Phase Portraits of Planar Vector Fields: Computer Proofs

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CONTENTS

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We present an algorithm for computer verification of the global structure of structurally stable planar vector fields. Constructing analytical proofs for the qualitative properties of phase portraits has been difficult. We try to avoid this barrier by augmenting numerical computations of trajectories of dynamical systems with error estimates that yield rigorous proofs. Our approach lends itself to high-precision estimates, because the proofs are broken into independent calculations whose length in floating-point operations does not increase with increasing precision. The algorithm is tested on a system that arises in the study of Hopf bifurcation of periodic orbits with 1:4 resonance.

1. INTRODUCTION

Henri Poincaré initiated the geometric analysis of phase portraits of planar vector fields in his dissertation [Poincaré 1880]. The concept of structural stability [Andronov and Pontryagin 1937] formalizes the idea that a phase portrait is qualitatively unchanged by perturbations. On compact orientable two-dimensional manifolds, structurally stable vector fields have a finite number of equilibrium points and periodic orbits, and all trajectories approach these invariant sets in forward and backwards time [Peixoto 1962]. The phase portraits of these structurally stable vector fields can be classified by graphs with vertices at the equilibria and periodic orbits, labeled by their stability type, and edges located near separatrices that connect saddles to sinks and sources. Numerical computation of the data contained in these graphs is often a routine matter and a reliable one, unless the vector fields are near structural instability. On the other hand, rigorous verification that numerically computed phase portraits are correct has seldom

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been accomplished, or even attempted. This paper presents a general approach to the problem of producing computer-generated proofs for the correctness of phase portraits for structurally stable vector fields. This approach was initiated by Salvador Malo [1993], and we develop it further. The algorithms we describe have been implemented and tested with a few examples. They have been effective in producing the desired proofs, perhaps far more so than other methods that have been attempted with these types of problems.

The difficulty with producing rigorous bounds on the location of trajectories lies in the growth of error estimates during highly iterative procedures. Numerical integration algorithms of a typical trajectory may involve thousands of time steps in such an iterative procedure. Controlling round-off errors of floating-point computations in such a process is problematic. Successful efforts to prove statements about the approximate location of trajectories have tended to rely upon very high numerical precision of the numerical integration as an antidote to the rapid growth of error estimates. Such precise numerical analysis has been based on the use of interval arithmetic. The basic operations of interval arithmetic produce intervals that contain the range of a function on rectangular domains. Our methods for proving properties of planar vector fields also make use of interval arithmetic, but they do so in a much more limited way. In particular, we refrain from iterative calculations with interval arithmetic. Geometric structures are computed that we expect to possess qualitative properties (like transversality) with respect to the original vector field. Validation of these properties utilizes interval arithmetic, but the computational complexity of each unit of interval arithmetic computation is independent of time steps or the resolution with which piecewise smooth objects are computed. When finer meshes are used to compute these piecewise smooth objects, the number of independent estimates that are performed with interval arithmetic increases, but the number of operations in each estimate remains unchanged.

The problems that we discuss have been very resistant to analysis. For example, Hilbert's sixteenth problem [Hilbert 1902] concerning bounds for the number of limit cycles of planar polynomial vector fields appears to be far from being solved, even for quadratic vector fields. There are many examples of vector fields whose phase portraits have been established through rigorous arguments, but the process of deriving a phase portrait from an analytic expression of a vector field remains mysterious. Indeed, the difficulty of the subject has been deceptive. Many published results and proofs have been later discovered to be flawed, including Dulac's work [1923] on singular cycles of analytic planar vector fields and the work of Petrovskii and Landis on Hilbert's sixteenth problem. Existing analytic arguments to reestablish the proof of Dulac's Theorem are complex and delicate [Il'yashenko 1991; Écalle 1993].

The logical structure of our arguments may appear to be somewhat confusing in relation to ideas of "constructive" mathematics, and a few comments at the outset may help put the matter into perspective. We regard the computer as an "oracle" which we ask questions. Questions are formulated as input data for sets of calculations. There are two possible outcomes to the computer's work: either the calculations rigorously confirm that a phase portrait is correct, or they fail to confirm it. The application of interval arithmetic to a given input data set involves a bounded number of operations that can be readily estimated. Thus, this phase of the analysis does not suffer from a halting problem. In the second case, one can change the input data and try again. The theory that we present states that if one begins with a structurally stable vector field, there is input data that will yield a proof that a numerically computed phase portrait is correct. However, this fails to be completely conclusive from an algorithmic point of view, because one has no way of verifying that a vector field is structurally stable in advance of a positive outcome. Thus, if one runs a set of trials of increasing precision, the computer will eventually produce a proof of the correctness of a phase portrait for a structurally stable vector field. Presented with a vector field that is not structurally stable, the computation will not confirm this fact; it will only fail in its attempted proof of structural stability. However, note that the numerical precursors to the interval arithmetic calculations can also be expected to fail to produce trial data as input for the interval arithmetic procedures. Pragmatically, we terminate the calculation when the computer produces a definitive answer or our patience is exhausted.

