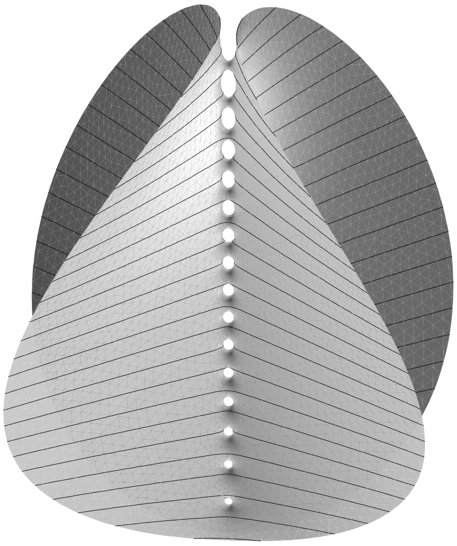
Higher-Genus Chen–Gackstatter Surfaces and The Weierstrass Representation for Surfaces of Infinite Genus

Edward C. Thayer

CONTENTS

1. The Chen–Gackstatter Surfaces and Their Generalizations
2. The Weierstrass Representation
3. The General Weierstrass Data and Period Problem for $M_{p,k}$
4. The Examples $M_{2,k}$
5. Numerical Results
6. A Global Representation of Karcher's Saddle Towers
Acknowledgements
References
Software Availability



Supported by the National Science Foundation under grants DMS-9011083 and DMS-9312087 and by the US Department of Energy under grant DE-FG02-86ER25015 of the Applied Mathematical Science subprogram of the Office of Energy Research.

Chen and Gackstatter [1982] constructed two complete minimal surfaces of finite total curvature, each having one Enneper-type end and all the symmetries of Enneper's surface. Karcher [1989] generalized the genus-one surface by increasing the winding order of the end. We prove that a similar generalization of the Chen–Gackstatter genus-two surface also exists. We describe a collection of immersed minimal surfaces that generalize both Chen–Gackstatter's and Karcher's surfaces by increasing the genus and the winding order of the end. The period problem associated with each of these surfaces is explained geometrically, and we present numerical evidence of its solvability for surfaces of genus as high as 35. We also make conjectures concerning these surfaces, and explain their motivation. Our numerical results led us to the Weierstrass data for several infinite-genus, one-ended, periodic minimal surfaces.

1. THE CHEN–GACKSTATTER SURFACES AND THEIR GENERALIZATIONS

C. C. Chen and F. Gackstatter [1982] constructed immersed minimal surfaces of genus one and two, each having one topological end of Enneper type and the same ambient symmetry group as the classical Enneper surface. These surfaces are illustrated in Figure 1.

We recall that the Enneper surface is given by the Weierstrass data (see Section 2 for definitions)

$$M = \mathbb{C}, \quad g = z, \quad \varphi_3 = dh = z\,dz.$$

Its symmetry group is generated by two planar reflectional symmetries ($z \to \bar{z}$ and $z \to -\bar{z}$ for the Weierstrass data just given) and two rotational symmetries ($z \to i\bar{z}$ and $z \to -i\bar{z}$). Each rotational

1058-6458/96 \$0.50 per page

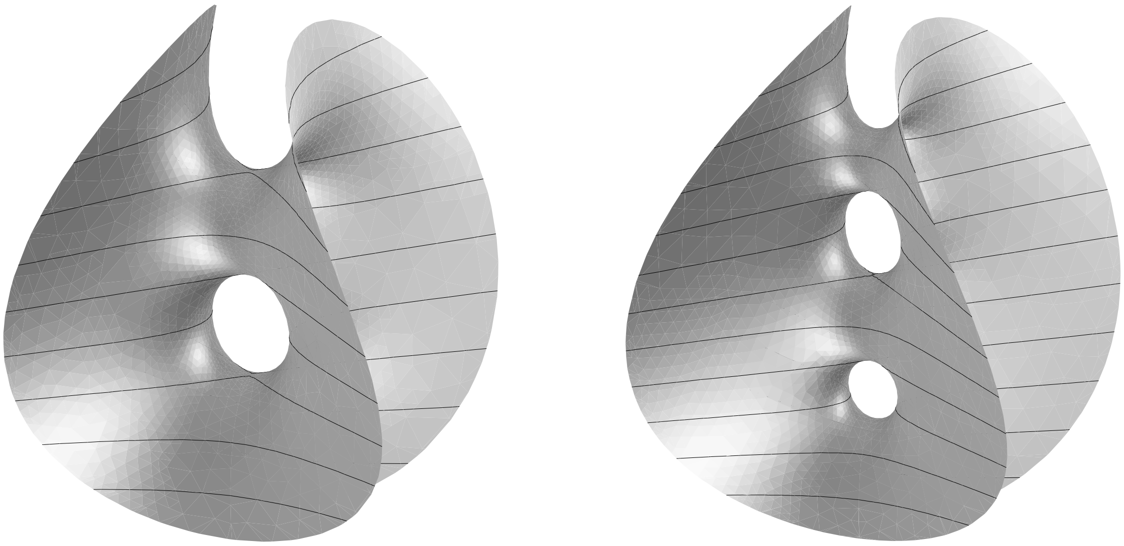


FIGURE 1. Chen and Gackstatter's surfaces of genus one (left) and genus two (right).

symmetry fixes a distinct straight line on the surface ($z = re^{\pm i\pi/4}$), and these lines meet at a single point ($z = 0$) and diverge to the end ($z = \infty$). The composition of the two reflectional symmetries is an orientation-preserving normal rotational symmetry of order two, fixing a line in $\mathbb{R}^3$ that is perpendicular to the surface and passes through the point $z = 0$.

