# On the Evaluation of Euler Sums

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**CONTENTS**



Euler studied double sums of the form

$$\zeta(r,s) = \sum_{1\le m<n} \frac{1}{n^s m^r}$$

for positive integers $r$ and $s$, and inferred, for the special cases $r = 1$ or $r+s$ odd, elegant identities involving values of the Riemann zeta function. Here we establish various series expansions of $\zeta(r, s)$ for real numbers $r$ and $s$. These expansions generally involve infinitely many zeta values. The series of one type terminate for integers $r$ and $s$ with $r + s$ odd, reducing in those cases to the Euler identities. Series of another type are rapidly convergent and therefore useful in numerical experiments.

## 1. INTRODUCTION

Following Euler, we consider the nested sum

$$\zeta(r,s) = \sum_{n=2}^{\infty} \frac{1}{n^s} \sum_{m=1}^{n-1} \frac{1}{m^r} = \sum_{m<n} \frac{1}{n^s m^r}, \tag{1.1}$$

where $r \ge 1$ and $s > 1$ are real numbers. By taking the sum over complementary pairs of summation indices we obtain a simple reflection formula

$$\zeta(r,s) + \zeta(s,r) = \zeta(r)\zeta(s) - \zeta(r+s), \tag{1.2}$$

where $\zeta(\cdot)$ is the Riemann zeta function.

A discussion of the precise region of convergence of (1.1), together with questions of analytic continuation, can be found in [Apostol and Vu 1984].

Euler discovered an identity for $\zeta(r, s)$ for $r$ even and $s$ odd:

$$\zeta(r,s) = -\tfrac{1}{2}\zeta(r+s) + \sum_{\substack{j=1\\ j\ \text{odd}}}^{r+s} \left( \binom{j-1}{s-1} + \binom{j-1}{r-1} \right) \zeta(j)\zeta(r+s-j). \tag{1.3}$$

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Note that a formula for the case of $r$ odd and $s$ even follows from this identity and the reflection formula. Euler also gave an identity for the case $r = 1$:

$$\zeta(1, s) = \tfrac{1}{2} s \zeta(s+1) - \tfrac{1}{2} \sum_{j=1}^{s-2} \zeta(j+1)\zeta(s-j), \quad (1.4)$$

regardless of the parity of the integer $s$. When $s$ is even this last identity is a special case of (1.2) and (1.3), but, as we shall see, the case of $s$ odd requires more work.

The function $\zeta(r, s)$ was investigated recently, both theoretically and numerically, in [Borwein et al. 1994]. The authors proved Euler's identities and evaluated what might be thought of as an outlying case:

$$\zeta(2, 4) = \zeta^2(3) - \tfrac{4}{3}\zeta(6). \quad (1.5)$$

This relation is also found in [Markett 1994].

To summarize, evaluations of $\zeta(r, s)$ for integers $r$ and $s$ are known when $r = 1$, when $r + s$ is odd, when $(r, s) = (2, 4)$ or $(4, 2)$, and, via (1.2), when $r = s$.

It is possible to derive some interesting relations involving several "unevaluated" Euler sums. For instance, it can be proved that

$$5\zeta(2, 6) + 2\zeta(3, 5) = 10\zeta(3)\zeta(5) - \tfrac{7}{5400}\pi^8 \quad (1.6)$$

and

$$\begin{aligned}\zeta(2, 6) &+ \zeta(2, 8) + \zeta(2, 10) + \cdots \\ &= -\tfrac{1}{2} + \tfrac{1}{12}\pi^2 - \tfrac{1}{120}\pi^4 + \tfrac{4}{2835}\pi^6 + \tfrac{1}{2}\zeta(3) - \zeta^2(3), \quad (1.7)\end{aligned}$$

but in neither of these identities has any of the individual Euler sums $\zeta(r, s)$ ever been evaluated as a finite series of zeta values.

In keeping with the fact that few Euler sums with $r+s$ even have been evaluated in closed form, Bailey, Borwein and Girgensohn [Bailey et al. 1994] suggest, on the basis of extensive numerical experiments with sophisticated variants of "lattice basis reduction" algorithms, that, for example, $\zeta(2, 6)$ and $\zeta(3, 5)$ are not individually expressible as a linear combination of products of values of the zeta function and related quantities.

In the same work the authors describe an Euler–Maclaurin scheme for the numerical evaluation of Euler sums. They have succeeded in numerical evaluating various sums to hundreds of digits, albeit at the expense of considerable computer time. In fact it was their observation that Euler–Maclaurin methods are not explicitly convergent that motivated the present treatment. Moreover, the lattice basis reduction algorithms require a lot of decimal digits of input, so rapidly convergent expansions are of interest.