The situation described in the previous paragraph is analogous to the question of producing a numerical proof that a continuous function has a zero. If a function changes sign, then computing values with sufficient precision will determine this fact. The intermediate value theorem completes the proof that the function has a zero. If there is a zero with a local maximum or minimum at a number that is not explicitly computable, we will not be able to determine the existence or nonexistence of a zero by a computation of fixed length. For example, the functions

$$f_\varepsilon(x) = (x-\pi)^2 + \varepsilon$$

will be indistinguishable by numerical calculations of fixed length for numbers ε of sufficiently small magnitude. Numerical proofs that a function vanishes can be expected to succeed only when the function has qualitative properties that can be verified with finite-precision calculations.

From an abstract perspective, the problem of verifying the correctness of the phase portrait of a structurally stable planar vector field seems trivial: simply increase the accuracy of error estimates for numerically computed trajectories. On further reflection, the matter appears more complicated. The numerical computations involve both truncation and round-off errors. As truncation errors are reduced by decreasing step sizes, the number of arithmetic operations and error estimates may grow, due to the larger number of operations. Thus, simultaneous increase in the number of step sizes and the floating-point precision of individual operations might not produce error estimates for trajectories that improve with increasing precision in the calculations. The number of arithmetic operations in the individual computer-generated estimates used in this paper do not increase in length with increasing precision. Thus, they are an improvement upon estimates for the accuracy of a numerical integration. With increasing precision, more estimates need to be made, but the number of arithmetic operations in each remains fixed. Still, there is a dependency of the proofs on numerical integration, since the input data for the verification routines comes from numerical integration. For our procedures to work with vector fields that approach the boundary of structural stability, the (unverified) accuracy of this data must improve with increasing precision. It is the procedures for producing proofs from accurate numerical integrations that are free from the uncontrolled growth of round-off errors with increasingly fine discretizations.

This paper treats only the case of planar vector fields, though the methods are applicable with increasing complexity to higher-dimensional systems. We chose this restriction for two reasons. First, structural stability is not a dense property in the space of vector fields on manifolds of dimension larger than two. Second, the topological and computational complexity increase rapidly with dimension, unless the dynamics of a high-dimensional system reduce to those of a low-dimensional system. This is particularly true when a system possesses chaotic invariant sets.

Faced with these difficulties, we have endeavored to present our strategy in the simplest possible setting. As is evident from the general lack of progress on Hilbert's sixteenth problem, the mathematical questions associated with this domain of problems are still formidable. Farzaneh [1995] has succeeded in using methods based on the strategy presented here to validate the existence of stable periodic orbits in the three-dimensional Lorenz system [Lorenz 1963].

2. BACKGROUND

We recall facts about planar vector fields in this section. Two flows are *topologically equivalent* if there is an orientation-preserving homeomorphism mapping trajectories of one onto trajectories of the other. A flow is said to be *structurally stable* if C^1 perturbations of the flow are topologically equivalent to the original flow. Structurally stable flows on orientable compact two-dimensional manifolds were characterized by Peixoto [1962], following earlier work for flows on the disk by DeBaggis [DeBaggis 1952; Peixoto and Peixoto 1959]:

Theorem 2.1. *A C^r vector field on a compact two-dimensional manifold is structurally stable if and only if*:

1. *there are a finite number of equilibrium points and periodic orbits, each hyperbolic*;
2. *no trajectories connect saddle points*;
3. *the nonwandering set of the flow consists entirely of equilibrium points and periodic orbits.*

Remark 2.2. We shall focus most of our attention on planar polynomial vector fields. Since the plane is not compact, one has the choice of studying the properties of these vector fields in a compact region [Peixoto and Peixoto 1959] or compactifying the vector fields to polynomial line fields on the projective plane or vector fields on the two-dimensional sphere [Lefschetz 1957]. Since the concept of structural stability only deals with the geometry of the singular foliations produced from a vector field or line field, we can apply Peixoto's theorem in this setting. In particular, we shall be concerned with planar vector fields that have a finite number of equilibria and periodic orbits, each hyperbolic, and have no saddle connections. Ignoring the issues of compactification, we shall call these vector fields structurally stable.

Definition 2.3. The *spine* of a planar vector field with hyperbolic equilibria is the set consisting of its equilibrium points, periodic orbits and the stable and unstable manifolds of the saddle points. Two spines are topologically equivalent if there is a homeomorphism of the plane mapping one to the other, preserving the stability types of equilibria and periodic orbits.

Theorem 2.4 [Peixoto 1973]. *If two planar vector fields with hyperbolic equilibria have topologically equivalent spines, they are topologically equivalent.*

In accord with this result, the main thrust of our work is to determine the topological equivalence class of a spine. The spine of a vector field is almost a graph. The limit sets of the saddle separatrices are either equilibrium points or periodic orbits. Spines fail to be graphs because separatrices tending to periodic orbits have infinite length. Still, a spine is a finite union of curves and equilibrium points that we shall regard as a combinatorial and topological object. We shall say that we have (rigorously) determined the phase portrait of a planar vector field if we have proved that its phase portrait lies in a specific topological equivalence class. From a computational point of view, what needs to be done is to verify the number of equilibria and their stability types, as well as the number and stability types of periodic orbits, and to confirm the topological location of the saddle separatrices in the complement of the equilibria and periodic orbits.

The least tractable part of determining the phase portrait of a planar vector field involves periodic orbits. Since all hyperbolic periodic orbits of a planar system are either stable or unstable, they can be located by forwards and backwards numerical integrations. The difficulty lies in confirming that these numerical integrations are correct. There are three results about planar vector fields that we shall use in these arguments: The Poincaré–Bendixson Theorem, Duff's theory of rotated vector fields, and Floquet theory for periodic orbits.