An end of a minimal surface is said to be of Enneper type if, when the limit normal at the end is vertical, the height differential has a pole of order at least three. Such an end has winding order at least three (see definition in Section 1.1).

Karcher [1989] generalized Chen and Gackstatter's genus-one surface by increasing the order of the normal rotational symmetry from two to arbitrary $k \in \mathbb{N}$ greater than two. This increases the genus from 1 to $k-1$ and the winding order of the end from 3 to $2k-1$. This technique of increasing the order of the normal rotational symmetry has led to several new minimal surfaces: see [Hoffman and Meeks 1985; 1988; 1989; 1990; Karcher 1988; 1989], to mention only a few.

As in the case of the Hoffman–Meeks generalization of Costa's three-ended, genus-one surface, the new surfaces have higher genus, but the period problem does not increase in complexity. This is because the rotational symmetry acts as a cyclic group on the generators of the fundamental group of the new surfaces, thus reducing the period problem for the new surfaces to, essentially, the original period problem. In the same way, Karcher's generalized Chen–Gackstatter surfaces, although of genus $k-1$, have the same number of period constraints as the original surface. Applying this generalization to the genus-two Chen–Gackstatter surface suggests surfaces of genus $2(k-1)$ with one end of Enneper type. We show in Section 4 that such surfaces exist.

Chen and Gackstatter describe the Weierstrass data for a genus-three anologue of their surfaces. Combining an extension of their idea to any genus with Karcher's generalization to allow for higher winding orders on the end, we produce the Weierstrass data for a countable collection of surfaces $M_{p,k}$, where $p \geq 0$ and $k \geq 2$ are integers. $M_{p,k}$ has genus $p(k-1)$, one Enneper-type end of winding order $2k-1$, the symmetries of Karcher's generalized Enneper surface with the same winding order, and $2p+1$ finite fixed points of the normal rotational symmetry. In this notation:

- $M_{0,2}$ is Enneper's surface;
- $M_{0,k}$ for $k > 2$ are Karcher's generalizations of Enneper's surface with winding order $2k-1$;
- $M_{1,2}$ and $M_{2,2}$ are the Chen–Gackstatter surfaces of genus one and two;
- $M_{1,k}$ with $k > 2$ are Karcher's generalizations of $M_{1,2}$ with winding order $2k-1$;
- $M_{2,k}$ with $k > 2$ are generalizations of $M_{2,2}$ with winding order $2k-1$.

We will refer to these surfaces collectively as CGK surfaces. In Section 3, we describe the Weierstrass data and period problem for $M_{p,k}$, and in Section 5 we present numerical results suggesting that the period problems are solvable for $p \leq 34$, $k \leq 9$. We prove in Section 4 that the period problem in the case $p = 2$ is solvable.

See Figures 2 and 3, as well as the figure on page 19, which represents $M_{15,2}$.

1.1. Conjectures Related to $M_{p,k}$

Let $M \subset \mathbb{R}^3$ be a complete, orientable minimal surface with r ends, genus p, and finite total curvature. Osserman [1986] proved that M is conformally equivalent to a compact Riemann surface $\bar{M}$ minus r points, each of which corresponds to an end of M.

Let Y_R denote the intersection of M with the sphere of radius $R > 0$, centered at the origin. Gackstatter [1976] proved that $X_R := R^{-1}Y_R$ converges smoothly, as R goes to infinity, to a collection of great circles on S^2. Each of these great circles corresponds to an end of M; the number of times the circle is covered is referred to as the winding order, and is denoted d_j for the j-th end of M. Gackstatter also improved an inequality contained in [Osserman 1986] by proving that

$$\int_M K\,dA = 2\pi\left(\chi(\bar{M}) - r - \sum_{j=1}^{r} d_j\right), \qquad (1.1)$$

which was also independently derived in [Jorge and Meeks 1983].

Osserman [1986] showed that if the surface has genus $p = 0$ and $r = 2$, with $d_1 = d_2 = 1$, then M is the catenoid; and that if $p = 0$ and $r = 1$, with $d_1 = 3$, then M is Enneper's surface $M_{0,2}$. Moreover, it follows from the Strong Halfspace Theorem [Meeks and Rosenberg 1990], without the assumption that the genus is zero, that if $r = 1$ and $d_1 = 1$ then M is the flat plane (see [Kusner 1987] for an alternate proof).

Hoffman [1982] observed that if one maximizes the genus while keeping the total curvature fixed, equation (1.1) implies

$$r + \sum_{j=1}^{r} d_j = 4,$$

since maximizing genus is equivalent to minimizing $r+\sum d_j$, which is an even integer greater than two. Hence there are only two possibilities for surfaces that maximize genus subject to the constraint of fixed total curvature:

(i) $r = 1$, with $d_1 = 3$,
(ii) $r = 2$, with $d_1 = d_2 = 1$.

Schoen [1983] showed that in case (ii) the genus must be zero, corresponding to the catenoid.