In this work we establish various formulas for $\zeta(r, s)$ for arbitrary real $r$ and $s$. One class of formulas generalizes the Euler identity (1.3). Formulas of another class converge more rapidly and are therefore of value in numerical work. The methods described herein also have application to other types of sums. After [Borwein et al. 1994] we can define four possible sums:

$$\zeta^{\pm\pm}(r, s) = \sum_{n=2}^{\infty} \frac{(\pm 1)^n}{n^s} \sum_{m=1}^{n-1} \frac{(\pm 1)^{m-1}}{m^r},$$

of which Euler's case (1.1) is just $\zeta^{++}$. (Other authors have used the notations $\sigma_h = \zeta^{++}$, $\sigma_a = \zeta^{+-}$, $\alpha_h = \zeta^{-+}$, $\alpha_a = \zeta^{--}$.) Again for integers $r$ and $s$ with $r + s$ odd, each of these sums can be given a finite zeta evaluation in the style of (1.3). The methods of this paper can be applied to these alternative sums, to yield corresponding converging series for each.

These methods will perhaps be applicable in the future to multiple zeta sums

$$\zeta(s_1, s_2, \ldots) = \sum_{0<n_1<n_2<\cdots} \frac{1}{n_1^{s_1} n_2^{s_2} \ldots}$$

or to the Witten zeta functions

$$W(r, s, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^r}\frac{1}{m^s}\frac{1}{(m+n)^t}. \quad (1.8)$$

These forms are described in Zagier's overview of the mathematical import of generalized zeta sums [Zagier 1994]; also given there are beautiful known evaluations for some of these sums.

## 2. THE PERIODIC ZETA FUNCTION

In this section we establish integral identities and series expansions involving the periodic zeta function $E$ [Apostol 1976, p. 257 and following]. The function is defined by

$$E(s,x) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s} = C(s,x) + iS(s,x),$$

with the cosine and sine parts $C$ and $S$ given by

$$C(s,x) = \sum_{n=1}^{\infty} \frac{\cos 2\pi n x}{n^s}, \quad S(s,x) = \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{n^s}.$$

The primary integral identities from which our results on Euler sums will follow are:

$$\zeta(s) = 2\int_0^{\frac{1}{2}} \cot \pi x \, S(s,x)\, dx, \tag{2.1}$$

$$\zeta(r,s) = -\tfrac{1}{2}\zeta(r+s) + 2\int_0^{\frac{1}{2}} \cot \pi x \, S(s,x) C(r,x)\, dx, \tag{2.2}$$

$$\zeta(r,s) = -\tfrac{1}{2}\zeta(r+s) + 2\int_0^{\frac{1}{2}} \cot \pi x \, S(r,x)(\zeta(s) - C(s,x))\, dx, \tag{2.3}$$

$$\zeta(r,s) = \zeta(r)\zeta(s) - \frac{1}{\Gamma(r)} \int_0^{\infty} \frac{x^{r-1} E\left(s, \frac{ix}{2\pi}\right)}{1 - e^{-x}}\, dx. \tag{2.4}$$

The first three integral identities follow from the fact that, for $n \ge 1$ and $m \ge 0$, the integral

$$\int_0^{\frac{1}{2}} \cot \pi x \sin 2\pi n x \cos 2\pi m x \, dx \tag{2.5}$$

is equal to 0, $\frac{1}{4}$, or $\frac{1}{2}$, depending on whether $m > n$, $m = n$, or $m < n$. To complete the derivations, one writes the $C$ and $S$ functions in (2.1)–(2.3) as trigonometric series, so that each integral becomes, after the evaluation of (2.5), the relevant one- or two-dimensional sum.

Identity (2.4) follows from series expansion of the term $(1 - e^{-x})^{-1}$ in powers of $e^{-x}$. In this case the $E$ function of imaginary argument becomes a real-valued polylogarithm function:

$$E\left(s, \frac{ix}{2\pi}\right) = \sum_{n=1}^{\infty} \frac{e^{-nx}}{n^s} = L_s(e^{-x}),$$

which is a case of the more general Lerch–Hurwitz zeta function.

Note that the cotangent function appears in the first three integral identities. A classic expansion that will prove quite useful, especially in the development of converging series evaluations, is

$$\cot \pi x = -\frac{2}{\pi} \sum_{k=0}^{\infty} \zeta(2k) x^{2k-1}. \tag{2.6}$$

The $E$ function admits of series expansion [Erdélyi 1953, vol. 1, p. 29] when $|x| < 1$ and $s$ is *not* a positive integer:

$$E(s,x) = \sum_{m=0}^{\infty} \zeta(s-m) \frac{(2\pi i x)^m}{m!} + \Gamma(1-s)(-2\pi i x)^{s-1}.$$

In the neighborhood of $s = n$, with $n$ a positive integer, the $\zeta$ singularity cancels the $\Gamma$ singularity in the following way. One may use the asymptotic relations, valid for small $\varepsilon$,

$$\zeta(1+\varepsilon) \approx \varepsilon^{-1} + \gamma + O(\varepsilon),$$
$$\Gamma(1-n-\varepsilon) \approx \frac{(-1)^n}{n!}(\varepsilon^{-1} - \psi(n) + O(\varepsilon))$$

to infer that, when the first argument $s = n$ is an integer,

$$E(n,x) = \sum_{\substack{m=0 \\ m \ne n-1}}^{\infty} \zeta(n-m) \frac{(2\pi i x)^m}{m!} + \frac{(2\pi i x)^{n-1}}{\Gamma(n)} (\psi(n) - \psi(1) + \tfrac{1}{2} i\pi - \log 2\pi x). \tag{2.7}$$