The Poincaré–Bendixson Theorem is a fundamental result concerning the limit sets of trajectories for planar vector fields. One statement of it is the following.

Theorem 2.5 [Hirsch and Smale 1974]. *A nonempty, compact limit set of a C^1 planar vector field that contains no equilibrium point is a periodic orbit.*

Our application of the Poincaré–Bendixson Theorem will be based upon surrounding a numerically computed (un)stable periodic orbit with an annulus that has the property that the vector field points transversally out of or into the annulus on its boundary. We formalize this concept as follows:

Definition 2.6. If X is a planar vector field, an annulus $A \subset \mathbb{R}^2$ is called a *transverse annulus* if

1. the boundary of A is a piecewise C^1 curve,
2. X is transverse to the boundary of A with X pointing out of A on both boundary components or pointing into A on both boundary components of A, and
3. X has no equilibrium points in A.

At a point where the boundary of A is not smooth, transversality with respect to the boundary is defined by the requirement that X or $-X$ lie in the sector bounded by the left and right tangents of the boundary curves that points towards the interior of the annulus. The Poincaré–Bendixson Theorem implies immediately that a transverse annulus contains a periodic orbit of X. We make a further definition:

Definition 2.7. A transverse annulus A for X that contains a single periodic orbit γ is called an *isolating annulus* for γ.

Our computations of transverse annuli will be accomplished in two different ways, both relying on additional results about vector fields in two dimensions. The first of these methods is based upon the theory of rotated vector fields [Duff 1953]. Consider the family of planar vector fields

$$\begin{aligned}\dot{x}_1 &= \cos\theta\, f_1(x_1,x_2) - \sin\theta\, f_2(x_1,x_2)\\ \dot{x}_2 &= \sin\theta\, f_1(x_1,x_2) + \cos\theta\, f_2(x_1,x_2),\end{aligned}$$

which is obtained by rotating the vector field $\dot{x} = f(x)$. Duff [1953] proves that the flow of the rotated vector fields has the following properties:

1. The equilibria of the rotated vector fields are independent of θ.
2. If θ is not a multiple of π, the flow of the rotated vector field is transverse to that of $\dot{x} = f(x)$, except at their equilibria.
3. Hyperbolic limit cycles of the rotated vector fields vary continuously and monotonically with θ.
4. If $\theta_1 - \theta_2$ is not a multiple of π, the limit cycles of the rotated vector fields for θ_1 and θ_2 are disjoint.

Rotated vector fields will be used in the computation of regions that bound saddle separatrices for structurally stable vector fields. Recall that the α- and ω-limit sets of a trajectory are its limit points as $t \to \mp\infty$. Each stable manifold of a saddle has α-limit set that is an unstable equilibrium or an unstable periodic orbit. Similarly, each unstable manifold of a saddle has ω-limit set that is a stable equilibrium or a stable periodic orbit. Our goal is to verify that the α-limit sets of stable manifolds and the ω-limit sets of unstable manifolds are the equilibrium points or periodic orbits found in a numerical computation. Assume that we have computed an isolating annulus for each periodic orbit and a disk surrounding each (un)stable equilibrium that lies in its (un)stable manifold. The eigendirections of a saddle point in a rotated family vary monotonically with the angle of rotation. Moreover, the separatrices of one member of a rotated family are transverse to the trajectories of other members of the family. Consider a segment σ_s of a stable manifold defined on the time interval $[-T,\infty)$ or a segment σ_u of an unstable manifold defined on the time interval $(-\infty,T]$ that extend to the isolating neighborhood U of its α- or ω-limit set. For small rotations, the corresponding separatrix segments of the rotated vector fields will be mutually disjoint and enter U. Therefore the separatrices of rotated vector fields with small positive and negative rotation angles bound a strip S in the complement of U so that $S \cup U$ contains the separatrix of the original vector field. Moreover, the trajectories of rotated fields lying on the boundary of S make a constant angle with the original vector field. Verifying the transversality of the boundary of S suffices to prove that the original saddle separatrix enters U. Use of rotated vector fields to rule out saddle connections of a planar vector field has been discussed extensively by Malo [1993].

Duff's theory implies that limit cycles of rotated vector fields for small positive and negative angles of rotation will form the boundary of an isolating annulus for a periodic orbit. Since the property of being a transverse annulus is an open property (with respect to the C^1 topology of its boundary components), we expect that numerical approximations to the limit cycles of rotated vector fields will form satisfactory boundaries for transverse annuli. However, verification that an annulus is isolating for a limit cycle seems to be more complex than verifying the transversality of the boundary components, so we use another technique to verify uniqueness of the limit cycles in annulus. The method entails the introduction of special coordinate systems that we call Floquet coordinates.

Definition 2.8. Let γ be a limit cycle for a planar vector field X. An annular coordinate system $\psi : S^1 \times \mathbb{R} \to \mathbb{R}^2$ with coordinates (u, v) is called a system of *Floquet coordinates* for γ if it has the following properties:

1. If $x \in \gamma$, then ψ is defined in a neighborhood of x, $v = 0$, $\dot v = 0$ and $\dot u = 1$.
2. $\partial\psi/\partial u$ and $\partial\psi/\partial v$ are perpendicular at $(u, 0)$.
3. For u fixed, $\psi(u, v)$ is an affine function of v.
4. The variational equations $(D\psi)^{-1}(DX)(D\psi)$ of X in the ψ coordinates are constant along the u axis $\psi^{-1}(\gamma)$.