Chen and Gackstatter's surfaces $M_{1,2}$ and $M_{2,2}$ demonstrate that there are surfaces that fall into case (i) and have genus greater than zero. Lopez [1992], and independently Bloß [1989], proved that $M_{1,2}$ is the unique, complete, oriented minimal surface of genus one with total curvature -8π. In particular, maximizing genus with fixed total curvature -8π gives a *unique* example $M_{1,2}$.

Hoffman and Meeks proposed the following conjecture:

Conjecture 1.1. *Each complete orientable minimal surface with maximal genus p and total curvature $-4\pi(p+1)$ is a Chen–Gackstatter surface $M_{p,2}$, and there are only finitely many surfaces of this type (each corresponding to a distinct period solution).*

For $p \leq 35$, only one solution has ever been found numerically for each surface. The initial work to establish this claim has been to prove the existence

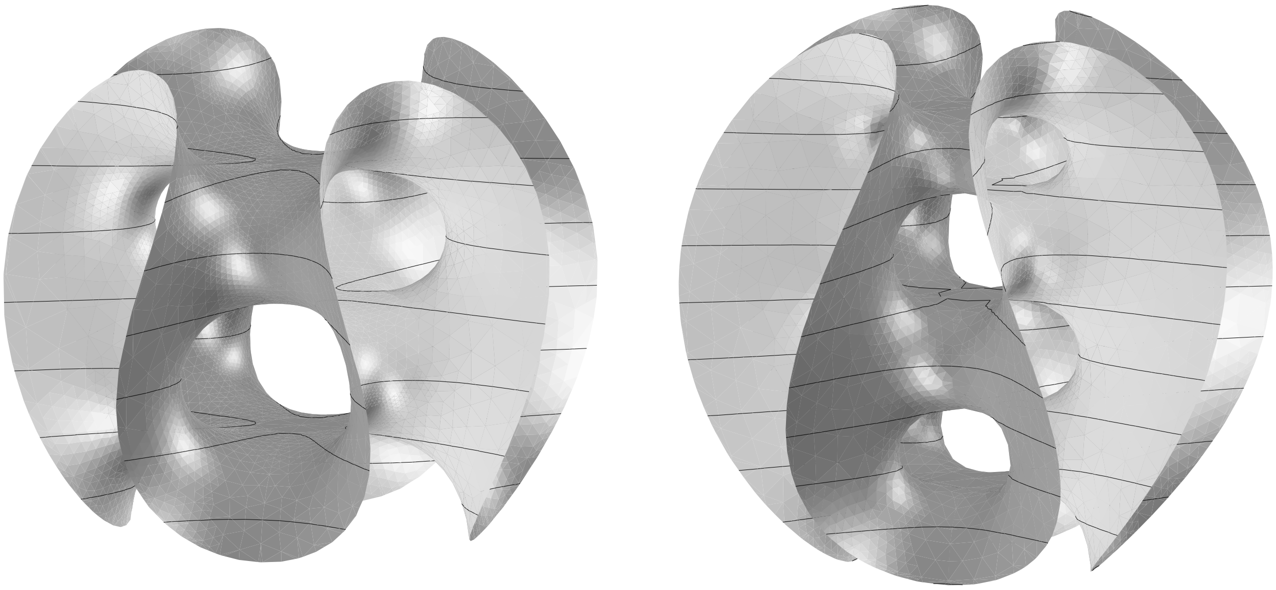


FIGURE 2. Surfaces $M_{1,4}$ (left) and $M_{2,4}$ (right). Compare the original Chen–Gackstatter surfaces in Figure 1, which are called $M_{1,2}$ and $M_{2,2}$ in our notation.

of at least one surface $M_{p,2}$, for each p. We explore this in Section 5.

Evidence gathered from the numerical experiments described in Section 5 suggests the following conjectures:

Conjecture 1.2. (i) *$M_{p,k}$ exists for all $p \geq 3$, $k \geq 2$.*

(ii) *Fix the order $k \geq 2$ of the normal rotational symmetry. Let $q_1, q_2, \ldots, q_p$ be the strictly positive heights above the origin in $\mathbb{R}^3$ of the horizontal points of $M_{p,k}$. Rescale each $M_{p,k}$ so that $q_1 = 1$. Then the parameters $(q_1, \ldots, q_p)$ that solve the period problems on $M_{p,k}$ converge to the integers as $p \to \infty$, that is, $\lim_{p\to\infty} q_j = j$.*

The observations that led to these conjectures also led to what we call the *global Weierstrass representation* of some complete periodic minimal surfaces. If $\Sigma \subset \mathbb{R}^3$ is a doubly or singly periodic minimal surface, let Λ be the maximal infinite abelian subgroup generated by its orientation-preserving symmetries. By *global data* for Σ, we mean the Weierstrass data for Σ proper—in other words, the image of the Weierstrass integral using global Weierstrass data will be the complete surface $\Sigma \subset \mathbb{R}^3$ of infinite total curvature. The classical embedded, periodic surfaces of Scherk viewed globally in $\mathbb{R}^3$ have one topological end and infinite genus. In general, Callahan, Hoffman, and Meeks [Callahan et al. 1990] proved that any embedded, doubly periodic, complete minimal surface in $\mathbb{R}^3$ has infinite genus and one topological end. Hence, in order to determine the global data for these examples, one must first understand these rather complicated covering spaces.