In particular, $C$ and $S$ have finite polynomial form in certain cases. In fact, for integer arguments of appropriate parity and real $x \in [0,1]$, these functions can be expressed in terms of Bernoulli polynomials $B_n$:

$$C(r,x) = -(-1)^{r/2} 2^{r-1} \pi^r \frac{B_r(x)}{r!} \quad \text{for } r \text{ even},$$
$$S(s,x) = -(-1)^{(s-1)/2} 2^{s-1} \pi^s \frac{B_s(x)}{s!} \quad \text{for } s \text{ odd}. \tag{2.8}$$

All coefficients of the Bernoulli polynomials are rational, so all coefficients of the $C$ or $S$ polynomials of appropriate parity belong to $\mathbb{Q}(\pi)$. It will be important for our derivation of converging series that even without the parity restrictions, $C$ and $S$ can at least be developed via (2.7) as an infinite series plus logarithm term.

It is fortuitous that a special-case elementary form exists beyond the odd/even restriction:

$$C(1,x) = -\log(2 \sin \pi x).$$

This can be shown by elementary means without recourse to $E$ expansions. Within the present context it may be imagined that Euler's closed forms for $\zeta(1,s)$ for any positive integer $s$ are possible because of this elementary form for $C(1,x)$. Note the equivalent elementary form for the index-one polylogarithm: $L_1(z) = -\log(1-z)$.

There is an alternative way to expand the periodic zeta function such that singularities do not appear in series terms. One develops a Taylor series around $x = \frac{1}{2}$:

$$E(s,x) = -\sum_{m=0}^{\infty} \eta(s-m) \frac{(2x-1)^m (\pi i)^m}{m!}, \tag{2.9}$$

where the eta function is defined by

$$\eta(s) = (1 - 2^{1-s})\zeta(s).$$

This function is entire; for example, $\eta(1) = \log 2$. The expansion (2.9) is certainly valid for all real $x \in [0,1]$. Again, $C$ and $S$ are finite polynomials in $(2x-1)$, if the first argument is an even or odd integer, respectively.

It is sometimes useful to invoke a polylogarithm analog of the eta function expansion (2.9). For a complex $\lambda$ with $\mathrm{Re}(\lambda) > 0$ we have

$$E\Big(s, \frac{ix}{2\pi}\Big) = L_s(e^{-x}) = \sum_{m=0}^{\infty} \frac{(\lambda - x)^m}{m!} L_{s-m}(e^{-\lambda}), \tag{2.10}$$

which becomes formally equivalent to the eta expansion as $\lambda$ approaches $i\pi$.

## 3. THE GENERALIZED EULER IDENTITY

Now we shall establish Euler's identity (1.3) and derive our generalization. To prove (1.3) we use the formalism of the previous section and the following expansion for the product of two Bernoulli polynomials.

**Lemma.** *Let $r$ and $s$ be nonnegative integers. Then the product $B_s(t)B_r(t)$ equals*

$$\sum_{\substack{j>0 \\ j \equiv r+s \bmod 2}} \frac{1}{j}\left( r\binom{s}{j-r} + s\binom{r}{j-s}\right) B_{r+s-j} B_j(t) + \tfrac{1}{2}((-1)^{r+s}+1) \frac{(-1)^r B_{r+s}}{\binom{r+s}{s}}.$$

*Proof.* See [Apostol 1976, p. 276, ex. 19], where a proof is outlined. Andrew Granville showed us another proof, whose essence we describe in the case of interest to us: $r$ is even and $s$ is odd. One compares coefficients in the readily verified identity

$$\begin{aligned} 2b^+(t,x)b^-(t,y) &= b^+(0,y)(b^-(t,x+y) - b^-(t,y-x)) \\ &\quad + b^+(0,x)(b^-(t,x+y) + b^-(t,y-x)), \end{aligned}$$

where

$$b^+(t,x) = \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} B_m(t) \frac{x^{m-1}}{m!} = \frac{e^{xt} + e^{x(1-t)}}{2(e^x - 1)}$$

and

$$b^-(t,x) = \sum_{\substack{m=0 \\ m \text{ odd}}}^{\infty} B_m(t)\frac{x^{m-1}}{m!} = \frac{e^{xt} - e^{x(1-t)}}{2(e^x - 1)}$$

are the generating functions for the even and odd Bernoulli polynomials, respectively. □

Now we use (2.8) when $r$ is even and $s$ is odd to cast the above lemma as a statement about the $S$ and $C$ functions. The result is

$$S(s,x)C(r,x) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{r+s} \left( \binom{j-1}{s-1} + \binom{j-1}{r-1} \right) S(j,x)\zeta(r+s-j). \tag{3.1}$$

From the integral identities (2.1) and (2.2) we immediately recover Euler's identity (1.3).

