not\cong \Delta \cap \Delta_i \not\cong \Delta \cap Y$ and $\Delta \cap Y \not\cong \Delta \cap X$. In that case, in Σ_i we have $d(\Delta, X) = d(\Delta, Y) = 2$, but there is no element of G taking (Δ, X) to (Δ, Y), and we can conclude that G does not act distance-transitively on Σ_i.

Remark. The calculations described above usually lead to an explicit rule for determining in which G-orbital a given ordered pair of elements of Δ^G lies. Such a rule enables us (at least in theory) to compute collapsed adjacency matrices for the orbital graphs for the action of G on $V = \Delta^G$. We have recently used such rules to compute collapsed adjacency matrices for the nontrivial orbital graphs of the two smallest valencies for almost all of the multiplicity-free representations we consider. Although not usually required for the proofs of our results, these matrices are of interest in their own right, say for the investigation of geometries related to the corresponding orbital graphs. Many of these matrices are published in [Ivanov et al. 1995], and we include the others in this paper. We note that the intersection matrices in [Ivanov et al. 1995] are the transposes of what we call collapsed adjacency matrices for orbital graphs, after a possible reordering of the suborbits. We have decided to retain the original proofs of our results, as these contain interesting information not available from collapsed adjacency matrices alone.

4. THE COMPUTATIONAL STUDY OF PERMUTATION CHARACTERS

Determining a Permutation Character

There are several methods one can apply in order to determine the permutation character of the permutation action of a finite group G on the cosets of a subgroup H. These methods are distinguished by the amount of information used by the methods. As a rule of thumb, the methods that require a detailed knowledge enable one to determine the permutation character exactly, but are only applicable for small groups. Other methods, which need much less information, do not always lead to a unique possibility, but can be used for very large groups. We will deal mainly with the second kind of methods, which are based on character theory. A good reference for the character theory used here is [Isaacs 1976].

For $g \in G$, the permutation character π of G, with respect to the subgroup H, has value $\pi(g)$ equal to the number of fixed points in the action of G on the (right) cosets of H. The permutation character π can also be interpreted as 1_H^G, the trivial character of H induced up to G. From this, we get a formula relating $\pi(g) = 1_H^G(g)$ to the H-conjugacy-classes lying in the G-conjugacy class of g, as follows. Let $h_1, \ldots, h_r$ be representatives for the conjugacy classes of elements in H contained in the G-conjugacy class of g. Then the value of the permutation character can be written in the following way:

$$1_H^G(g) = |C_G(g)| \sum_{i=1}^{r} \frac{1}{|C_H(h_i)|}.$$

Thus, the permutation character can be derived from the knowledge of the H-conjugacy classes and the knowledge in which G-conjugacy classes they are contained. The map that attaches to each H-conjugacy class the G-conjugacy class it is contained in is called the *fusion map* from H to G.

The group theory system GAP contains a powerful function, written by T. Breuer, which supports the determination of the fusion map given information, like that which can be found in a GAP character table, about the H-conjugacy classes and the G-conjugacy classes. This information usually contains the orders of the representatives, the power maps and the orders of the centralizers.

For all sporadic simple groups other than Fi_{24}', the baby monster group B, and the monster group M, the conjugacy classes and the character tables for all maximal subgroups have now been determined. These tables are publicly available as part of the GAP character table library, which forms part of the GAP system [Schönert et al. 1994].

We give a short outline of how one proceeds to determine a permutation character using GAP and its character table library. More information on the use of the functions described below can be obtained using the online help system of GAP.

One first reads in the character table of the chosen finite simple group G using the command CharTable supplied with the library name of the character table of G. The GAP character table of G is a so-called GAP record, and one component of this record is a list (*maxes*) containing the names under which the character tables of the maximal subgroups of G can be found in the library. Using the name for the chosen maximal subgroup H, we read in the character table of H.

The function SubgroupFusions, when supplied with the character tables of H and G as the main arguments, returns the possible fusion maps consistent with all the restrictions. Since fusion maps are only determined up to automorphisms of the character tables, the function RepresentativesFusions can be used to get a list of representatives for the fusion maps. For each of the representatives in turn we can determine the permutation character of G on the cosets of H via the function Induced supplied with the fusion map and the trivial character of H. Since we are interested in the multiplicities of the irreducible characters in the resulting permutation character, we determine the decomposition of the permutation character into ordinary irreducible characters using the function MatScalarProducts, applied to the irreducible characters of G and the permutation character. It is then trivial to derive from the decomposition whether the permutation character is multiplicity-free or not. Observe that even though there might be several possible fusion maps, it is still possible that the putative permutation characters corresponding to these maps coincide. A more detailed account of the basic theory is contained in [Breuer 1991]; see also [Neubüser et al. 1984]. We remark that many fusion maps are now explicitly stored in the GAP character table library.

Useful Tricks to Show That Certain Characters Are Not Multiplicity-Free

Let H and K be subgroups of a finite group G. Then the number of orbits of H acting on the cosets of K in G is equal to the scalar product $[1_H^G, 1_K^G]$ of the permutation characters corresponding to H and K (in particular, the permutation rank of G on H is $[1_H^G, 1_H^G]$). Thus, if 1_K^G is the sum of at most m irreducible characters (counting multiplicities), and we can show that H has more than m orbits on the cosets of K, then 1_H^G cannot be multiplicity-free.

As an application, we record the following well-known result.

Lemma 4.1. *Let a and b be elements of G, in respective G-conjugacy classes $\mathcal{A}$ and $\mathcal{B}$, and let $C(a)$ and $C(b)$ denote the centralizers in G of a and b. Let m be the number of conjugacy classes $\mathcal{C}$ of G such that the $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ structure constant in G is nonzero. Then $C(a)$ has at least m orbits on the cosets of $C(b)$, and thus, if $1_{C(b)}^G$ is the sum of fewer than m irreducible characters, then $1_{C(a)}^G$ is not multiplicity-free.*

Now each of our Fischer groups F has a rank 3 action on a class D of 3-transpositions, which induces a group of automorphisms of $\langle D\rangle$. Often we can show that the stabilizer H in F of a small subset S of elements of D has more than three orbits on D, by showing that there are more than three isomorphism classes of groups $\langle S, d\rangle$, as d ranges over D. For example, if $S \subseteq D$, $\langle S\rangle \cong 2^2$, then the isomorphism types of groups of the form $\langle S, d\rangle$ ($d \in D$) are 2^2, 2^3, $2 \times S_3$, and S_4. If $S \subseteq D$, $\langle S\rangle \cong S_3$, then the isomorphism types of groups of the form $\langle S, d\rangle$ ($d \in D$) are S_3, $2 \times S_3$, S_4, and $3^2{:}\, S_3$ [Fischer 1969]. We thus have:

Lemma 4.2. *Let F be a Fischer group (as defined on* page 235*), acting on a class D of 3-transpositions. Let H be the stabilizer in F of a set S of 3-transpositions, such that $\langle S\rangle \cong 2^2$ or $\langle S\rangle \cong S_3$. Then the action of F on the cosets of H is not multiplicity-free.*

On the Permutation Characters of $G.2$

Now let G be a finite simple group having an outer automorphism of order 2, and $G.2$ be the extension of G by this outer automorphism. The irreducible characters of $G.2$ and their relationship with the irreducible characters of G are explicitly described by Clifford's theorem. We first note that since the outer automorphism acts on G, it also acts naturally on the conjugacy classes and the irreducible characters of G. The irreducible characters of $G.2$ fall into two sets, namely the ones that are extensions of the irreducible characters of G invariant under the outer automorphism, and those that are the induction of the irreducible characters of G not invariant under the automorphism. There are always two extensions of a given invariant character, and the induced characters of the two noninvariant characters in the same orbit are identical.