Our investigations with $M_{p,k}$ for large p yielded insights into the algebraic structure of some one-ended surfaces of infinite genus. Specifically, we observed that with fixed k, and with the vertical height of the handle on $M_{p,k}$ nearest the straight lines held constant for increasing values of p, in a neighborhood of the origin, $M_{p,k}$ converge to Karcher's singly periodic, symmetric saddle tower with normal rotational symmetry of order k. The saddle towers with normal rotational symmetry of

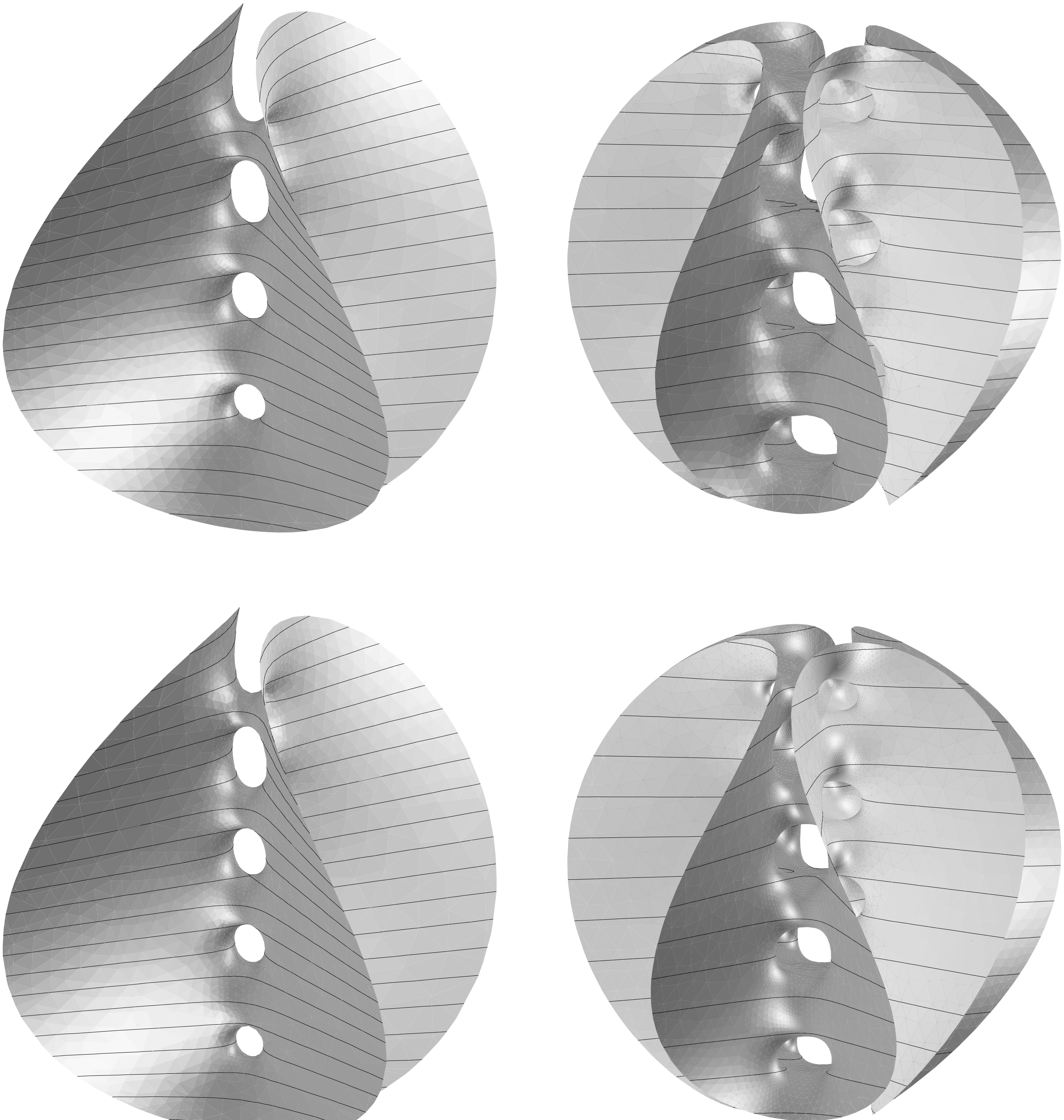


FIGURE 3. Surfaces $M_{3,2}$, $M_{3,4}$, $M_{4,4}$, and $M_{4,2}$ (clockwise from top left).

order k generalize Scherk's singly periodic surface and might be thought of as a desingularization of k planes in $\mathbb{R}^3$. This suggested that these periodic surfaces could be understood globally, but their algebraic equations would involve infinite products. Assuming Conjecture 1.2(ii), we simplified these infinite products to expressions involving the tangent function.

Once the global data for the *symmetric* saddle towers was understood in terms of the tangent function, we were able to generalize these results to give the global data of the deformed saddle towers, as well as the doubly periodic surfaces of Scherk. We describe these examples and their new representations in Section 6.