But we can go further. From the eta expansion (2.9) for the periodic zeta function we can give a general series without restrictions on integer $r$ and $s$. We summarize the algebraic steps. First separate the eta expansion (2.9) into real and imaginary parts and multiply these parts to express the $SC$ product of the integral representation (2.2) as a power series in $2x-1$. Then consider the following lemma:

**Lemma.** *Let $n$ be a positive odd integer. Then*

$$(2x-1)^n = 2\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{(-1)^{(k+1)/2}}{\pi^k}\frac{n!}{(n+1-k)!}S(k,x).$$

*Proof.* The assertion is equivalent, using (2.8), to

$$(2x-1)^n = \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{n!\,2^k}{(n+1-k)!}B_k(x).$$

Multiplying by $u^n$ and summing over odd $n$ shows that the result follows from comparing coefficients in the identity

$$u\sinh(u(2x-1)) = b^-(x,2u)\sinh u,$$

which is a restatement of the earlier formula for the generating function of the odd Bernoulli polynomials. □

Using this lemma, the integral in (2.2) can be performed formally, with the help of (2.1), to resolve the general Euler sum as follows:

**Theorem 3.1.** *Define constants $\Phi_k$, for $k > 1$ an odd integer, by*

$$\Phi_k = -\frac{2}{\pi}\sum_{\substack{d=1 \\ d \text{ odd}}}^{k-2} (-1)^{(d-1)/2}\frac{\pi^d}{d!}\zeta(k-d+1). \tag{3.2}$$

*Then, for real $r \ge 1$ and $s > 1$, we have*

$$\zeta(r,s) = -\tfrac{1}{2}\zeta(r+s) + \sum_{\substack{k=3 \\ k \text{ odd}}}^{\infty} \Phi_k \sum_{\substack{j=0 \\ j \text{ even}}}^{k-1} \binom{k}{j}\eta(r-j)\eta(s-k+j). \tag{3.3}$$

This general series terminates, of course, when $r$ is an even integer and $s$ is an odd integer, due to the vanishing of the eta function for negative even arguments. The result in these terminating cases is equivalent to the finite Euler identity (1.3).

The constants $\Phi_k$ themselves present an interesting computational problem. The asymptotic behavior for large odd indices $n$ is, as we shall see,

$$|\Phi_n| \approx \frac{2\pi^{n-1}\log n}{n!}.$$

In spite of this rapid decay, the summands in (3.2) vary radically in magnitude and it is difficult to maintain precision. Computing $\Phi_k$ using this alternating series is similar to computing zero by evaluating the sine power series at $\pi$.

A second computational problem is that even when we know numerical values of the constants $\Phi_k$, the series (3.3) generally exhibits slow convergence. In the next section we develop means for addressing such convergence problems.

## 4. CONVERGENCE

Even though each $\Phi_k$ can be written as a finite zeta series, we find that the following infinite series is better behaved numerically, in that terms do not vary radically in magnitude:

$$\Phi_k = 2(-1)^{(k-1)/2}\frac{\pi^{k-1}}{k!} \times \left(\sum_{j=1}^{k}\frac{1}{j} - \log\pi + \sum_{n=1}^{\infty}\frac{\zeta(2n)}{n\,4^n\binom{k+2n}{k}}\right).$$

This relation is obtained by integrating the power series of the product $SC$ in the variable $(2x-1)$, and applying the cotangent expansion (2.6).

It is perhaps of interest that, in the course of this work in resolving high-precision values for the $\Phi_k$, we noticed an efficient means for evaluating the Riemann zeta function itself for odd arguments. The idea is to use one of the various expansions for a function such as $S(3,x)$ and integrate (2.1) termwise via the cotangent expansion (2.6). In this way one can evaluate $\zeta(3)$—or $\zeta(n)$ for any odd integer $n$—to $D$ good digits in $O(D\log D)$ arithmetic operations. In fact the implied $O$ constant can be made conveniently small. A typical such series is:

$$\frac{36}{\pi^2}\zeta(3) = 707 + 144\log\frac{5159780352}{678223072849} - 12\sum_{n=1}^{\infty}\frac{(\zeta(2n)-1-4^{-n}-9^{-n})(2n+5)}{4^n(2n+1)(2n+2)(2n+3)},$$

which yields about two good decimal digits per summand. One may yet improve the convergence by peeling off longer partial sums from $\zeta(2n)$, expressing the necessary correction as extra logarithmic terms.

We have mentioned that the general eta series (3.3) converges poorly (except of course when it terminates). By trading off the elegance of the singularity-free expansion (2.9) for the more complicated but numerically efficient logarithmic expansion (2.7), we get a rapidly converging series.

The steps run as follows. First, by multiplying the real and imaginary parts of expansion (2.7), develop an $SC$ product as a power series for real $x$, plus possible logarithmic terms. Place this $SC$ series into the cotangent integral (2.2), and use the expansion (2.6) to integrate term by term.