Let us now consider a not necessarily irreducible character χ of G and an extension χ' of χ to $G.2$. It follows from Frobenius reciprocity that the multiplicity of an induced irreducible character of $G.2$ in χ' is the same as the multiplicity of the original (noninvariant) irreducible character of G in χ. Also, the sum of the multiplicities of the extensions of a given invariant irreducible character ψ equals the multiplicity of ψ in χ. We thus have the following result.

Lemma 4.3. *Let M be a subgroup of $G.2$ such that $|M : M \cap G| = 2$. If the permutation character $1_{M\cap G}^G$ is multiplicity-free, then the permutation character $1_M^{G.2}$ is again multiplicity-free. If $1_{M\cap G}^G$ has a noninvariant irreducible constituent having multiplicity at least 2, or $1_{M\cap G}^G$ has any irreducible consituent having multiplicity at least 3, then $1_M^{G.2}$ is not multiplicity-free.*

The maximal subgroups of $G.2$ also fall into two sets. As defined in [Wilson 1985], a *nonnovelty* M is a maximal subgroup of $G.2$ whose intersection $M \cap G$ is a maximal subgroup of G, and a *novelty* is a maximal subgroup whose intersection with G is not a maximal subgroup of G. In both cases, M contains $M \cap G$ as a normal subgroup of index 2.

In order to decide whether the permutation character $1_M^{G.2}$ is multiplicity-free, we first consider the the permutation character of G on $M \cap G$. If the permutation character of G on this intersection is multiplicity-free, then so is the permutation character $1_M^{G.2}$. If the permutation character for G contains either an invariant character with multiplicity at least 3 or a noninvariant character with multiplicity at least 2 then the permutation character for $G.2$ is not multiplicity-free. If none of these cases hold, we may determine the extended permutation character of the given one for G, using the fact that the extended permutation character on the maximal subgroup M of $G.2$ is a summand of the permutation character of $G.2$ on $M \cap G$. This poses a strong restriction on the irreducible characters of $G.2$ that may appear in the extended permutation character. In order to determine the permutation character for $G.2$ on the maximal subgroup M, we have written a GAP program that lists the subsums of the constituents of the permutation character of $G.2$ on $M \cap G$, which fulfill certain necessary conditions of being a permutation character (for $G.2$ on M). In the cases we had to consider we were always led to a unique solution.

5. THE MULTIPLICITY-FREE PRIMITIVE PERMUTATION REPRESENTATIONS OF THE FISCHER GROUPS

We now classify the multiplicity-free primitive permutation representations of the Fischer groups and determine their characters. Each character turns out to contain only irreducible constituents with Frobenius–Schur indicator +1, so each of these multiplicity-free representations has all its orbitals self-paired. Each of the rank 3 representations is distance-transitive, and we shall show in the next section that each multiplicity-free primitive representation of a Fischer group of rank greater than 3 is not distance-transitive.

Since the character tables for the maximal subgroups of Fi_{22} and Fi_{23} have been determined and are accessible via the character table database contained in GAP, it is a straightforward exercise to determine the permutation characters belonging to the actions of Fi_{22} and Fi_{23} on the cosets of their maximal subgroups.

Theorem 5.1. *The multiplicity-free primitive permutation characters for* Fi_{22} *are*:

1. $1_{2\cdot U_6(2)}^{\mathrm{Fi}_{22}} = 1a + 429a + 3080a$
2. $1_{O_7(3)}^{\mathrm{Fi}_{22}} = 1a + 429a + 13650a$
3. $1_{O_7(3)}^{\mathrm{Fi}_{22}} = 1a + 429a + 13650a$
4. $1_{O_8^+(2):S_3}^{\mathrm{Fi}_{22}} = 1a + 3080a + 13650a + 45045a$
5. $1_{2^{10}:M_{22}}^{\mathrm{Fi}_{22}} = 1a + 78a + 429a + 1430a + 3080a + 30030a + 32032a + 75075a$
6. $1_{2^6:S_6(2)}^{\mathrm{Fi}_{22}} = 1a + 429a + 1430a + 3080a + 13650a + 30030a + 45045a + 75075a + 205920a + 320320a$
7. $1_{{}^2F_4(2)'}^{\mathrm{Fi}_{22}} = 1a + 1001a + 1430a + 13650a + 30030a + 289575a + 400400ab + 579150a + 675675a + 1201200a$

Theorem 5.2. *The multiplicity-free primitive permutation characters for* Fi_{23} *are*:

1. $1_{2\cdot\mathrm{Fi}_{22}}^{\mathrm{Fi}_{23}} = 1a + 782a + 30888a$
2. $1_{O_8^+(3):S_3}^{\mathrm{Fi}_{23}} = 1a + 30888a + 106743a$
3. $1_{S_8(2)}^{\mathrm{Fi}_{23}} = 1a + 782a + 3588a + 30888a + 60996a + 106743a + 274482a + 812889a + 1951872a + 5533110a + 21348600a + 26838240a + 29354325a$
4. $1_{2^{11}\cdot M_{23}}^{\mathrm{Fi}_{23}} = 1a + 782a + 3588a + 30888a + 60996a + 274482a + 789360a + 812889a + 1677390a + 1951872a + 5533110a + 7468032a + 21348600a + 28464800a + 29354325a + 97976320a$

The determination of the primitive multiplicity-free representations of the simple group Fi_{24}' differs from that for Fi_{22} and Fi_{23}, since not all character tables of the maximal subgroups of Fi_{24}' are known. However, there is an obvious bound for the index of a subgroup whose permutation character is multiplicity-free, namely the sum of the degrees of all irreducible characters of Fi_{24}', which is 7824318655674. This already implies that we only have to consider the permutation characters on the first nine maximal subgroups of Fi_{24}' given in the Atlas. For all but the sixth and the ninth maximal subgroup the character tables have been determined and we can proceed in the same way as for Fi_{22} and Fi_{23}. For the sixth maximal subgroup, $N(3B)$, the conjugacy classes and their fusion into Fi_{24}' have been determined by U. Schiffer, a diploma student at RWTH Aachen, and an account of this work will appear in [Schiffer 1995]. The decomposition of the permutation character follows immediately from this information; it is:

$$\begin{aligned}&1a+57477a+249458a+1666833a+35873145aa\\&\quad+40536925a+79452373a+112168056a\\&\quad+281380736a+1264015025a+1540153692a\\&\quad+3208653525aa+3283490925a+5775278080a\\&\quad+8529641472a+9100908180a+17068369920a\\&\quad+17161712568a+25027497495a+45049495491a\\&\quad+54234085491a+63831063582a.\end{aligned}$$

Thus, the permutation character $1_{N(3B)}^{\mathrm{Fi}_{24}'}$ is not multiplicity-free.

The ninth maximal subgroup is the $2B$-centralizer. In the proof of Theorem 5.5 we show that the permutation character of Fi_{24} acting on the class $2B$ is not multiplicity-free, which implies that the action of Fi_{24}' on the class $2B$ is not multiplicity-free as well.