The first three of these conditions state that ψ is a normal bundle to γ.

Theorem 2.9 [Hartman 1982, Theorem 6.1, p. 60]. *A periodic orbit of a smooth vector field has a system of Floquet coordinates.*

Dulac's Criterion [Andronov et al. 1966, p. 305] states that a two-dimensional vector field can only have a single limit cycle in an annulus on which the divergence of the vector field is constant. Thin strips surrounding the coordinate axis in a system of Floquet coordinates for a hyperbolic periodic orbit satisfy this criterion and form isolating annuli for the periodic orbit.

In the next section, we describe an algorithm for numerically computing Floquet coordinates.

3. NUMERICAL COMPUTATION OF FLOQUET COORDINATES

As input for the interval arithmetic computations we describe later, we need a representation of an approximate Floquet coordinate system. Let $g : S^1 \to \mathbb{R}^2$ be a periodic orbit of the planar vector field X defined by the equation $\dot x = f(x)$. Denote by $f^\perp(x)$ the vector obtained by rotating $f(x)$ by $\pi/2$. A parametrization of the normal bundle of g is given by $\psi(u, v) = g(u) + k(u) f^\perp(u) v$ for any smooth positive function $k : S^1 \to \mathbb{R}$. Suitable choices of k will make ψ^{-1} a Floquet system of coordinates in a neighborhood of g. These k are determined by the following differential equation.

Theorem 3.1. *If $\dot x = f(x)$ defines a planar vector field X with periodic orbit g and $\psi(u, v) = g(u) + v\exp(\beta(u)) f^\perp(u)$ is a parametrization of a tubular neighborhood of g, the divergence of X in the (u, v) coordinates is given by*

$$-\frac{d\beta}{du} + \frac{f^\perp Df f^\perp}{f \cdot f} - \frac{f Df f}{f \cdot f}$$

along the periodic orbit $v = 0$.

Proof. The vector field X in the (u, v) coordinates is $(D\psi)^{-1} f \circ \psi$. Setting $h = \exp(\beta) f^\perp$, we compute

$$D\psi = \begin{pmatrix} g_1'(u) + v h_1'(u) & h_1(u) \\ g_2'(u) + v h_2'(u) & h_2(u) \end{pmatrix}$$

and

$$(D\psi)^{-1} = \frac{1}{\det} \begin{pmatrix} h_2(u) & -h_1(u) \\ -(g_2'(u) + v h_2'(u)) & g_1'(u) + v h_1'(u) \end{pmatrix},$$

with

$$\det = (g_1'(u) + v h_1'(u)) h_2(u) - (g_2'(u) + v h_2'(u)) h_1(u).$$

Thus $(D\psi)^{-1}f$ equals

$$\frac{1}{\det}\begin{pmatrix} h_2(u)f_1 - h_1(u)f_2 \\ -(g_2'(u)+vh_2'(u))f_1 + g_1'(u) + vh_1'(u)f_2 \end{pmatrix}$$

and $\operatorname{div}((D\psi)^{-1}f\circ\psi)$ equals

$$\frac{\partial}{\partial u}\Big(\frac{h_2(u)f_1 - h_1(u)f_2}{\det}\Big) + \frac{\partial}{\partial v}\Big(\frac{-(g_2'(u)+vh_2'(u))f_1 + (g_1'(u)+vh_1'(u))f_2}{\det}\Big).$$

Along the periodic orbit $v=0$,

$$\begin{pmatrix} g_1' \\ g_2' \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

and $\det = h_2f_1 - h_1f_2 = \exp(\beta)f\cdot f$. Therefore $v=0$ implies $-g_2'f_1 + g_1'f_2 = 0$ and

$$\begin{aligned}
\operatorname{div}&((D\psi)^{-1}f\circ\psi)\\
&= \frac{\partial}{\partial v}\Big(\frac{-g_2'(u)f_1 + g_1'(u)f_2}{\det}\Big) - \frac{h_2'f_1 - h_1'f_2}{\det}\\
&= \frac{-g_2'(\partial f_1/\partial v) + g_1'(\partial f_2/\partial v)}{\exp(\beta)f\cdot f} - \frac{h_2'f_1 - h_1'f_2}{\det}\\
&= \frac{f^\perp\cdot Dff^\perp}{f\cdot f} - \frac{f\cdot Dff}{f\cdot f} - \beta'.
\end{aligned}$$

The last calculation uses the fact that

$$\frac{\partial}{\partial v}(f\circ\psi) = Dfh$$

and

$$h' = (\exp(\beta)f^\perp\circ\psi)' = \beta'h + \exp(\beta)(Dff)^\perp$$

along the periodic orbit $v=0$. □

This theorem gives a variational equation that we use for the computation of Floquet coordinates. In the Floquet coordinates, $\operatorname{div}((D\psi)^{-1}f\circ\psi)$ is constant along the periodic orbit. Since the Lyapunov exponent λ is the integral of this quantity along the periodic orbit, we must have

$$\beta' = \frac{f^\perp\cdot Dff^\perp}{f\cdot f} - \frac{f\cdot Dff}{f\cdot f} - \frac{\lambda}{T},$$

where T is the period of the periodic orbit. If values for λ and T are known, the data needed for the computation of Floquet coordinates is obtained by integration of this differential equation along with the equation $\dot{x} = f(x)$. If numerical integration of $\dot{x} = f(x)$ together with its variational equation $\dot{\xi} = Df(\xi)$ produces an approximation of the limit cycle, its period and Lyapunov exponent, then a further numerical integration of the equation for β produces all of the data required for construction of a coordinate system that approximates the Floquet coordinates.