The procedure is somewhat tedious and the resulting formula is rather unwieldy: see Theorem 4.1 on page 281. In spite of its complexity, however, the formula has the advantage that much of the calculation uses arithmetic involving only rational numbers and values of the zeta function. We have checked it numerically over many pairs $(r,s)$.

Note that as a byproduct of this work with cotangent integrals we get formulas for general cases of the integral

$$I_n = \int_0^{\frac{1}{2}} x^n \cot\pi x\,dx.$$

Such values (except for $n=1$) seem not to appear in published tables. It turns out that, for every positive integer $n$,

$$I_n \in \mathbb{Q}(\pi,\ \log 2,\ \zeta(3),\ \zeta(5),\ \ldots).$$

One may prove this by expressing monomials $x^n$ in terms of $S$ functions and the related functions $SO(s,x) = S(s,x) - S(s,2x)/2^s$. Actually the $SO$ functions, which are sine parts of a periodic function like $E$, but developed over odd summation indices $n$, become, for odd $s$, proportional to standard Euler polynomials. One inserts expansions of the monomial into the integral representation (2.1) to obtain the finite series evaluation

$$I_n = \frac{n!}{2^n}\sum_{\substack{k=1\\ k\,\mathrm{odd}}}^{n}\frac{(-1)^{(k-1)/2}}{\pi^k}\frac{\eta(k)}{(n-k+1)!} + \tfrac{1}{2}((-1)^n+1)\frac{4n!\,(1-2^{-n-1})}{(2\pi)^{n+1}}\zeta(n+1).$$

For example, we have

$$I_5 = \frac{\log 2}{32\pi} - \frac{15}{32\pi^3}\zeta(3) + \frac{225}{64\pi^5}\zeta(5).$$

**Theorem 4.1.** *For any real $r \ge 1$ and real $s > 1$, we have*

$$\zeta(r,s) = -\tfrac{1}{2}\zeta(r+s) - \frac{4}{\pi}\sum_{k=0}^{\infty}\zeta(2k)\Biggl(\sum_{\substack{j=1\\ j\text{ odd}}}^{\infty} L_j(2k) \sideset{}{'}\sum_{\substack{m=0\\ m\text{ even}}}^{j-1} \binom{j}{m}\zeta(r-m)\zeta(s-j+m)$$

$$+ A(r)\sideset{}{'}\sum_{\substack{m=1\\ m\text{ odd}}}^{\infty} L_m(2k+r-1)\zeta(s-m)(\Delta_{r-1}J(r,\, m+r+2k-1)+1)$$

$$+ B(s)\sideset{}{'}\sum_{\substack{m=0\\ m\text{ even}}}^{\infty} L_m(2k+s-1)\zeta(r-m)(\Delta_s J(s,\, m+s+2k-1)+1)$$

$$+ \frac{A(r)B(s)}{q2^q}\biggl((\Delta_{r-1}J(r,q)+1)(\Delta_s J(s,q)+1) + \frac{\Delta_{r-1}\Delta_s}{q^2}\biggr)\Biggr),$$

*where the primed sums indicate that terms involving singularities of the zeta function are omitted, and where we define*

$$q = r+s+2k-2; \qquad \Delta_n = \begin{cases} 1 & \text{if } n \text{ is an even integer,}\\ 0 & \text{otherwise;}\end{cases}$$

$$J(n,a) = \sum_{j=2}^{\lfloor n\rfloor - 1}\frac{1}{j} - \log\pi + \frac{1}{a}; \qquad L_j(a) = \frac{(-1)^{\lfloor j/2\rfloor}(2\pi)^j}{j!\,2^{j+a}(j+a)};$$

$$A(r) = \frac{\pi^r 2^{r-2}}{\Gamma(r)\cos(\frac{1}{2}\pi r)}(1-\Delta_{r-1}) + \Delta_{r-1}\frac{(-1)^{(r-1)/2}(2\pi)^{r-1}}{\Gamma(r)};$$

$$B(s) = \frac{\pi^s 2^{s-2}}{\Gamma(s)\sin(\frac{1}{2}\pi s)}(1-\Delta_s) - \Delta_s\frac{(-1)^{s/2}(2\pi)^{s-1}}{\Gamma(s)}.$$

(Because of the $\Delta$ factors, one never need compute $J(n,a)$ for noninteger $n$, so as a practical matter the greatest integer notation $\lfloor\,\rfloor$ in the definition of $J$ is superfluous.)

## 5. EULER'S IDENTITY FOR $\zeta(1, n)$

The Euler formula (1.4), for even integers $s$, follows immediately from the integral identity (2.3) and the finite expansion (3.1), because in such cases $S(1,x)$ and $C(s,x)$ are polynomials. But the identity for $\zeta(1,s)$ with $s$ an odd integer is more problematic.