We have thus proved:

Theorem 5.3. *The multiplicity-free primitive permutation characters for* Fi_{24}' *are*:

1. $1_{\mathrm{Fi}_{23}}^{\mathrm{Fi}_{24}'} = 1a + 57477a + 249458a$
2. $1_{O_{10}^-(2)}^{\mathrm{Fi}_{24}'} = 1a + 8671a + 57477a + 249458a + 555611a + 1666833a + 35873145a + 48893768a + 79452373a + 415098112a + 1264015025a + 1540153692a + 2346900864a + 3208653525a + 10169903744a + 13904165275ab + 17161712568a$
3. $1_{3^7\cdot O_7(3)}^{\mathrm{Fi}_{24}'} = 1a + 57477a + 249458a + 35873145a + 40536925a + 79452373a + 112168056a + 281380736a + 1069551175a + 1264015025a + 3208653525a + 3283490925a + 5775278080a + 10776585600ab + 17068369920a + 17161712568a + 54234085491a$

(The character in the third case is stated incorrectly in [Ivanov et al. 1995].)

We now turn our attention to $\mathrm{Fi}_{22}{:}2$ and Fi_{24}.

Theorem 5.4. *The faithful multiplicity-free primitive permutation characters for* $\mathrm{Fi}_{22}{:}2$ *are as follows*:

1. $1_{2\cdot U_6(2).2}^{\mathrm{Fi}_{22}:2} = 1a + 429a + 3080a$
2. $1_{O_8^+(2):S_3\times 2}^{\mathrm{Fi}_{22}:2} = 1a + 3080a + 13650a + 45045a$
3. $1_{2^{10}:M_{22}:2}^{\mathrm{Fi}_{22}:2} = 1a + 78a + 429a + 1430a + 3080a + 30030a + 32032a + 75075a$
4. $1_{2^7:S_6(2)}^{\mathrm{Fi}_{22}:2} = 1a + 429a + 1430a + 3080a + 13650a + 30030a + 45045a + 75075a + 205920a + 320320a$
5. $1_{^2F_4(2)}^{\mathrm{Fi}_{22}:2} = 1a + 1001a + 1430a + 13650a + 30030a + 289575b + 800800a + 579150a + 675675b + 1201200c$

Proof. We first consider the nonnovelties amongst the maximal subgroups of $\mathrm{Fi}_{22}{:}2$. In Fi_{22}, only the first through sixth and the ninth maximal subgroups lead to a multiplicity-free primitive permutation character of Fi_{22}. (We order the maximal subgroups as in the Atlas, where the list of maximal subgroups of Fi_{22} and $\mathrm{Fi}_{22}{:}2$ is complete [Kleidman and Wilson 1987].) All of these except the second and the third extend to nonnovelties, and hence lead to multiplicity-free permutation characters for $\mathrm{Fi}_{22}{:}2$.

The nonnovelty corresponding to the 7th maximal subgroup of Fi_{22} is the (setwise) stabilizer of a pair of commuting 3-transpositions, and that corresponding to the eighth maximal subgroup is the stabilizer of a set of three 3-transpositions generating an S_3. By Lemma 4.2 the corresponding permutation characters are not multiplicity-free.

We explicitly constructed the permutation character π of $\mathrm{Fi}_{22}{:}2$ on the extension $H.2$ of the tenth maximal subgroup H of Fi_{22} (using the fact that

π is a summand of the permutation character of $\mathrm{Fi}_{22}{:}\,2$ on H), and showed that π is not multiplicity-free.

Next, we observe that the permutation character of Fi_{22} on its eleventh maximal subgroup has an irreducible constituent with multiplicity 3, and so the extension of this character to $\mathrm{Fi}_{22}{:}\,2$ is not multiplicity-free.

The twelfth and thirteenth maximal subgroups of Fi_{22} do not extend to nonnovelties, and the fourteenth maximal subgroup of Fi_{22} has index greater than the sum of the character degrees of the irreducible characters of $\mathrm{Fi}_{22}{:}\,2$.

We are now left with the novelties in $\mathrm{Fi}_{22}{:}\,2$. There are exactly two novelties (up to conjugacy) of $\mathrm{Fi}_{22}{:}\,2$, namely $G_2(3){:}\,2$ and $3^5{:}\,(U_4(2){:}\,2\times 2)$.

In the case of the novelty $G_2(3){:}\,2$ the permutation character can be calculated using GAP and the functions explained in Section 4, since the character table for $G_2(3){:}\,2$ is an Atlas table, and therefore contained in the GAP character table library. The permutation character we obtain is

$$\begin{aligned}1_{G_2(3):2}^{\mathrm{Fi}_{22}:2} = 1a &+ 429ab + 10725b + 13650aab\\ &+ 48048b + 50050c + 75075ae\\ &+ 81081b + 579150ab + 675675a\\ &+ 1164800a + 1201200ac + 1360800ab\\ &+ 1441792ab + 1791153a + 2027025b,\end{aligned}$$

and thus is not mutliplicity-free.

For the second novelty $3^5{:}\,(U_4(2){:}\,2\times 2)$, we compute that

$$\begin{aligned}1_{3^5:(U_4(2)\times 2)}^{\mathrm{Fi}_{22}} = 1a &+ 429aa + 3080a + 13650aaa\\ &+ 45045a + 75075a + 81081a\\ &+ 150150a + 289575a + 320320a\\ &+ 360855aa + 675675a + 1360800aa.\end{aligned}$$

It follows that the permutation character of $\mathrm{Fi}_{22}{:}\,2$ on $3^5{:}\,(U_4(2){:}\,2\times 2)$, being an extension of the permutation character given above, is not multiplicity-free since the corresponding permutation character for the simple group has an invariant irreducible constituent of multiplicity 3. ☐

Theorem 5.5. *The faithful primitive multiplicity-free permutation characters of* Fi_{24} *are precisely the extensions of the primitive multiplicity-free permutation characters of* Fi_{24}', *and are as follows*:

1. $1_{2\times \mathrm{Fi}_{23}}^{\mathrm{Fi}_{24}} = 1a + 57477a + 249458a$
2. $1_{O_{10}^-(2):2}^{\mathrm{Fi}_{24}} = 1a + 8671b + 57477a + 249458a + 555611b + 1666833a + 35873145a + 48893768b + 79452373a + 415098112b + 1264015025a + 1540153692a + 2346900864b + 3208653525a + 10169903744b + 13904165275a + 17161712568a$
3. $1_{3^7.O_7(3):2}^{\mathrm{Fi}_{24}} = 1a + 57477a + 249458a + 35873145a + 40536925a + 79452373a + 112168056a + 281380736a + 1069551175b + 1264015025a + 3208653525a + 3283490925a + 5775278080a + 17068369920a + 17161712568a + 21553171200a + 54234085491a$

Proof. We consider the maximal subgroups of Fi_{24}, the automorphism group of Fi_{24}', and the sum of the degrees of the ordinary irreducible characters of Fi_{24} gives an upper bound for the indices of the maximal subgroups we have to consider. It follows from the list of the maximal subgroups given in [Linton and Wilson 1991] that we only have to deal with first nine nonnovelties amongst the maximal subgroups of Fi_{24} listed in the Atlas. The indices of all novelties are greater than the bound. There are exactly three primitive multiplicity-free permutation characters for Fi_{24}', namely the ones on Fi_{23}, $O_{10}^-(2)$ and $3^7{\cdot}O_7(3)$, and they lead to multiplicity-free permutation characters of Fi_{24} on the nonnovelties $2\times \mathrm{Fi}_{23}$, $O_{10}^-(2){:}\,2$, and $3^7{\cdot}O_7(3){:}\,2$.

The permutation characters of Fi_{24} on the nonnovelties $(2\times 2{\cdot}\mathrm{Fi}_{22}){:}\,2$ and $S_3\times O_8^+(3){:}\,S_3$ can be seen not to be multiplicity-free by applying Lemma 4.2. The permutation characters of Fi_{24}' on $2^{11}{\cdot}M_{24}$ and on $2^2.U_6(2){:}\,S_3$ contain an invariant character with multiplicity 3, and so the permutation characters on the corresponding nonnovelties are not multiplicity-free.