4. INTERVAL ARITHMETIC

The numerical data generated by numerical integrations of the original vector field, its variational equations, rotations and the associated equation for constructing Floquet coordinates, are discrete representations of continuous objects. Proofs of global properties for planar vector fields based on transversality rely upon smooth functions. The transition from discrete data to smooth functions is made by interpolation. Piecewise polynomial and rational functions will be created from the numerical data, and their properties investigated with interval arithmetic. In all cases, the desired proofs will be reduced to a set of computations showing that the ranges of the constructed functions do not contain zero. This is *the* standard problem of interval arithmetic, and there is substantial theory and practice in organizing these range computations to be as accurate and efficient as possible. The rigorous bounds of the range computations achievable in the implementation of the algorithms described in this paper are important, but incidental to the theory. This section assumes that if F is a rational function that is positive on a rectangle in the plane, this fact can be verified using interval arithmetic computations.

The numerical implementation of interval arithmetic is based upon a set of functions $F_j(I_1,\ldots,I_k)$ whose arguments and values are intervals of real numbers. The intervals may have zero length. For each F_j, there is a real-valued function $\bar{F}_j:\mathbb{R}^k\to\mathbb{R}$ with the property that if $x_i\in I_i$ for $i=1,\ldots,k$, then $\bar{F}_j(x_1, \dots x_k) \in F_j(I_1, \dots, I_k)$. For the interval arithmetic computations used in this paper, it suffices to take F_j that correspond to the basic arithmetic operations of addition, subtraction, multiplication and division. Directed rounding operations from the ANSI/IEEE 754-1985 Arithmetic Standard are used in an implementation of the functions F_j. To program the interval arithmetic calculations, a lexical analysis of compound arithmetic expressions written in the computer language C (produced by the Unix utility functions lex) is transformed (with the Unix utility yacc) to programs built upon a small library of interval arithmetic evaluations of the basic arithmetic operations. The libraries and lex/yacc programs that we use were written by Salvador Malo [1993].

We perform two types of calculations with interval arithmetic. In the first, the transversality of a vector field $\dot{x} = f(x)$ to a piecewise polynomial curve is computed. The piecewise polynomial curve $\gamma(u)$ comes either from numerical integration of a rotated vector field, or from an explicitly defined curve in a piecewise polynomial coordinate system. In either case the transversality calculation for a polynomial vector field amounts to verifying that the functions $\gamma'(u) \cdot f(\gamma(u))$ do not vanish. If f is polynomial, this requires computation of the range of the piecewise polynomial function $\gamma'(u) \cdot f(\gamma(u))$. Algebraic methods could be used as an alternative to interval arithmetic computation for this purpose, but we do not explore these alternatives here.

Our second type of interval arithmetic computation begins with piecewise polynomial coordinate transformation of a vector field, and then evaluates expressions involving the derivatives of the vector field in the transformed coordinates. If $\dot{x} = f(x)$ is a polynomial vector field and $(\psi)^{-1}$ is a polynomial coordinate system for a region of the plane, the transformed vector field $((D\psi)^{-1} f \circ \psi)$ and its divergence are rational functions. Therefore, interval arithmetic evaluation of these expressions can be carried out as a sequence of interval arithmetic evaluations of basic arithmetic operations. The goal is to compute the integral of the divergence of a vector field in Floquet coordinates. We encounter a small technical problem here: if the transformed vector field is only piecewise smooth, its divergence may be singular. Discontinuous changes in the vector field may contribute to the stability of the limit cycle. To obtain a continuous transformed vector field having divergence with only simple jump discontinuities, the coordinate transformation should be C^1.

To reduce the complexity of the divergence calculations, we rescale the vector field slightly so that $\dot{v}$ is identically 1. Thus a continuous line field transverse to a periodic orbit is preserved, and the divergence of the vector field becomes $\partial\dot{v}/\partial v$. Using this approach, we can use C^0 coordinate systems, since all that needs to be calculated is whether the flow expands or contracts in the v coordinate direction.