Until the present work, the Weierstrass data for all known periodic surfaces apart from the helicoid had been constructed for the minimal surface in the flat three-manifold resulting from the quotient of $\mathbb{R}^3$ by Λ. For any known doubly or singly periodic surface Σ, the quotient Σ/Λ has finite topological type. In [Meeks and Rosenberg 1993] it was shown that finite topological type is equivalent to finite total curvature. Moreover, either the stereographic projection of the Gauss map g extends to a meromorphic function, or the logarithmic differential of g extends to a meromorphic differential on the compactification. But by working in $\mathbb{R}^3/\Lambda$, one implicitly assumes that the lift of the surface to $\mathbb{R}^3$ is periodic or has a screw-motion symmetry.

It had long been an open question whether there are nonperiodic embedded minimal surfaces of infinite total curvature. The question was answered in the affirmative in [Hoffman et al. 1993]. The authors constructed a genus-one helicoid, that is, a minimal torus with one topological end of infinite total curvature asymptotic to a helicoid. Crucial to their discovery was the characterization of the Gauss map's essential singularity at the end of the helicoid, that is, the global representation of the helicoid. Their success with the helicoid has produced great interest in characterizing other ends of infinite total curvature. We hope that, by understanding the global data for other periodic surfaces, we will ultimately be able to construct new nonperiodic examples asymptotic to the original periodic surfaces. The ends we describe are topologically more complicated than one of helicoidal type; a helicoidal end is conformally a punctured disc, while any representative of a Scherk-type end in $\mathbb{R}^3$ has infinite genus.

In all cases, the surfaces on which we define the global Weierstrass data will have infinite genus and one topological end. These examples were studied in the 1950's in research on the classification problem for open Riemann surfaces. Using the results from that period due to M. Heins [1952], we recover in Section 6.1 Osserman's result: *Scherk's doubly periodic surface has one topological end and infinite genus.*

2. THE WEIERSTRASS REPRESENTATION

One of the principal tools used to construct minimal surfaces in $\mathbb{R}^3$ is the Weierstrass Representation Theorem:

Given a discrete subset $\{p_j\}$ of a Riemann surface M, a meromorphic function $g : M\setminus\{p_j\} \to \mathbb{C}$, and a holomorphic one-form φ_3 on $M \setminus \{p_j\}$, the following mapping $X : M \to \mathbb{R}^3$ is a conformal minimal immersion:

$$X(z) = \operatorname{Re}\int_p^z \Phi, \tag{2.1}$$

where

$$\Phi = (\varphi_1, \varphi_2, \varphi_3) = \left(\tfrac{1}{2}(g^{-1}-g)\varphi_3,\ \tfrac{1}{2}i(g^{-1}+g)\varphi_3,\ \varphi_3\right)$$

and p is a point fixed on M. The map X is regular away from poles of g and is regular at a pole of g of order m provided φ_3 has a zero there of order m. In order for X to be well-defined on M, we must require that

$$\operatorname{Re}\oint_\gamma \Phi = 0 \tag{2.2}$$

for any closed curve $\gamma \subset M$. The Riemann surface M, meromorphic function g, and one-form φ_3 are referred to as the *Weierstrass data*. Here, g is the stereographic projection from $(0,0,1)$ of the

Gauss map of $X(M)$. Since φ_3 is the integrand in the third component, it is referred to as the *height differential.*

Given the representation of the minimal surface as above, the first and second fundamental forms on the immersion are given by

$$\mathrm{I} = \tfrac{1}{4}(|g|^{-1} + |g|)^2|\varphi_3|^2$$
$$\mathrm{II} = \mathrm{Re}\Big(\frac{dg}{g}\,\varphi_3\Big).$$

This means, for example, that

$$\mathrm{II}(\nu) = \mathrm{Re}\Big(\frac{dg(\nu)}{g(p)}\,\varphi_3(\nu)\Big),$$

for $\nu \in T_pX(M)$.

From the equation for the second fundamental form, one establishes the following result:

Lemma 2.1 [Karcher 1989]. *Let $\mu(t)$ be a geodesic on M.*

(i) *$X(\mu(t))$ is a planar curve provided that*

$$\Big(\frac{dg}{g}\,\varphi_3\Big)(\mu'(t)) \in \mathbb{R}.$$

(ii) *$X(\mu(t))$ is a straight line provided that*

$$\Big(\frac{dg}{g}\,\varphi_3\Big)(\mu'(t)) \in i\mathbb{R}.$$

3. THE GENERAL WEIERSTRASS DATA AND PERIOD PROBLEM FOR $M_{P,K}$

We start by deriving the Weierstrass data for the CGK surfaces $M_{p,k}$, where $p \geq 0$ and $k \geq 2$. We recall that $M_{p,k}$ is an immersed minimal surface with one Enneper-type end of winding order $2k-1$, genus $p(k-1)$, and the same symmetry group as Karcher's generalized Enneper's surface $M_{0,k}$. This symmetry group is generated by reflection in k planes and rotation about k lines on the surface. The lines meet at a single point $\mathcal{O}$ on the surface and pass through the end, which is represented by a point $E \in \bar{M}_{p,k}$. The reflectional symmetries generate a cyclic subgroup R_k of order k, consisting of rotations, which fixes $2p+1$ points and the end. The Riemann–Hurwitz formula

$$\begin{aligned}\chi(\bar{M}_{p,k}) &= 2 - 2p(k-1)\\ &= k\chi(\bar{M}_{p,k}/R_k) - (k-1)(2p+2)\end{aligned}$$

implies that $\chi(\bar{M}_{p,k}/R_k) = 2$, so $\bar{M}_{p,k}/R_k$ is a sphere.