Since we already know the permutation character of Fi_{24}' on the normalizer of a $3B$ in Fi_{24}', it is straightforward to derive the permutation character of Fi_{24} on the normalizer of a $3B$ in Fi_{24}, using GAP. The decomposition of the permutation character for Fi_{24} is

$$\begin{aligned}&1a+57477a+249458a+1666833a+35873145aa\\&\quad+40536925a+79452373a+112168056a\\&\quad+281380736a+1264015025a+1540153692a\\&\quad+3208653525aa+3283490925a+5775278080a\\&\quad+8529641472a+9100908180a+17068369920a\\&\quad+17161712568a+25027497495a+45049495491a\\&\quad+54234085491a+63831063582a,\end{aligned}$$

and hence this permutation character is not multiplicity-free.

For the ninth maximal subgroup, the $2B$-centralizer in Fi_{24}, we shall use Lemma 4.1. We calculate the permutation character of Fi_{24} on its $2A$-centralizer and obtain

$$\begin{aligned}&1a + 57477aa + 249458a + 555611b\\&\quad + 35873145a + 79452373a + 112168056a\\&\quad + 159402880a + 1264015025a + 3208653525a,\end{aligned}$$

which is the sum of exactly 11 irreducible characters. Using the GAP command ClassMultCoeffsCharTable, we find that there are exactly 16 Fi_{24} conjugacy classes $\mathcal{C}$ for which the $(2A, 2B, \mathcal{C})$ structure constant is nonzero, and conclude that the permutation character of Fi_{24} on the class $2B$ is not multiplicity-free. □

6. ANALYSIS OF THE MULTIPLICITY-FREE PRIMITIVE REPRESENTATIONS OF RANK GREATER THAN 3

In this section we present a case by case analysis of the multiplicity-free primitive representations of rank greater than 3 of the Fischer groups. We give detailed information on each such representation, including its subdegrees, and show that each is not a DTR, to complete the proof of Theorem 2.1.

In the statements below, expressions in parentheses such as $(:2)$ and $(\times 2)$ give alternate statements: thus Theorem 6.1 covers the representation of Fi_{22} on the cosets of $O_8^+(2){:}\,S_3$ and the representation of $\mathrm{Fi}_{22}{:}\,2$ on the cosets of $O_8^+(2){:}\,S_3 \times 2$, and so on.

$\mathrm{Fi}_{22}(:2)$ on $\mathrm{O}_8^+(2){:}\,\mathrm{S}_3(\times 2)$

Theorem 6.1. *The subdegrees of the representation of* $\mathrm{Fi}_{22}(:2)$ *on the cosets of* $O_8^+(2){:}\,S_3(\times 2)$ *are* 1, 1575, 22400, 37800, *and the representation is not distance-transitive.*

Proof. We reproduce from [Praeger and Soicher] the collapsed adjacency matrices for the orbital graphs corresponding to the two smallest subdegrees greater than 1:

0	1575	0	0		0	0	22400	0
1	198	512	864		0	512	8064	13824
0	36	567	972		1	567	8224	13608
0	36	576	963		0	576	8064	13760

We now observe that each of these graphs has diameter 2, and so the above representations are not distance-transitive. (But, as noted in [Praeger and Soicher], each of these graphs is distance-regular.) □

$\mathrm{Fi}_{22}(:2)$ on $2^{10}{:}\,\mathrm{M}_{22}(:2)$

Let $\Gamma(\mathrm{Fi}_{22})$ be the graph whose vertex-set is the conjugacy class of 3510 3-transpositions of Fi_{22}, two 3-transpositions being joined if and only if their product has order 2. Then this graph has just one Fi_{22}-orbit of maximal cliques, each having size 22. The stabilizer of a maximal clique is $2^{10}{:}\,M_{22}$ in Fi_{22}, and $2^{10}{:}\,M_{22}{:}\,2$ in $\mathrm{Fi}_{22}{:}\,2$.

Theorem 6.2. *The subdegrees of the representation of* $\mathrm{Fi}_{22}(:2)$ *the cosets of* $2^{10}{:}\,M_{22}(:2)$ *are* 1, 154, 1024, 3696, 4928, 11264, 42240, 78848, *and the representation is not distance-transitive.*

Proof. We perform the following sequence of calculations using GRAPE and GAP. The method is based on the approach described in Section 3.

We first use GRAPE to construct the graph $\Gamma = \Gamma(\mathrm{Fi}_{22})$ from the degree 3510 representation of Fi_{22} on its 3-transpositions (this representation was constructed via a coset enumeration, using a presentation of $Y_{332} \cong 2^2.\mathrm{Fi}_{22}$ and enumerating over the centralizer $2^3.U_6(2)$ of a 3-transposition [Conway et al. 1988]). Then a clique K of size 22 is found in Γ, and the stabilizer of this clique computed. Next, representatives $K_1 = K, K_2, \dots, K_8$ for the eight orbits of H on the maximal cliques of Γ are calculated (using the GRAPE function CompleteSubgraphsOfGivenSize) and the stabilizers of these eight cliques are determined (using GAP). The subdegrees above are then obtained.

Now to each maximal clique M of Γ there corresponds the set $\bar{M}$ of the 1024 vertices of Γ joined to no vertex of M. Ordering $K_1, \dots, K_8$ to represent suborbits in increasing order of length, we find that $|\bar{K} \cap \bar{K}_i| = 1024$, 512, 232, 384, 256, 352, 288, 296, for $i = 1, \dots, 8$, respectively.

Now if Fi_{22} on $2^{10}{:}\,M_{22}$ (or $\mathrm{Fi}_{22}{:}\,2$ on $2^{10}{:}\,M_{22}{:}\,2$) is a distance-transitive representation, $\mathrm{Fi}_{22}{:}\,2 \cong \mathrm{Aut}\,\Gamma$ acts distance-transitively on Σ_{512} or Σ_{232}, where Σ_n is defined to be the graph having vertices the maximal cliques of Γ, with two vertices X, Y joined in Σ_n if and only if $|\bar{X} \cap \bar{Y}| = n$.

We show that $\mathrm{Fi}_{22}{:}\,2$ does not act distance-transitively on Σ_{512} by finding maximal cliques X, Y of Γ, such that both X and Y are at distance 2 from K in Σ_{512}, but $|\bar{K} \cap \bar{X}| = 256$ and $|\bar{K} \cap \bar{Y}| = 384$, and so no element of $\mathrm{Fi}_{22}{:}\,2$ takes (K, X) to (K, Y). (Alternatively, a collapsed adjacency matrix for Σ_{512} is calculated in [Rowley and Walker 1993], and we see that there are exactly two suborbits at distance 2 from a given vertex of that graph.)

We complete the proof by showing that $\mathrm{Fi}_{22}{:}\,2$ does not act distance-transitively on Σ_{232}. We find maximal cliques X, Y of Γ, such that both X and Y are joined to K_3 in Σ_{232}, but $|\bar{K} \cap \bar{X}| = 296$ and $|\bar{K} \cap \bar{Y}| = 384$. □

Remark. collapsed adjacency matrices for Σ_{512} and Σ_{232} are now available in [Ivanov et al. 1995].