We use coordinate patches that are based upon cubic interpolations of the vector field and have total degree four in both coordinates. The cubic interpolations employed here rely upon the availability of the tangent direction to trajectories from evaluation of the vector field. Let $\dot{x} = f(x)$ be a smooth vector field, and let $p_0, \dots, p_n$ be $n+1$ points obtained from application to f of a numerical integration procedure with constant time step Δ. Construct a C^1 cubic interpolating polynomial that passes through the computed points p_n and has derivatives $f(p_n)$ at these points. If the time interval of this trajectory segment is translated so that its origin becomes $\frac{1}{2}(2n+1)\Delta$, the domain of the function interpolating between p_n and p_{n+1} will be $[-\frac{1}{2}\Delta, \frac{1}{2}\Delta]$. The coefficients of this interpolating polynomial $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ are

$$\begin{aligned}
a_0 &= \tfrac{1}{2}(p_{n+1} + p_n) - \tfrac{1}{8}\Delta(f(p_{n+1}) - f(p_n)),\\
a_1 &= \tfrac{3}{2}\Delta^{-1}(p_{n+1} - p_n) - \tfrac{1}{4}(f(p_{n+1}) + f(p_n)),\\
a_2 &= \tfrac{1}{2}\Delta^{-1}(f(p_{n+1}) - f(p_n)),\\
a_3 &= -2\Delta^{-3}(p_{n+1} - p_n) + \Delta^{-2}(f(p_{n+1}) + f(p_n)).
\end{aligned}$$

This interpolation formula is used to compute a piecewise cubic curve $g(u)$ approximating the numerically computed periodic orbit of the vector field $\dot{x} = f(x)$. A piecewise polynomial coordinate system $\psi(u,v) = g(u) + v\,h(u)$ is then constructed, where the curve $h(u)$ has the form $c(u)\,(g^{\perp})'(u)$ and $c(u)$ is the linear interpolation between the numerically computed values for $\exp\beta(p_n)$ and

$$\exp\beta(p_{n+1}).$$

Thus h and ψ are cubic functions of u, and ψ is affine in v. When computing the transformed vector field $(D\psi)^{-1}f \circ \psi$, the common denominator of this rational function is $\det((D\psi))$. To simplify our calculations, the vector field is scaled so that the u component is identically 1. Thus the rescaled vector field is $\tilde{f} = (1,\, dv/du)$. The trajectories can be parametrized by u and the lines given by constant u and varying v are preserved by the flow of $\tilde{f}$. Note that $\tilde{f}$ is still a rational vector field if f is polynomial. The degree of $\tilde{f}$ is bounded by $b = (\deg f + 1)(\deg \psi) - 1$. To demonstrate the existence of a periodic orbit of $\tilde{f}$, it suffices to determine the sign of the v component of $\tilde{f}$ along curves defined by $v = \pm\varepsilon$. If this component never vanishes on the two components but has opposite sign on the two, the Poincaré–Bendixson Theorem implies the existence of a limit cycle in the annulus bounded by these two curves.

Since the functions $h(u)$ have values determined by $f(p_n)$, the derivatives $h'(u)$ may be discontinuous as we move from one patch to the next. Thus the transformed vector field may be discontinuous, and the derivative of its return map might have contributions coming from the discontinuities in addition to the integral of the divergence along trajectories. Nonetheless, the integral of the divergence of $\tilde{f}$ determines whether segments parallel to the v axis are expanded or contracted. Note that the divergence of $\tilde{f}$ is $\delta(u,v) = \partial(dv/du)/\partial v$. Flowing from the boundary of one coordinate patch to the next, the integral of $\delta(u,v)$ calculates the rate at which infinitesimal trajectory segments in the v direction are contracted or expanded. If δ does not change sign in a neighborhood of the limit cycle, all initial conditions are contracted ($\delta < 0$) or expanded ($\delta > 0$) towards each other in the v direction. In particular, only one limit cycle can occur in the neighborhood.

5. AN EXAMPLE

We discuss the application of the procedure described in the previous section to prove the existence and uniqueness of a limit cycle for the cubic vector field

$$\dot{z} = \exp(i\theta)z - (\gamma + i\alpha)z\,|z|^2 + \bar{z}^3,$$

with parameter values $(\theta,\gamma,\alpha) = (1.0553, 0.09, 1.2)$. This family of vector fields arises in the study of Hopf bifurcation of periodic orbits with characteristic multipliers $\pm i$ [Arnold 1977]. There have been intensive numerical studies of the dynamics displayed by this family [Berezovskaia and Khibnik 1980; Krauskopf 1994a; 1994b], but the system has resisted analytical attempts to fully characterize its dynamics. In one region of the (θ,γ,α) parameter space, the system has a pair of concentric limit cycles. Malo [1993] used rotated vector fields to produce a pair of rigorously verified transverse annuli for these limit cycles at the parameter values $(\theta,\gamma,\alpha) = (1.0703, 0.1, 1.2)$. Uniqueness of the limit cycles in each of these transverse annuli was not proved. Malo also used rotated vector fields to prove the nonexistence of limit cycles for parameter values $(\theta,\gamma,\alpha) = (\pi/4, 2, 3)$ by proving that there are strips containing trajectories that connect the four sinks to infinity and to the source at the origin.

We study this family further by exhibiting an isolating annulus for a periodic orbit at the parameter values $(\theta,\gamma,\alpha) = (1.0553, 0.09, 1.2)$ that are close to those where there are a pair of limit cycles. The single limit cycle at these parameter values has period 3.394537 and characteristic multiplier -2.101615095. There are large segments in regions where the divergence of the vector field is positive and large segments in regions where the divergence of the vector field is negative. Figure 1