Position $M_{p,k}$ in space so that $\mathcal{O}$ lies at the origin and R_k fixes the x_3-axis. Then the $2p+1$ finite fixed points of R_k on $M_{p,k}$ will lie on the x_3-axis and will have vertical normals. Label the p vertical points of $M_{p,k}$ above the origin $Q_1, \dots, Q_p$, in order of increasing height. Analogously, label the vertical points below the origin $Q_{-1}, \dots, Q_{-p}$, in order of increasing depth. Since R_k also fixes the end, our positioning of $M_{p,k}$ in space implies that the end has a vertical normal.

Taking $z : M_{p,k} \to \mathbb{C} \cup \{\infty\}$ to be the lift of a coordinate on the sphere $M_{p,k}/R_k$, normalized so that $z(\mathcal{O}) = 0$, $z(Q_1) = 1$, and $z(E) = \infty$, we define $q_j := z(Q_j)$ for $-p \leq j \leq p$, $j \neq 0$; in particular, $q_1 = 1$. The symmetries of $M_{p,k}$ imply that $q_j \in \mathbb{R}$ and $q_{-j} = -q_j$ for $j = 1, \dots, p$. Also, since $z : M_{p,k} \to \mathbb{C} \cup \{\infty\}$ is conformal, we have $1 = q_1 < q_2 < \cdots < q_p$.

Specifying a rotational position of $M_{p,k}$ in $\mathbb{R}^3$, we have the normalized Weierstrass data

$$\begin{aligned} w^k &= \begin{cases} zH_p(z, q_1, \dots, q_p), & \text{if } p = 2m,\\ \dfrac{z}{z^2 - q_p^2} H_p(z, q_1, \dots, q_p) & \text{if } p = 2m+1,\end{cases}\\ g &= B\,w^{k-1},\\ \varphi_3 &= dz,\end{aligned} \tag{3.1}$$

where $B > 0$ is a real number, the q_j are real numbers satisfying $1 = q_1 < q_2 < \cdots < q_p$, and

$$H_p(z, q_1, q_2, \dots, q_p) = \prod_{l=1}^{m} \frac{z^2 - q_{2l}^2}{z^2 - q_{2l-1}^2}.$$

To summarize what is known about the $M_{p,k}$:

(i) The $M_{0,k}$ are genus-zero generalizations of Enneper's surface with winding order $2k-1$.

(ii) $M_{1,2}$ and $M_{2,2}$ are the original surfaces discovered by Chen and Gackstatter, of genus one and two.
(iii) Karcher [1989] extended Chen and Gackstatter's original existence proof of $M_{1,2}$ to prove $M_{1,k}$ exists for $k > 2$.
(iv) In Section 4, we show that Chen and Gackstatter's existence proof for $M_{2,2}$ can be modified to give the existence of $M_{2,k}$ for $k > 2$.
(v) Espírito Santo [1993] proved that $M_{3,2}$ exists by estimating the path integrals involved in the period problem using Riemann sums.
(vi) In general, $M_{p,k}$ has genus $p(k-1)$, and by the Jorge–Meeks–Gackstatter equality, its total curvature is

$$-4\pi(\text{genus} - k + 1) = -4\pi(p-1)(k-1).$$

3.1. The General Period Problem

Given the Weierstrass data (3.1), the minimal surface $X : M_{p,k} \to \mathbb{R}^3$ given by

$$X(q) = \operatorname{Re} \int_{q_0}^{q} \Phi \tag{3.2}$$

(where $q_0 \in M_{p,k}$ is fixed) and

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(g^{-1} - g)\varphi_3 \\ i\frac{1}{2}(g^{-1} + g)\varphi_3 \\ \varphi_3 \end{pmatrix}$$

is well-defined, provided that $\oint_\gamma \Phi \in i\mathbb{R}$ for γ any closed curve on $M_{p,k}$. Since $\varphi_3 = dz$ is an exact form, $\oint_\gamma \varphi_3 = 0$ for any closed curve γ, so we need only be concerned with the first and second components of the mapping X. Determining the values of B, q_2, q_3, …, q_p in such a way that this condition is satisfied is referred to as solving the *period problem*.

The symmetries of $M_{p,k}$ simplify the period problem significantly, reducing the expected number of homotopy classes to be considered from $2p(k-1)$ to p. Choosing representatives γ_j, for $-p \le j \le p$, $j \ne 0$, as in Figure 4, we will see that we need only check the periods of the first or second components of X along the curves γ_1, γ_2, …, γ_p.

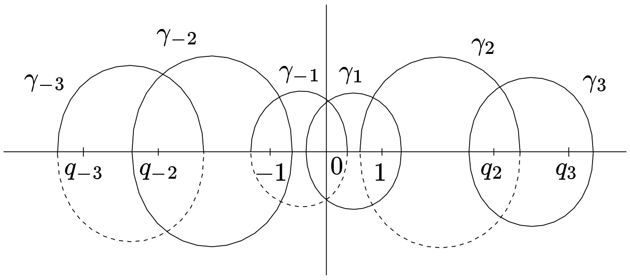


FIGURE 4. Values of z with homotopy class representatives indicated. The change from dotted lines to solid lines indicates that the curve changes sheets on $M_{p,k}$.