$\mathrm{Fi}_{22}(:2)$ on $2^6{:}\,\mathrm{S}_6(2)(.2)$

Theorem 6.3. *The representation of $\mathrm{Fi}_{22}(:2)$ on the cosets of $2^6{:}\,S_6(2)(.2)$ has subdegrees* 1, 135, 1260, 2304, 8640, 10080, 45360, 143360, 241920^2, *and the representation is not distance-transitive.*

Proof. From the permutation characters, we see that the ranks are the same for the two permutation representations of the theorem.

We construct the degree 694980 representation of Fi_{22} on the cosets of $2^6{:}\,S_6(2)$ by coset enumeration of the cosets of $Y_{331} \cong 2^2.2^6{:}\,S_6(2)$ in $Y_{332} \cong 2^2.\mathrm{Fi}_{22}$ [Conway et al. 1988]. We then calculate the collapsed adjacency matrices for the orbital graphs for this representation, and record below the collapsed adjacency matrices for the orbital graphs of the smallest two valencies greater than 1 (the suborbits are ordered in nondecreasing order of length):

$$
\begin{array}{rrrrrrrrrr}
0 & 135 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 14 & 56 & 0 & 64 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 9 & 0 & 48 & 0 & 72 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 30 & 0 & 0 & 0 & 105 & 0 \\
0 & 1 & 7 & 8 & 21 & 0 & 42 & 0 & 56 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 36 & 0 & 0 & 96 \\
0 & 0 & 2 & 0 & 8 & 8 & 21 & 0 & 64 & 32 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 27 & 54 & 54 \\
0 & 0 & 0 & 1 & 2 & 0 & 12 & 32 & 40 & 48 \\
0 & 0 & 0 & 0 & 0 & 4 & 6 & 32 & 48 & 45
\end{array}
$$

$$
\begin{array}{rrrrrrrrrr}
0 & 0 & 1260 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 56 & 84 & 0 & 448 & 0 & 672 & 0 & 0 & 0 \\
1 & 9 & 82 & 64 & 144 & 96 & 288 & 0 & 576 & 0 \\
0 & 0 & 35 & 35 & 0 & 0 & 315 & 560 & 315 & 0 \\
0 & 7 & 21 & 0 & 126 & 0 & 210 & 0 & 560 & 336 \\
0 & 0 & 12 & 0 & 0 & 84 & 108 & 384 & 576 & 96 \\
0 & 2 & 8 & 16 & 40 & 24 & 162 & 256 & 336 & 416 \\
0 & 0 & 0 & 9 & 0 & 27 & 81 & 360 & 405 & 378 \\
0 & 0 & 3 & 3 & 20 & 24 & 63 & 240 & 475 & 432 \\
0 & 0 & 0 & 0 & 12 & 4 & 78 & 224 & 432 & 510
\end{array}
$$

The result follows. □

$\mathrm{Fi}_{22}(:2)$ on ${}^2\mathrm{F}_4(2)'(.2)$

Theorem 6.4. *The subdegrees of the representation of Fi_{22} on the cosets of ${}^2F_4(2)'$ are* 1, 1755, 11700, 14976, 83200^2, 140400, 187200, 374400, 449280, 2246400. *For the representation of $\mathrm{Fi}_{22}{:}\,2$ on the cosets of ${}^2F_4(2)$ the suborbits of equal length are fused. Neither of these representations is distance-transitive.*

Proof. We proceed along the lines described in Section 3. We use GAP and GRAPE to compute with Fi_{22} as a group of permutations of 14080 points.

In this representation, a subgroup $H \cong {}^2F_4(2)'$ has two orbits, of 1600 and 12480 points. Fixing such a subgroup and letting Δ be its smaller orbit, we look for elements $\{g_1, \dots, g_{11}\}$ such that

$$\{\Delta^{g_i} \mid i \leq i \leq 11\}$$

is a set of representatives for the H-orbits on Δ^G.

If Δ^g and $\Delta^{g'}$ lie in the same H-orbit, $|\Delta \cap \Delta^g|$ must equal $|\Delta \cap \Delta^{g'}|$, and so we first test random elements g of Fi_{22} to see how many different values of $|\Delta \cap \Delta^g|$ we can find. A search of 5000 random elements gives nine values: 196, 176, 180, 208, 192, 256, 1600, 100 and 320 (our "random" elements deliberately included the identity).

We let $g_1, \dots, g_9$ be elements giving rise to these values, and we then compute (using GAP) the orders of the subgroups $\mathrm{stab}_H(\Delta^{g_i})$ and so obtain the sizes of the nine orbits represented. These sizes are, respectively: 374400, 2246400, 449280, 187200, 140400, 11700, 1, 14976 and 1755. These leave 166400 points unaccounted for, or about 5% of the total of $|\mathrm{Fi}_{22} : {}^2F_4(2)'| = 3592512$ points. It seems unlikely that our random search would simply have missed the two orbits containing these points, so we surmise that we must have failed to distinguish them from the nine orbits we have.

Accordingly, we perform a second search, using not just the size, but the exact graph isomorphism type (as computed by *nauty*) of an orbital graph of Fi_{22} on the 14080 points, restricted to $\Delta \cap \Delta^g$, to distinguish between orbits. This is much slower, but only a few dozen random elements need to be searched to find the two missing orbits, which have $\Delta \cap \Delta^g$ of cardinality 196, and which both have size 83200. We conclude that these two orbits must be fused under the action of $\mathrm{Fi}_{22}{:}2$ since the permutation character implies that the rank is smaller in that case.

Having obtained the suborbit structure, it now remains to check for distance-transitivity. We only need to check the orbital graphs corresponding to the two suborbits of smallest length (greater than 1). We do this as described in Section 3. In the valency 1755 graph, we find suborbits of sizes 187200 and 449280 at distance 2 from a fixed vertex, and in the valency 11700 graph we find suborbits of sizes 449280 and 2246400 at distance 2 from a fixed vertex. $\square$

We remark that a collapsed adjacency matrix for the orbital graph of valency 1755 is published in [Ivanov et al. 1995], and we record below a collapsed adjacency matrix for the orbital graph of valency 11700, for the action of Fi_{22} on the cosets of ${}^2F_4(2)'$:

0	0	11700	0	0	0	0	0	0	0	0
0	80	100	0	0	0	640	640	5120	0	5120
1	15	516	0	1024	1024	576	96	0	3840	4608
0	0	0	300	0	0	0	0	1800	1200	8400
0	0	144	0	612	576	864	216	648	1944	6696
0	0	144	0	576	612	864	216	648	1944	6696
0	8	48	0	512	512	812	272	1088	1344	7104
0	6	6	0	96	96	204	1308	1440	1248	7296
0	24	0	72	144	144	408	720	1980	960	7248
0	0	100	40	360	360	420	520	800	2060	7040
0	4	24	56	248	248	444	608	1208	1408	7452

Fi_{23} on $\mathrm{S}_8(2)$

Theorem 6.5. *The subdegrees of the permutation representation of* Fi_{23} *on the cosets of* $S_8(2)$ *are* 1, 2295, 13056, 24192, 107100, 261120, 1285200, 2203200, 3046400, 3290112, 12337920, 20844800 *and* 32901120. *The representation is not distance-transitive.*

Proof. We construct (a compressed form of) the degree 86316516 representation of Fi_{23} on the cosets of $S_8(2)$ by double coset enumeration of the double cosets of $Y_{431} \cong S_8(2) \times 2$ and $Y_{430} \cong S_9$ in $Y_{432} \cong \mathrm{Fi}_{23} \times 2$ [Linton 1991], using a new GAP double coset enumeration program written by the first author.