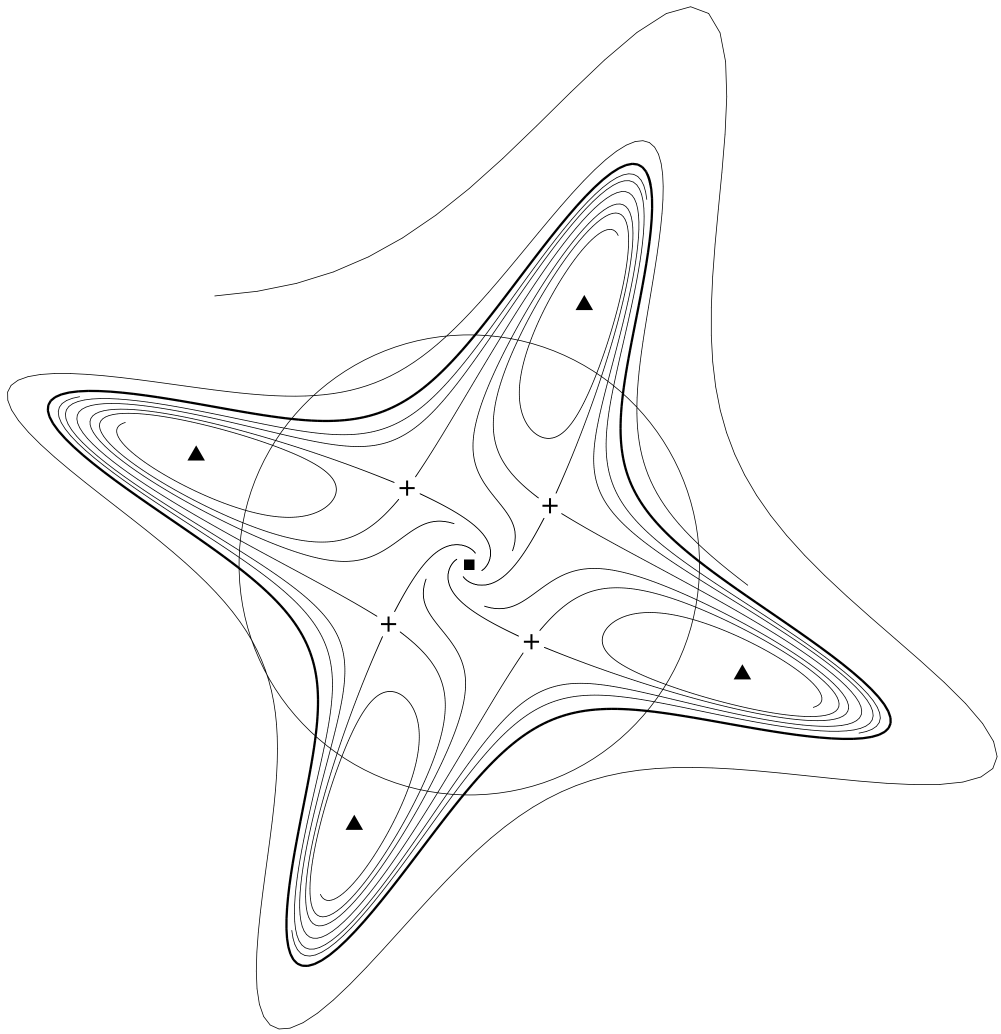

FIGURE 1. The phase portrait of the vector field $\dot{z} = \exp(i1.0553)z - (0.09 + i1.2)z|z|^2 + \bar{z}^3$, showing the saddle point separatrices and one trajectory accumulating at the limit cycle. Triangles denote sinks and the square is the source at the origin. The divergence of the vector field vanishes on the circle.

shows the limit cycle together with the circle on which the divergence of the vector field is zero. Thus, establishing the stability properties of the limit cycle requires careful analysis. We perform this analysis by computing a piecewise polynomial system of approximate Floquet coordinates. The numerical integration of the vector field is accompanied by solution of the variational equation that determines the function β used in the construction of Floquet coordinates as well as the solution of the "standard" variational equations that are used to determine the characteristic multiplier of the trajectory. The numerical integrations were performed with a fourth order Runge–Kutta algorithm with the step size $\Delta = 0.0001697265519$ adjusted so that the length of the limit cycle was 20,000 time steps. The data from these numerical calculations provided the input for the calculation of the Floquet coordinate system.

The transformation to Floquet coordinates and calculation of the divergence of the rescaled vector field was implemented in a C program that used interval arithmetic. Each segment of the numerically computed limit cycle was approximated by a cubic curve $g(u)$ whose tangent was the (numerically) computed value of the vector field at the end points of the segment. The normal bundle to this computed curve was parametrized in the form $g(u) + vk(u)g'(u)$ where $k(u)$ is a linear function that interpolates the values of $\exp\beta(u)$ numerically computed from the variational equations for the vector field. In each segment of length Δ, we performed interval arithmetic calculations to estimate the divergence of the vector field in the annulus defined by $|v| < 10^{-5}$. To estimate the range of the divergence function more accurately, we partitioned the domain $[-\frac{1}{2}\Delta, \frac{1}{2}\Delta]$ of u into ten subintervals I_j and performed a single interval calculation of the divergence for the rectangle $(u,v) \in I_j \times [-10^{-5}, 10^{-5}]$. Over much of the limit cycle these produce estimates of the divergence that lie in the range $[-0.63, -0.61]$. Near the points where the limit cycle is farthest from the origin, the estimated range of the divergence becomes larger, with extreme values in the interval $[-0.780663, -0.444572]$. Nowhere does the divergence of the transformed vector field come close to the origin relative to its average divergence, which equals -0.6191168619. We conclude that the vector field has at most one limit cycle in the annulus $|v| < 10^{-5}$. To prove the existence of a limit cycle in the annulus $|v| < 10^{-5}$, we performed interval arithmetic calculations of the v component of the transformed vector field on the boundary components of the annulus. To obtain verification of the transversality conditions in this coordinate system, we had to finely partition each segment of the the piecewise cubic curve defining the boundary between two of its knot points. Each segment was subdivided into 2000 subsegments. The computed value of the u component of the transformed vector field before rescaling is approximately 40, so the expected length of its v component is approximately $0.62 \cdot 40 \cdot 10^{-5} = 0.000248$. The computed intervals lie in the range $[-0.000381, -0.0000836]$ on the boundary $v = 0.00001$ and $[0.0000821, 0.000380]$ on the boundary $v = -0.00001$. Thus, the coordinate transformation to Floquet coordinates produces an annulus in which the interval arithmetic calculations prove that the vector field of this example has a unique limit cycle in the isolating annulus.