Recall that generically we have positioned $M_{p,k}$ in such a way that $\mathcal{O}$ lies at the origin and the points Q_j are horizontal. Additionally, our choice of B in (3.1) positions $M_{p,k}$ so that the planar curve corresponding to $z \in [0,1]$ and $w \in \mathbb{R}$ lies in the $x_2 = 0$ plane.

While one may write out each of the curves γ_j and examine their periods, we prefer to present a more geometric description of these periods. We start by considering the image of one sheet of w under the mapping X defined in (3.2), where w is restricted to vary over one sheet, say the sheet where the phase of w lies between $-\pi/k$ and π/k. This portion of $M_{p,k}$ is bounded by planar curves and contains two horizontal rays starting at the origin and passing to the end. If the period problem were solved, the planar curves would lie in either of two planes, both of which contain the x_3-axis and make an angle of π/k with the plane $x_2 = 0$. See Figure 5 (bottom right). Since $M_{p,k}$ is symmetric about the straight rays eminating from $\mathcal{O}$, once the planar curves above the origin lie in the appropriate planes through the origin, the curves below the origin will also. Hence we need only consider the periods around $\gamma_1, \ldots, \gamma_p$.

Each γ_j is homotopic to a piecewise smooth curve made up of two smooth planar curves joined at horizontal points on the surface, with the angle between the planes in which these curves lie being equal to π/k. The two constituent smooth planar curves, moreover, are congruent by a reflectional

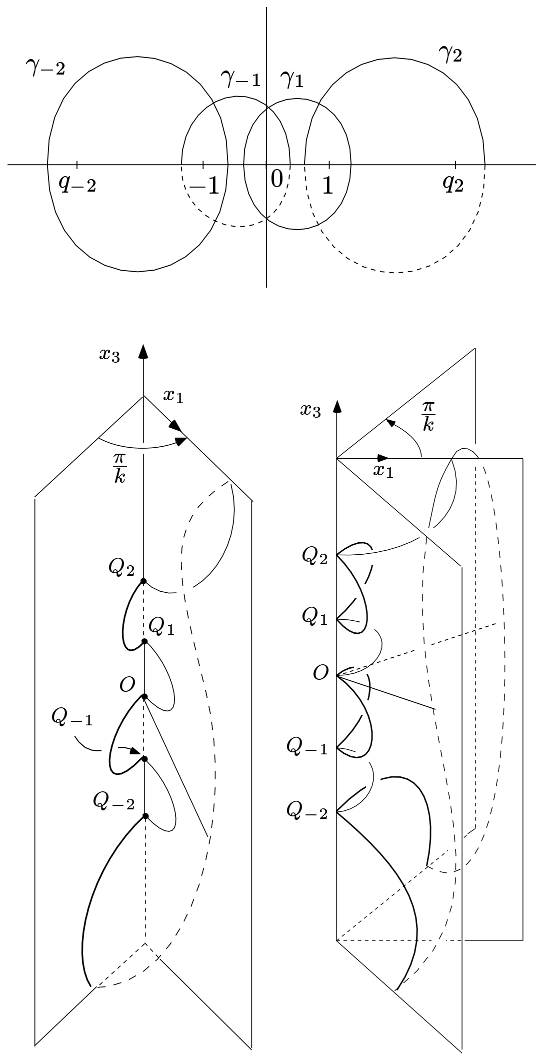


FIGURE 5. Three views of $M_{2,k}$ once the periods have been solved. Top: z-plane with homotopy classes on $M_{2,k}$. Bottom left: sketch of the boundary curves of $M_{2,k}$ in $\mathbb{R}^3$ corresponding to z such that $\operatorname{Im}(z) \geq 0$ and w lies on one sheet. Bottom right: the same boundary curves after reflection through the $x_2 = 0$ plane.

symmetry. We modify our notation so that γ_j refers to this piecewise smooth curve. Hence

$$\operatorname{Re} \oint_{\gamma_j} (\varphi_1, \varphi_2) = 2 \operatorname{Re} \int_{z=q_{j-1}}^{z=q_j} (\varphi_1, \varphi_2),$$

for $j = 1, 2, \ldots, p$, where w is restricted so that the phase of w lies between $-\pi/k$ and π/k.

Our normalization of the Weierstrass data ensures that the planar curve joining $\mathcal{O}$ and Q_1 lies in the $x_2 = 0$ plane, so that

$$\operatorname{Re} \oint_{\gamma_1} \varphi_2 = 2 \operatorname{Re} \int_{z=0}^{z=1} \varphi_2 = 0.$$

Hence we need only consider $\operatorname{Re} \oint_{\gamma_1} \varphi_1$.

A careful examination of the other periods reveals a similar simplification is possible for them too. In particular, γ_2 starts at the point Q_1, climbs to Q_2, and lies in a vertical plane parallel to a plane rotated $\pm\pi/k$ from the $x_2 = 0$ plane. The real parts of the periods about γ_2 are zero when Q_2 lies on the x_3-axis, which is true when the second component of Q_2 is zero. Therefore, provided that

$$\operatorname{Re} \int_{z=0}^{z=1} (\varphi_1, \varphi_2) = 0,$$

it is sufficient to know that

$$\operatorname{Re} \int_{z=1}^{z=q_2} \varphi_2 = 0$$

to ensure that the real parts of the periods about γ_2 are zero.