Since $Y_{430} < Y_{431}$ the suborbits must be unions of double cosets, and it is easy to calculate them all. We can then compute the collapsed adjacency matrices corresponding to the orbital graphs of the two smallest valencies greater than 1. With the suborbits in increasing order of length, these matrices are:

$$\begin{array}{rrrrrrrrrrrrr}
0 & 2295 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 30 & 0 & 0 & 280 & 1024 & 0 & 960 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1350 & 0 & 0 & 945 & 0 & 0 \\
0 & 0 & 0 & 85 & 0 & 0 & 850 & 0 & 0 & 0 & 0 & 0 & 1360 \\
0 & 6 & 0 & 0 & 9 & 0 & 216 & 144 & 0 & 768 & 0 & 1152 & 0 \\
0 & 9 & 0 & 0 & 0 & 135 & 0 & 135 & 0 & 126 & 0 & 1890 & 0 \\
0 & 0 & 0 & 16 & 18 & 0 & 61 & 72 & 0 & 0 & 576 & 144 & 1408 \\
0 & 1 & 8 & 0 & 7 & 16 & 42 & 149 & 224 & 112 & 56 & 1008 & 672 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 162 & 81 & 108 & 324 & 648 & 972 \\
0 & 0 & 0 & 0 & 25 & 10 & 0 & 75 & 100 & 135 & 600 & 450 & 900 \\
0 & 0 & 1 & 0 & 0 & 0 & 60 & 10 & 80 & 160 & 160 & 960 & 864 \\
0 & 0 & 0 & 0 & 4 & 16 & 6 & 72 & 64 & 48 & 384 & 789 & 912 \\
0 & 0 & 0 & 1 & 0 & 0 & 55 & 45 & 90 & 90 & 324 & 855 & 835
\end{array}$$

$$\begin{array}{rrrrrrrrrrrrr}
0 & 0 & 13056 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 7680 & 0 & 0 & 5376 & 0 & 0 \\
1 & 0 & 210 & 0 & 1575 & 0 & 0 & 0 & 5600 & 0 & 5670 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4896 & 0 & 0 & 8160 \\
0 & 0 & 192 & 0 & 192 & 0 & 1728 & 0 & 4608 & 0 & 1728 & 4608 & 0 \\
0 & 0 & 0 & 0 & 0 & 120 & 0 & 0 & 840 & 0 & 0 & 7560 & 4536 \\
0 & 0 & 0 & 0 & 144 & 0 & 336 & 288 & 1152 & 0 & 4608 & 3456 & 3072 \\
0 & 8 & 0 & 0 & 0 & 0 & 168 & 1008 & 0 & 1792 & 672 & 4032 & 5376 \\
0 & 0 & 24 & 0 & 162 & 72 & 486 & 0 & 1944 & 0 & 3888 & 4536 & 1944 \\
0 & 0 & 0 & 36 & 0 & 0 & 0 & 1200 & 0 & 720 & 1200 & 2700 & 7200 \\
0 & 1 & 6 & 0 & 15 & 0 & 480 & 120 & 960 & 320 & 2130 & 5760 & 3264 \\
0 & 0 & 0 & 0 & 16 & 64 & 144 & 288 & 448 & 288 & 2304 & 4608 & 4896 \\
0 & 0 & 0 & 6 & 0 & 36 & 120 & 360 & 180 & 720 & 1224 & 4590 & 5820
\end{array}$$

The result follows. □

Fi_{23} on $2^{11}\cdot M_{23}$

Let $\Gamma(\mathrm{Fi}_{23})$ be the graph whose vertex-set is the conjugacy class of 31671 3-transpositions of Fi_{23}, two 3-transpositions being joined if and only if their product has order 2. Then this graph has just one Fi_{23}-orbit of maximal cliques, each having size 23. The stabilizer of a maximal clique is $2^{11}\cdot M_{23}$.

Theorem 6.6. *The subdegrees of the representation of* Fi_{23} *on the cosets of* $2^{11}\cdot M_{23}$ *are* 1, 506, 23552, 28336, 113344, 129536, 971520, 1036288, 1813504, 4533760, 8290304, 21762048, 31088640, 31653888, 36270080, 58032128, *and the representation is not distance-transitive.*

Proof. This proof is similar to the proof that Fi_{22} on the cosets of $2^{10}{:}\,M_{22}$ is not a distance-transitive representation.

We first construct the graph $\Gamma = \Gamma(\mathrm{Fi}_{23})$ from the degree 31671 representation of Fi_{23} on its 3-transpositions (this representation was constructed via a coset enumeration, using a presentation of $Y_{432} \cong 2 \times \mathrm{Fi}_{23}$ and enumerating over the centralizer $Y_{332} \cong 2^2.\mathrm{Fi}_{22}$ of a 3-transposition [Conway et al. 1988]). Then a clique K of size 23 is found in Γ, and the stabilizer of this clique computed. Next, representatives $K_1 = K, K_2, \ldots, K_{16}$ for the sixteen orbits of H on the maximal cliques of Γ are calculated (using the GRAPE functions CompleteSubgraphsOfGivenSize and OrbitRepresentatives), and the stabilizers of these sixteen cliques determined. The subdegrees above are then obtained.

Ordering $K_1, \ldots, K_{16}$ to represent suborbits in increasing order of length, we find $|K \cap K_i| = 23$, 7, 1, 3, 1, 2, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, for $i = 1, \ldots, 16$.

Now define Σ_i to be the orbital graph whose vertices are the maximal cliques of Γ, and edge-set is the orbit of $\{K, K_i\}$ under Fi_{23}. We need only show that Fi_{23} does not act distance-transitively on Σ_2 or Σ_3, to complete the proof of the theorem.

In Σ_2, we find vertices X, Y joined to K_2, such that $|K \cap X| = 3$ and $|K \cap Y| = 1$.

In Σ_3, we find vertices X, Y joined to K_3, such that $|K \cap X| = 3$ and $|K \cap Y| = 0$. □

We remark that collapsed adjacency matrices for Σ_2 and Σ_3 are now available in [Ivanov et al. 1995].

$\mathrm{Fi}_{24}'(:2)$ on $\mathrm{O}_{10}^-(2)(:2)$

Let $\Gamma(\mathrm{Fi}_{24})$ be the graph whose vertex-set is the conjugacy class of 306936 3-transpositions of Fi_{24}, two 3-transpositions being joined if and only if their product has order 2. We shall use this graph to apply the method of Section 3.

Theorem 6.7. *The permutation representation of* Fi_{24}' *on the cosets of* $O_{10}^-(2)$, *and that of* Fi_{24} *on the cosets of* $O_{10}^-(2){:}2$, *have subdegrees* 1, 25245, 104448, 157080, 12773376, 45957120, 67858560, 107233280, 193881600, 263208960, 579059712, 1085736960, 5147197440, 5428684800, 7238246400, 12634030080 *and* 17371791360. *Neither of these representations is distance-transitive.*

Proof. We apply the general method of Section 3, computing in the graph $\Gamma = \Gamma(\mathrm{Fi}_{24})$. We construct permutations generating the action of $\mathrm{Fi}_{24} \cong Y_{442}/O_3(Y_{442})$ on this graph by (double) coset enumeration, using the presentation of Y_{442} given in [Conway and Pritchard 1992]. A subset of these generators give a subgroup $H \cong O_{10}^-(2){:}\,2 \cong Y_{441}$. This has three orbits on the vertices of Γ, having sizes 528, 104448, 201960. We call the smallest of these orbits Δ_1, and the second-smallest Δ_2.