To complete verification of the global properties of this phase portrait, there are three remaining tasks:

1. prove that the α-limit set of the stable manifold of a saddle point is the origin,
2. prove that the ω-limit set of the unstable manifold of a saddle point has one separatrix tending to the periodic orbit and one separatrix tending to a sink, and
3. prove that there is a trajectory connecting infinity to the periodic orbit.

For each of these tasks, we use rotated vector fields to find curves transverse to the vector field that form corridors trapping the vector field in the appropriate region. In addition to the interval arithmetic calculations that proceed in a similar fashion to those for establishing the existence of an isolating annulus, it is also necessary to make local arguments about the properties of the equilibrium points, and to estimate a region at region at infinity which contains no periodic orbits.

The radial component $\frac{1}{2}(\dot{z}\bar{z} + z\dot{\bar{z}})$ of the vector field is positive inside the disk of radius 0.67 centered at the origin, so all trajectories entering this disk have the origin as α-limit set. The derivative of the vector field at the saddle point located at approximately $(0.567825, 0.43735)$ is approximately

$$\begin{pmatrix} 1.38 & -1.33 \\ -2.05 & -0.58 \end{pmatrix}.$$

The eigenvalues are approximately 2.32 and -1.52, and the corresponding eigenvectors are approximately $(0.82, -0.58)$ and $(0.42, 0.93)$. For rotation angles in the interval $(-0.1, 0.1)$, the eigenvalues remain well away from the imaginary axis and the eigenvectors stay far from the coordinate axes.

This implies that initial conditions that are vertically above and below the saddle point and initial conditions that are right and left of the saddle point can be used for the computation of corridors that will bound the saddle separatrices. Points whose distance from the saddle is approximately 0.01 suffice for these computations. Next we note that the divergence of the vector field is symmetric with respect to rotations, and a decreasing quadratic function of r that vanishes on a circle near $r = 1.655$. Therefore, any limit cycle must intersect the disk of this radius. Finally, we need an estimate for how wide the isolating annulus is, so that we can determine when points are inside the annulus. The highest point on the limit cycle occurs near $(1.153, 2.96481)$. Here the value of the vector field is approximately $(4, 0)$ and the factor $\exp \beta$ is about 0.375. Therefore the width of the isolating annulus is about 0.000015. These estimates provide sufficient data for creating corridors that connect the saddle separatrices to their limit sets and infinity to the limit cycle.

In carrying out the interval arithmetic calculations, it is helpful to consider the angles of rotation that can be used in computing the trajectories that will form the corridor boundaries. The stringent limitations that we encounter are those associated with trajectories approaching the limit cycle. To obtain trajectories that enter or leave the isolating neighborhood, the limit cycles of the rotated vector field cannot separate the isolating neighborhood from the trajectory. With a rotation angle of 10^{-5}, the limit cycle intersects the vertical line $x = 1.153$ near $y = 2.96498$. The difference is about an order of magnitude larger than the width of the isolating neighborhood that has been determined, so it is necessary to work with rotation angles that are of the order of 10^{-6} to obtain trajectories that enter the isolating neighborhood. With these small rotation angles the transversality verification for a curve $g(u)$ computed from numerical integration of the rotated vector field requires small steps. Performing these calculations by direct interval arithmetic evaluation of the cross product of the tangent vector to g with the vector field requires very small steps along the curve. If a piecewise linear curve g is used, then the variation in the direction of f along an individual curve segment should be comparable to the rotation angle ψ. For small Δ, $f(x + \Delta)$ is approximately $Df_x \cdot \Delta$. From this we estimate the change in angle along a curve segment to be

$$\Bigl(\frac{f^\perp Df_x f}{f \cdot f}\Bigr)\Delta.$$

Thus we expect that we need to take Δ comparable to $|Df|^{-1}\psi$. At $(1.153, 2.96481)$ near the limit cycle, Df is roughly 35. These estimates lead us to expect that each circuit around the limit cycle for the corridor will require on the order of 10^8 interval arithmetic evaluations. The transversality calculations along the boundary of the isolating annulus used 8×10^7 steps, so the work is comparable. The truncation and roundoff errors associated with these calculations are still small relative to the transversality condition, but the computations are lengthy. Thus, it would be helpful to find techniques that will be more effective in determining the range of the functions whose zeros signal the loss of transversality, but this was not necessary in confirming the global structure of the specific vector field studied in this paper.

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