In a similar fashion, we may now argue that the conditions

$$\operatorname{Re} \oint_{\gamma_3} (\varphi_1, \varphi_2) = 0 \quad \text{and} \quad \operatorname{Re} \oint_{\gamma_4} (\varphi_1, \varphi_2) = 0$$

are satisfied, provided that

$$\operatorname{Re} \int_{z=q_2}^{z=q_3} \varphi_1 = 0 \quad \text{and} \quad \operatorname{Re} \int_{z=q_3}^{z=q_4} \varphi_2 = 0.$$

In general, the period problem reduces to showing that

$$\begin{aligned} \operatorname{Re} \int_{z=q_{j-1}}^{z=q_j} \varphi_1 = 0 \quad \text{for } j \text{ odd}, \\ \operatorname{Re} \int_{z=q_{j-1}}^{z=q_j} \varphi_2 = 0 \quad \text{for } j \text{ even}. \end{aligned} \tag{3.3}$$

Although this period problem involves finding the zero of a function $\Omega : \mathbb{R}^p \to \mathbb{R}^p$, which is generically difficult, more can be said. In fact, an observation in [Chen and Gackstatter 1982] generalizes to this case and allows us to determine the value of B, given the values of $q_2, q_3, \dots, q_p$ such that the first of these period conditions is satisfied. That is,

$$\operatorname{Re}\int_{z=0}^{z=1}\varphi_1 = \operatorname{Re}\int_0^1\Big(\frac{1}{Bw^{k-1}} - Bw^{k-1}\Big)\,dz = 0$$

provided that

$$B^2 = \operatorname{Re}\frac{\int_0^1 dz/w^{k-1}}{\int_0^1 w^{k-1}\,dz}. \tag{3.4}$$

One must confirm that the right hand side is positive, since we need $B\in\mathbb{R}$, $B \neq 0$; this is immediate since w is a strictly decreasing negative real-valued function for $z\in(0,1)$, as seen from equation (3.1). Hence the first period condition imposes a constraint on B, given the values $q_2,\dots,q_p$.

One typically looks to degree-theory arguments when searching for zeros of functions $\Omega : \mathbb{R}^{p-1} \to \mathbb{R}^{p-1}$, and this approach has been successful for period problems arising from other minimal surfaces [Wohlgemuth 1993; Traizet]. Known zeros, representing lower-genus examples, severely complicate these methods in the present case. Specifically, the boundary of the domain $1 < q_2 < \dots < q_p$, over which one searches, contains known zeros for the period functions. For example, if, while searching for the solution to the period problem for $M_{3,2}$, we encounter $q_j = q_l$ for $j \neq l$, the solution found corresponds to a solution of the period problem for $M_{2,2}$. Should several parameters coalesce, the surface may actually degenerate to Enneper's surface $M_{0,2}$.

4. THE EXAMPLES $M_{2,K}$

In this section we solve the period problem for the new surfaces $M_{2,k}$, where $k \geq 3$. The proof presented is an extension of the original proof given by Chen and Gackstatter for $M_{2,2}$, with the substitution of the more general Weierstrass data and some additional observations.

Recall from (3.1) that the Weierstrass data for $M_{2,k}$ is

$$\begin{aligned} w^k &= \frac{z(z^2-q_2^2)}{(z^2-1)},\\ g &= B\,w^{k-1},\\ \eta &= dz, \end{aligned} \tag{4.1}$$

where $B > 0$ and $q_2 > 1$ are reals. The general period problem (3.3) for $M_{p,k}$ reduces to the following two-dimensional problem:

Period Problem. *Show there exist $B > 0$ and $q_2 > 1$ such that:*

$$\operatorname{Re}\int_0^1 \frac{1}{2}\Big(\frac{1}{Bw^{k-1}} - Bw^{k-1}\Big)\,dz = 0$$

and

$$\operatorname{Re}\int_1^{q_2} \frac{i}{2}\Big(\frac{1}{Bw^{k-1}} + Bw^{k-1}\Big)\,dz = 0.$$

Denoting the real k-th root of a positive real number r by $\sqrt[k]{r}$, we see that, for $z\in(0,1)$,

$$w^{k-1} = \sqrt[k]{\Big(\frac{z(z^2-q_2^2)}{z^2-1}\Big)^{k-1}},$$

and for $z\in(1,q_2)$,

$$w^{k-1} = e^{i\pi(k-1)/k}\sqrt[k]{\Big(\frac{z(q_2^2-z^2)}{z^2-1}\Big)^{k-1}}.$$

For convenience we define the following quantities:

$$F_1 = \int_0^1 \sqrt[k]{\Big(\frac{z(z^2-q_2^2)}{z^2-1}\Big)^{k-1}}\,dz;$$

$$G_1 = \int_0^1 \sqrt[k]{\Big(\frac{z^2-1}{z(z^2-q_2^2)}\Big)^{k-1}}\,dz;$$

$$F_2 = \int_1^{q_2} \sqrt[k]{\Big(\frac{z(q_2^2-z^2)}{z^2-1}\Big)^{k-1}}\,dz;$$

$$G_2 = \int_1^{q_2} \sqrt[k]{\Big(\frac{z^2-1}{z(q_2^2-z^2)}\Big)^{k-1}}\,dz.$$