We now aim to find the orbits of H on Δ_1^G, and we proceed by computing, for random elements $g \in G$ the numbers

$$n_1(g) = |\Delta_1 \cap \Delta_1^g| \quad \text{and} \quad n_2(g) = |\Delta_2 \cap \Delta_1^g|.$$

Each of these is an H-orbit invariant. We find distinct pairs of values $(n_1(g_i), n_2(g_i))$ for $i = 1, \dots, 15$.

We would like to compute $S_i = \mathrm{stab}_H(\Delta_1^{g_i})$ for each i, but computing set stabilisers in a representation of degree 306936 is too hard for GAP on available computers, so we instead compute

$$S_i' = \mathrm{stab}_H(\Delta_2 \cap \Delta_1^{g_i}),$$

which must contain S_i as a subgroup. The order of S_i' is then a multiplicative upper bound for $|S_i|$, giving rise to a lower bound for $|\Delta_1^{g_i H}|$. We will later show that all these bounds are exact.

The results obtained so far are shown in Table 1.

i	$n_1(g_i)$	$n_2(g_i)$	$\lvert H : S_i'\rvert$
1	528	0	1
2	66	462	104448
3	36	384	1570800
4	3	120	45957120
5	10	272	67858560
6	15	270	107233280
7	6	168	193881600
8	0	132	263208960
9	3	180	579059712
10	6	222	1085736960
11	0	177	5147197440
12	3	192	5428684800
13	0	186	7238246400
14	0	165	12634030080
15	1	182	17371791360

TABLE 1. Pairs $(n_1(g), n_2(g))$, and corresponding indices, for the permutation representation of Fi_{24} on the cosets of $O_{10}^-(2)$.

Assuming that all our bounds are exact, we see that the two remaining orbits (we know from the permutation character that the rank is 17) contain just 12798621 points. Since this number is odd, we see that one of the two remaining orbits must have odd size. Relatively few subgroups of H have odd index, and for most such subgroups K, the difference $12798621 - |H : K|$ does not divide $|H|$. A few calculations suggest that the orbit sizes might be 25245 and 12773376.

Based on this conjecture, we attempt to find a representative of the orbit of size 25245. The point stabilizer in this orbit would be

$$K \cong 2^{6+8}{:}\,(A_8 \times S_3),$$

a subgroup of index 2 of the octad stabilizer in Fi_{24}. It thus seems reasonable to look at large cliques in Δ_1 in the hope of finding a structure stabilised by K. The Fi_{24}-stabiliser of this structure will then contain a representative of the desired orbit.

Using GRAPE we can compute in the subgraph of Γ induced on Δ_1 and find a clique C of size 16, whose stabilizer in H can be seen to be a subgroup of index 3 in our desired group K. Looking now in Γ, we find just eight points joined to all of C, which form an octad O. Using the ProbablyStabilizer function of GRAPE, we can find the pointwise stabilizer of five points from O, which is a group of order $2^7.3$. A randomly chosen element g_{16} of this group has $n_1(g_{16}) = 48$, $n_2(g_{16}) = 0$, and

$$|H : S_{16}'| = 25245.$$

This demonstrates (up to the strictness of our bounds) that the subdegrees are as claimed. If one of the bounds were not strict, then one of the subdegrees would have to be a proper multiple of the bound, and the unexhibited orbit, which we claim to have size 12773376, would be accordingly smaller. It is easy to check this cannot happen, since all the subdegrees must be the indices of subgroups of H.

Finally, it is easy to check, as described in Section 3, that neither of the two suborbits of smallest length greater than 1 gives rise to a distance-transitive graph. In the valency 25245 graph, the suborbits numbered 3 and 4 are both at distance 2 from a fixed vertex, and in the valency 104448 graph, the suborbits numbered 3 and 6 are both at distance 2 from a fixed vertex. □

We have since computed collapsed adjacency matrices for the orbital graphs corresponding to the two smallest subdegrees greater than 1, for the action of Fi_{24}' on $O_{10}^{-}(2)$, and record them below.

0	25245	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	60	0	1120	0	16384	0	0	7680	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	14850	0	0	10395	0	0	0	0	0
0	18	0	27	0	0	1296	0	864	0	9216	0	0	13824	0	0	0
0	0	0	0	85	0	850	0	0	0	0	0	0	2550	20400	0	1360
0	9	0	0	0	405	0	0	135	0	3906	0	0	5670	0	15120	0
0	0	0	30	160	0	151	0	360	0	0	2880	6144	480	960	0	14080
0	0	0	0	0	0	0	243	810	0	540	1620	0	4860	0	2592	14580
0	1	8	7	0	32	126	448	375	1344	224	168	0	6048	1680	10752	4032
0	0	0	0	0	0	0	0	990	891	0	0	1584	1980	0	7920	11880
0	0	0	25	0	310	0	100	75	0	285	1800	6000	1350	0	7200	8100
0	0	1	0	0	0	180	160	30	0	960	450	640	5040	5400	5760	6624
0	0	0	0	0	0	81	0	0	81	675	135	1728	3375	5265	6615	7290
0	0	0	4	6	48	6	96	216	96	144	1008	3200	2421	2208	5760	10032
0	0	0	0	36	0	9	0	45	0	0	810	3744	1656	3753	4896	10296
0	0	0	0	0	55	0	22	165	165	330	495	2695	2475	2805	7128	8910
0	0	0	0	1	0	55	90	45	180	270	414	2160	3135	4290	6480	8125

0	0	104448	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	61440	0	0	43008	0	0	0	0	0
1	0	462	5775	0	0	0	30800	0	5040	0	62370	0	0	0	0	0
0	0	384	384	0	0	10368	18432	0	0	0	10368	0	55296	9216	0	0
0	0	0	0	272	0	2720	0	0	0	4896	0	0	20400	40800	0	35360
0	0	0	0	0	120	0	840	0	5040	0	0	1680	22680	15120	45360	13608
0	0	0	240	512	0	752	2304	1440	0	0	17280	12288	15360	7680	0	46592
0	0	30	270	0	360	1458	4968	0	0	0	17010	7776	29160	6480	7776	29160
0	8	0	0	0	0	504	0	2688	0	7168	2016	0	20160	10080	21504	40320
0	0	2	0	0	880	0	0	0	2112	0	1782	19008	7920	9240	39744	23760
0	0	0	0	108	0	0	0	2400	0	1440	4200	10800	8100	5400	18000	54000
0	1	6	15	0	0	1080	1680	360	432	2240	5082	4608	24480	18000	15360	31104
0	0	0	0	0	15	162	162	0	972	1215	972	9006	12690	19800	29322	30132
0	0	0	16	48	192	192	576	720	384	864	4896	12032	13584	11136	20736	39072
0	0	0	2	72	96	72	96	270	336	432	2700	14080	8352	14580	24192	39168
0	0	0	0	0	165	0	66	330	828	825	1320	11946	8910	13860	30558	35640
0	0	0	0	26	36	182	180	450	360	1800	1944	8928	12210	16320	25920	36092

Collapsed adjacency matrices for the orbital graphs corresponding to the two smallest subdegrees greater than 1, for the action of Fi_{24}' on $O_{10}^{-}(2)$. Suborbits are ordered in increasing order of length.