Our method of proof is based on the observation of [Borwein et al. 1994] that certain generating functions are tractable. We start by defining the $S$ function with three arguments as a sum reminiscent of the usual $S$ functions:

$$S(s,x;z) = \sum_{n=1}^{\infty}\frac{\sin 2\pi n x}{n^s\Bigl(1-\dfrac{z^2}{n^2}\Bigr)}.$$

The following generating function involves at once all the $\zeta(1,x)$ with $x$ odd:

$$g(z) := \sum_{\substack{n=3\\ n\text{ odd}}}^{\infty} z^{n-3}(\zeta(1,n)+\tfrac{1}{2}\zeta(1+n))$$

$$= 2\int_0^{\frac{1}{2}} \cot\pi x\, S(3,x;z)C(1,x)\,dx,$$

To evaluate the integral in the expression for $g(z)$, we observe that the augmented $S$ function can be given an elementary form, for example through Poisson summation:

$$S(3,x;z) = \frac{\pi}{2z^2}\Big(2x-1-\frac{\sin(\pi z(2x-1))}{\sin \pi z}\Big).$$

Therefore our generating function is

$$g(z) = \frac{1}{z^2}H(0) - \frac{\pi H(z)}{z \sin \pi z}, \tag{5.1}$$

where $H$ is the somewhat forbidding integral

$$H(z) = \int_0^{\frac{1}{2}} \log^2(2\sin\pi x)\cos(\pi z(1-2x))\,dx.$$

It turns out that such integrals can be resolved in terms of derivatives of beta functions:

$$\begin{aligned} H(z) &= \frac{1}{2\pi}\frac{\partial^2}{\partial\nu^2}2^\nu \int_0^{\frac{\pi}{2}} \cos^{\nu-1} t\cos 2zt\,dt\,\bigg|_{\nu=1} \\ &= \frac{1}{2}\frac{\partial^2}{\partial\nu^2}\frac{\Gamma(\nu)}{\Gamma(\frac{1}{2}(\nu+1)+z)\Gamma(\frac{1}{2}(\nu+1)-z)}\,\bigg|_{\nu=1}. \end{aligned}$$

On taking derivatives with respect to $\nu$ we obtain psi functions $\Psi = \Gamma'/\Gamma$, then use known zeta expansions of such psi functions to arrive at

$$H(z) = \frac{\sin\pi z}{2\pi z}\Bigg(\bigg(\sum_{\substack{k=3\\ k\text{ odd}}}^{\infty}\zeta(k)z^{k-1}\bigg)^2 + \zeta(2) - \tfrac{1}{2}\sum_{\substack{k=0\\ k\text{ even}}}^{\infty}\zeta(k+2)(k+1)z^k\Bigg).$$

Finally, we use this last form for $H$ in (5.1) to recover the coefficient of $z^{s-3}$, namely, $\zeta(1,s) + \frac{1}{2}\zeta(1+s)$ for odd $s$. This results in a form for $\zeta(1,s)$ equivalent to the Euler form (1.4).

This analytical derivation gives us a byproduct analogous to the exact cotangent integral evaluations (5.1); namely, we now know any integral of the form

$$J_q = \int_0^{\frac{1}{2}} \log^2(2\sin\pi x)(2x-1)^q\,dx,$$

where $q$ is an even integer. Each such integral belongs to $\mathbb{Q}(\pi^2, \zeta(3), \zeta(5), \ldots)$. For example,

$$J_6 = \frac{11\pi^2}{360} + \frac{60}{\pi^4}\zeta^2(3) - \frac{720}{\pi^6}\zeta(3)\zeta(5).$$

There may be some hope for using such logarithmic integrals to establish some outlying cases, such as (1.5), or relations such as (1.6), although we have not carried out such derivations. Such identities involving unevaluated sums follow from a partial-fraction algebraic method of [Borwein et al. 1994] that can be traced back to Euler. One may also find (1.6) in the guise of [Markett 1994, eq. (1.8)]. For example, the algebraic method yields (1.5) in the guise of the identity

$$\zeta(6) = 12\zeta(1,5) + 6\zeta(2,4).$$

Such an identity also follows from the fact that the integral

$$\int_0^{\frac{1}{2}} \big(6S(5,x)C(1,x) + 3S(4,x)C(2,x) - 5S(6,x)\big) \times \cot\pi x\,dx$$

vanishes (even though the integrand generally does not). But it is not clear how to establish such results via integral calculus alone. It is possibly relevant that the term involving $C(1,x)$ can be integrated by parts to yield some logarithmic forms $J_q$.

## 6. EXPANSIONS WITH FREE PARAMETER

We have developed a generalized Euler series and a rapidly converging series. There is yet another type of series, this time involving incomplete gamma function values. For the identity (2.4), we observe that the integral may be split in the classic style due to Riemann in his studies of the zeta functional equation:

$$\zeta(r,s) = \zeta(r)\zeta(s) - \frac{1}{\Gamma(r)}\Big(\int_0^\lambda + \int_\lambda^\infty\Big)\frac{x^{r-1}}{1-e^{-x}}L_s(e^{-x})\,dx. \tag{6.1}$$

A rapidly converging expansion can be developed as follows. Let the free parameter be $\lambda < 2\pi$. In the first integral above, the factor $1/(1-e^{-x})$ may be developed in a converging Bernoulli series, with the polylogarithm developed also in a series, of the type (2.7). The second integral can be expressed by expanding the same $1/(1-e^{-x})$ factor, but this time in powers of $e^{-x}$, to yield incomplete gamma function terms. Define functions $H_\nu$ and $G_\nu$ for real indices $\nu$ by

$$H_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m B_m z^{m+\nu}}{m!\,(m+\nu)},$$
$$G_\nu(z) = \frac{d}{d\nu} H_\nu(z).$$

The result of these manipulations of (6.1) can be expressed as follows.