$\mathrm{Fi}_{24}'(:2)$ on $3^7{\cdot}O_7(3)(:2)$

Theorem 6.8. *The permutation representation of* Fi_{24}' *on the cosets of* $3^7{\cdot}O_7(3)$ *has the following subdegrees*: 1, 1120, 49140, 275562, 816480, 21228480, 57316896, 62178597, 286584480, 429876720, 2901667860, 5158520640, 6964002864, 9183300480^2, 15475561920, 23213342880 *and* 52230021480. *For the representation of* Fi_{24} *on the cosets of* $3^7{\cdot}O_7(3){:}\,2$, *the suborbits of equal length are fused. Neither representation is distance-transitive.*

Proof. A geometric argument in [Ivanov et al.] shows that each of these representations has suborbits of sizes 1120 and 49140. This argument makes use of a certain rank 4 extended dual polar space $\mathcal{G}$ on which Fi_{24}' acts flag-transitively, with "point" stabilizer $3^7{\cdot}O_7(3)$. We compute the remaining (nontrivial) subdegrees below. The subdegrees 1120 and 49140 turn out to be the smallest nontrivial ones. In [Ivanov et al. 1995] it is also shown, using the geometry $\mathcal{G}$, that neither Fi_{24}' nor Fi_{24} acts distance-transitively on the orbital graphs corresponding to these two smallest nontrivial subdegrees.

To compute the remaining subdegrees, we once again consider the action of Fi_{24}' on the class of 306936 3-transpositions in Fi_{24}. The first problem is to construct permutations generating a subgroup $H \cong 3^7{\cdot}O_7(3)$. We do this in a somewhat roundabout manner. First we obtain elements t and s of Fi_{24}' of classes $2B$ and $3E$ respectively. Searching at random through the conjugates of t (as described in [Linton and Wilson 1991]) we find some conjugates which, together with s, generate subgroups isomorphic to $L_2(7)$. In each of these there is an involution inverting s. Taking a number of these involutions we obtain generators for $N_{\mathrm{Fi}_{24}'}(s) \cong 3^2{:}\,2 \times G_2(3)$. The normal subgroup 3^2 of this group contains an element r of class $3A$, which can easily be computed. This element r, together with s and one of the conjugates of t that generates an $L_2(7)$ with s (of a particular class) generate the required subgroup H.

There are just three orbits of H on the 306936 transpositions, of sizes 1134, 30240 and 275562. We let Δ_1 be the smallest orbit and Δ_2 the second-smallest. As above we let $n_1(g) = |\Delta_1 \cap \Delta_1^g|$ and $n_2(g) = |\Delta_2 \cap \Delta_1^g|$. We now test a number of random elements g of Fi_{24} and record the values of $(n_1(g), n_2(g))$ that arise: see Table 2. We also record how many times each pair is encountered. We find 13 distinct pairs.

i	$n_1(g_i)$	$n_2(g_i)$	$\lvert H : S_i'\rvert$	#enc.	#exp.
1	120	0	275562	1	0
2	3	429	816480	3	1
3	18	198	21228480	49	33
4	30	60	57316896	80	91
5	42	0	62178597	96	99
6	9	165	286584480	470	457
7	15	96	2901667860	5234	4636
8	1	140	5158520640	8217	8242
9	13	80	6964002864	11047	11127
10	0	119	9183300480	29443	14673
11	6	102	15475561920	24877	24727
12	3	117	23213342880	36868	37091
13	4	112	52230021480	83615	83455

TABLE 2. Pairs $(n_1(g), n_2(g))$ for the representation of Fi_{24}' on the cosets of $3^7{\cdot}O_7(3)$, the corresponding indices, and the number of times each pair is encountered (fifth column). The last column lists the "expected" number of encounters, $200000\,|H : S_i'|/|\mathrm{Fi}_{24}' : H|$.

The known orbits, together with the ones in the table, leave 9613177200 points unaccounted for, which is about 7% of the total. It seems most unlikely that our search (of 200000 elements) would have missed orbits containing this many points, so we can presume that we have failed to discriminate them from some of the orbits that we have found. That is to say, some pairs (n_1, n_2) correspond to two or more orbits. To form a conjecture as to which pairs this might be we look at how often each pair was encountered, compared to the size of the orbit known to correspond to it. If each pair corresponded to just one orbit we would expect to find pair i about $200000\,|H : S_i'|/|\mathrm{Fi}_{24}' : H|$ times. We tabulate these numbers in Table 2 as well.

These numbers show clearly that pairs 7 and 10 deserve further attention. In each case we generate a number (say 10) of elements g with the appropriate $(n_1(g), n_2(g))$ and use backtrack methods to see whether or not the corresponding sets Δ_1^g actually lie in the same orbits of H. We find two new orbits by this method, accounting for 429876720 (pair 7) and 918330048 (pair 10) points.

This accounts for all remaining points, so the bounds we have are exact. The fusion of suborbits of equal length in the case of Fi_{24} can be seen from the permutation character, or by observing that the corresponding 2-point stabilisers are subgroups $L_2(13)$, which are conjugate in $3^7{\cdot}O_7(3){:}\,2$. □

0	1120	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	39	351	0	729	0	0	0	0	0	0	0	0	0	0	0	0
0	8	32	0	216	864	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	80	0	0	0	1040	0	0	0	0	0	0	0	0
0	1	13	27	65	312	0	0	702	0	0	0	0	0	0	0	0
0	0	2	0	12	80	54	0	324	162	0	486	0	0	0	0	0
0	0	0	0	0	20	20	0	0	0	0	540	0	540	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1120	0
0	0	0	1	2	24	0	0	139	36	243	432	0	0	0	243	0
0	0	0	0	0	8	0	0	24	80	0	144	0	216	0	648	0
0	0	0	0	0	0	0	0	24	0	96	192	0	32	0	200	576
0	0	0	0	0	2	6	0	24	12	108	176	0	90	0	216	486
0	0	0	0	0	0	0	0	0	0	0	0	80	80	240	240	480
0	0	0	0	0	0	2	0	0	6	6	30	36	272	216	120	432
0	0	0	0	0	0	0	0	0	0	0	0	91	182	210	182	455
0	0	0	0	0	0	0	3	3	12	25	48	72	80	144	301	432
0	0	0	0	0	0	0	0	0	0	32	48	64	128	160	192	496

0	0	49140	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	351	1404	0	9477	37908	0	0	0	0	0	0	0	0	0	0	0
1	32	534	729	1728	8208	5832	0	23328	8748	0	0	0	0	0	0	0
0	0	130	260	0	4160	0	0	8320	4680	31590	0	0	0	0	0	0
0	13	104	0	897	3900	0	0	14742	4212	0	25272	0	0	0	0	0
0	2	19	54	150	1152	432	0	4806	1944	6561	15066	0	5832	0	13122	0
0	0	5	0	0	160	240	0	1080	1350	5265	2700	2430	10800	0	3240	21870
0	0	0	0	0	0	0	280	0	1680	3500	0	10080	4480	0	8960	20160
0	0	4	8	42	356	216	0	2371	972	4860	9936	0	1944	0	10935	17496
0	0	1	3	8	96	180	243	648	882	2187	3312	3402	6264	3888	12960	15066
0	0	0	3	0	48	104	75	480	324	3362	3840	3312	4960	2880	8800	20952
0	0	0	0	4	62	30	0	552	276	2160	5052	972	4446	5184	9828	20574
0	0	0	0	0	0	20	90	0	210	1380	720	4770	6760	6240	8640	20310
0	0	0	0	0	8	40	18	36	174	930	1482	3042	7944	7560	7764	20142
0	0	0	0	0	0	0	0	0	91	455	1456	2366	6370	8918	8918	20566
0	0	0	0	0	12	8	24	135	240	1100	2184	2592	5176	7056	10381	20232
0	0	0	0	0	0	24	24	96	124	1164	2032	2708	5968	7232	8992	20776

Collapsed adjacency matrices for the orbital graphs corresponding to the two smallest subdegrees greater than 1, for the action of Fi_{24} on $3^7{\cdot}O_7(3){:}\,2$. Suborbits are ordered in increasing order of length.

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