**Theorem 6.1.** *For $r > 1$ real, $s > 1$ noninteger, and $\lambda \in (0, 2\pi)$, we have*

$$\zeta(r,s) = \zeta(r)\zeta(s) - \frac{1}{\Gamma(r)} \sum_{m=1}^{\infty} \sum_{m=1}^{m} \frac{\Gamma(r, m\lambda)}{m^r n^s}$$
$$- \frac{1}{\Gamma(r)} \sum_{\mu=0}^{\infty} \frac{\zeta(s-\mu)(-1)^\mu}{\mu!} H_{\mu+r-1}(\lambda)$$
$$- \frac{\Gamma(1-s)}{\Gamma(r)} H_{r+s-2}(\lambda).$$

*For $s$ integer, the equality must be modified as follows: replace the third line by*

$$\frac{(-1)^s}{\Gamma(r)\Gamma(s)} \left( H_{s+r-2}(\lambda) \sum_{m=1}^{s-1} \frac{1}{m} - G_{s+r-2}(\lambda) \right)$$

*and, on the second line, omit the summand involving the zeta singularity.*

The expansion in this theorem has several features of interest. For one thing, a computer program can be tested strenuously by altering the free parameter $\lambda$, in which case one expects of course an invariant numerical result. Note also that for $r$ an integer, the incomplete gamma functions are elementary. But perhaps the most important feature is that the incomplete gamma sum is not fundamentally two-dimensional as it might first appear. In fact, one may keep track of the partial sum of $n^{-s}$, and by so doing evaluate the double sum up to $m = M$ with $O(M)$ evaluations of the $m$-dependent part.

If means for fast polylog evaluation are available, an interesting option is to use the polylog expansion (2.10) in representation (6.1) to obtain the following alternative series. For any real $r > 1$ and $s > 1$, and any $\lambda \in (0, 2\pi)$,

$$\zeta(r,s) = \zeta(r)\zeta(s) - \frac{1}{\Gamma(r)} \sum_{m=1}^{\infty} \sum_{n=1}^{m} \frac{\Gamma(r, m\lambda)}{m^r n^s}$$
$$- \frac{\lambda^{r-1}}{\Gamma(r)} \sum_{m=0}^{\infty} \lambda^m L_{s-m}(e^{-\lambda}) \sum_{n=0}^{\infty} \frac{(-\lambda)^n \Gamma(n+r-1) B_n}{n!\,\Gamma(n+r+m)}.$$

Aside from the development of converging expansions, we note that formal manipulations of the polylogarithm integral representation can yield interesting identities. Equation (1.7) and many like it may be obtained by summing appropriately inside the integral of (2.4).

Finally, we observe that the Witten zeta function (1.8) admits of straightforward integral representations:

$$\begin{aligned} W(r,s,t) &= \int_0^1 E(r,x)E(s,x)\overline{E(t,x)}\,dx \\ &= \frac{1}{\Gamma(t)} \int_0^\infty x^{t-1} E\Big(r, \frac{ix}{2\pi}\Big) E\Big(s, \frac{ix}{2\pi}\Big)\,dx. \end{aligned} \tag{6.2}$$

Various algebraic relations, such as Zagier's triangle recurrence

$$W(r,s,t) = W(r-1,\, s,\, t+1) + W(r,\, s-1,\, t+1),$$

follow immediately upon integration by parts of the second, polylogarithm representation in (6.2). As for numerical work, it is evident that the expansion methods of this treatment may be applied to these integral representations to cast the Witten zeta function as a converging series.

## 7. NUMERICAL RESULTS

Using the converging series given in Theorems 4.1 and 6.1, we tested several known Euler identities and established numerical values for such oddities as $\zeta(\frac{3}{2},2)$ and $\zeta(\frac{5}{2},\frac{5}{2})$. We show on Table 1 some of the results found.

To ensure software reliability, one has various options. First, the free-parameter expansions of Section 6 should give invariant results as the parameter $\lambda$ is varied. Second, "check-sum" identites abound; for example, the first relation of (1.6) is unlikely to hold numerically if either $\zeta(2,6)$ or $\zeta(3,5)$ is off the mark.

Often, given a specific pair $(r,s)$, one may further streamline a converging series. For example, we computed $\zeta(2,6)$ to a little beyond 1000 decimal digits using the following modification of the series of Theorem 4.1 for the equivalent case of $\zeta(6,2)$:

$$\begin{aligned}\zeta(6,2) &= -\tfrac{1}{2}\zeta(8)\\ &\quad -\frac{8\pi^6}{945}\sum_{j=1}^{7}\frac{a_j}{2^j}\Biggl(\sum_{n=0}^{\infty}\frac{\zeta(2n)}{4^n}\frac{1+(2n+j)(1-\log\pi)}{(2n+j)^2}\\ &\qquad\qquad +\sum_{n=1}^{\infty}\frac{1}{4^n(2n+j)}\sum_{k=1}^{n}\frac{\zeta(2k)\zeta(2n-2k)}{k(2k+1)}\Biggr),\end{aligned}$$

where the constants $a_j$, for $j=1,\dots,7$, are equal to 1, 0, $-21$, 0, 105, $-126$, 42 (they are related to the coefficients of the sixth Bernoulli polynomial).

This run was performed with Pari [Batut et al. 1992], and consumed a few hours on a common workstation; by comparison, 100-digit accuracy requires just a few seconds.

To effect a rigorous numerical check on such a high-precision run, we used a different expansion—this time the free-parameter expansion of Theorem 6.1—to calculate $\zeta(3,5)$ also to a little more than 1000 digits. Then we verified a checksum relation (1.6) on our 1000-digit values of $\zeta(2,6)$ and $\zeta(3,5)$. Both 1000-digit values appear correct on the basis of this test. A subset of the digits found for $\zeta(2,6)$ and $\zeta(3,5)$ is shown in Table 1.

Just prior to publication, our numerical evaluation of the above formula for $\zeta(6,2)$ was checked independently by David Bailey, who employed an FFT-based convolution scheme for the final sum, and a scheme for rapid evaluation of the $\zeta(2n)$. He reports that his 1200-digit run is in agreement with our first 1000 digits for $\zeta(2,6)$.

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## REFERENCES

[Apostol 1976] T. Apostol, *Introduction to Analytic Number Theory*, Springer, New York, 1976.

[Apostol and Vu 1984] T. Apostol and T. Vu, "Dirichlet series related to the Riemann zeta function", *J. Number Theory* **19** (1984), 85–102.

[Batut et al. 1992] C. Batut, D. Bernardi, H. Cohen and M. Olivier, *User's Guide to Pari-GP*. This manual is part of the program distribution, available by anonymous ftp from the host pari@ceremab.u-bordeaux.fr.

[Borwein et al. 1994] D. Borwein, J. Borwein, and R. Girgensohn, "Explicit evaluation of Euler sums", to appear in *Proc. Edinburgh Math. Soc.*

[Bailey et al. 1994] D. Bailey, J. Borwein, and R. Girgensohn, "Experimental evaluation of Euler resums", *Experimental Mathematics* **3** (1994), 17–30.

[Erdélyi 1953] A. Erdélyi, *Higher Transcendental Functions*, McGraw-Hill, New York, 1953–55.

[Markett 1994] C. Markett, "Triple sums and the Riemann zeta function", *J. Number Theory* **48** (1994), 113–132.

[Zagier 1994] D. Zagier, "Values of zeta functions and their applications," preprint (Max-Planck Institut).

| $r$ | $s$ | Approximate value of $\zeta(r,s)$ |
|---|---|---|
| $\frac{3}{2}$ | 2 | 0.93182449042503409855161511070364305170750579463468769957662770 |
| $\frac{5}{2}$ | $\frac{5}{2}$ | 0.38133015311160926057188187543098929328088653813490311664930381 |
| 2 | 6 | 0.0178197404168359883626595302487246121687131371102911884188 . . . |
| | | . . . 473292693586748663898283792081659511950953195 |
| 3 | 5 | 0.0377076729848475440113047822936599148226013194152775240126 . . . |
| | | . . . 177265769807079032483793747603319517196244996 |
| 2 | 8 | 0.0041224696783998322240469568386942088558126273584685692852453 |
| 3 | 7 | 0.00841966850309633242396857971467065063691787506395809227257446 |
| 4 | 6 | 0.0174551947508350247357406393866684137318592829095214310061565 |
| 2 | 10 | 0.00099920678720969184043380148821583760914101923281940968488203 |
| 3 | 9 | 0.00201547801088202946783053145858135503874776651437449337609272 |
| 4 | 8 | 0.0040882961515893033313621992830912734634204960410691654540421 |
| 5 | 7 | 0.0083663991887686780781702994259187088925622914932741007840123 |
| 4 | 10 | 0.00099571474274251309551098420825772106634908875663822487087287 |
| 5 | 9 | 0.00201013899185484762781776097488739981176241063120855599011704 |
| 6 | 8 | 0.0040800562712988267651149954426882196454448363888123113382069 |
| 6 | 10 | 0.00099485808496876276081122728160355874986899821530337158555764 |
| 7 | 9 | 0.0020088266878980288790660543386219286731728325200883185439044 |
| 8 | 10 | 0.0009946456558278109016146529677459396783051618067192814141506 |

**TABLE 1.** First sixty or so digits of some values of $\zeta(r,s)$ for which no finite evaluation is known. For $\zeta(2,6)$ and $\zeta(3,5)$ we also have given digits 955–1000 out of the respective 1000-digit expansions.

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