On Artin L-Functions for Octic Quaternion Fields

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CONTENTS

We study the Artin L-function $L(s, \chi)$ associated to the unique character χ of degree 2 in quaternion fields of degree 8. We first explain how to find examples of quaternion octic fields with not too large a discriminant. We then develop a method yielding a quick computation of the order n_χ of the zero of $L(s, \chi)$ at the point $s = \frac{1}{2}$. In all our calculations, we find that n_χ only depends on the sign of the root number $W(\chi)$; indeed $n_\chi = 0$ when $W(\chi) = +1$ and $n_\chi = 1$ when $W(\chi) = -1$. Finally we give some estimates on n_χ and low zeros of $L(s, \chi)$ on the critical line in terms of the Artin conductor $\mathfrak{f}_\chi$ of the character χ.

1. INTRODUCTION

The well known conjecture that the zeros of the Riemann zeta function are simple can be also stated for a more general class of Dirichlet L-series and Artin L-functions associated to one-dimensional characters of number fields. Conjecturally when the base field is $\mathbb{Q}$, these functions never vanish at the central critical point [Murty and Murty 1997]. More particularly, a question of J.-P. Serre is to know whether the order n_χ of a zero of $L(s, \chi)$ at the point $s = \frac{1}{2}$ is the smallest possible with respect to the constraints imposed by the properties of the character χ, in particular those imposed by the sign of the root number $W(\chi)$ when χ is real-valued.

A precise form of this conjecture is stated in [Goss 1996, p. 324]. In this paper, we study the case of two-dimensional characters χ arising from quaternion fields $N/\mathbb{Q}$ of degree 8. Recall that the explicit computation of values of Artin L-functions done in [Tollis 1997] based on a formula due to A. F. Lavrik and E. Friedman (see [Cohen 2000, Section 10.3]) becomes very lengthy from degree 7 onwards. The expected running time is roughly $O(\sqrt{\mathfrak{f}_\chi})$. However, for the method we develop here, the required time is $O(\ln \mathfrak{f}_\chi)$, which allows us to deal with degree-eight fields. We also give faster

AMS Subject Classification: 11R42
Keywords: Artin L-functions, zeros, quaternion fields

1058-6458/2001 $0.50 per page

algorithms that depend on the Generalized Riemann Hypothesis.

2. DEFINITIONS AND NOTATION

Let N/K be a Galois extension of a number field with Galois group $G = \mathrm{Gal}(N/K)$, let (ρ, V) be a representation of G and χ its character. Then the Artin L-function attached to χ is defined:

$$L(N/K, \chi, s) = \prod_{\mathfrak{p} \text{ finite}} \frac{1}{\det(1 - \rho(\varphi_{\mathfrak{P}})|V^{I_{\mathfrak{P}}} N(\mathfrak{p})^{-s})},$$

where the product is over all finite primes $\mathfrak{p}$ of K. Here $\varphi_{\mathfrak{P}}$ is the Frobenius automorphism of one $\mathfrak{P}$ above an unramified $\mathfrak{p}$. For ramified $\mathfrak{p}$, see [Martinet 1977]. The Artin L-series converges uniformly in half-planes $\mathrm{Re}\, s > 1 + \delta$ (with $\delta > 0$) and defines an analytic function on the half-plane $\mathrm{Re}\, s > 1$. Using basic properties of representations, one can prove that

$$\zeta_N(s) = \zeta_K(s) \prod_{\chi \neq 1} L(N/K, \chi, s)^{\chi(1)},$$

where χ varies over the nontrivial irreducible characters of G. The positive integer $\chi(1)$ arises from the decomposition of the the regular representation reg_G of G into $\mathrm{reg}_G = \sum_\chi \chi(1)\, \chi$; see [Serre 1978].

In order to obtain an L-function with a functional equation, it is necessary to introduce Euler factors for the infinite primes of K. For every infinite place $\mathfrak{p}$ of K, define

$$L_{\mathfrak{p}}(N/K, \chi, s) = \begin{cases} L_{\mathbb{C}}(s)^{\chi(1)}, & \mathfrak{p} \text{ complex}, \\ L_{\mathbb{R}}(s)^{n^+} L_{\mathbb{R}}(s+1)^{n^-}, & \mathfrak{p} \text{ real}, \end{cases}$$

where

$$L_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s), \quad L_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(s/2)$$

and

$$n^+ = \frac{\chi(1) + \chi(\varphi_{\mathfrak{P}})}{2}, \quad n^- = \frac{\chi(1) - \chi(\varphi_{\mathfrak{P}})}{2}.$$

Define the enlarged Artin function $\Lambda(N/K, \chi, s)$ by

$$\Lambda(N/K, \chi, s) = c(N/K, \chi)^{s/2} L_\infty(N/K, \chi, s) L(N/K, \chi, s),$$

where

$$c(N/K, \chi) = |d_K|^{\chi(1)} N_{K/\mathbb{Q}}(\mathfrak{f}(N/K, \chi))$$

and

$$L_\infty = \prod_{\mathfrak{p}|\infty} L_{\mathfrak{p}}(N/K, \chi, s);$$

this function has a meromorphic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(N/K, \chi, 1 - s) = W(\chi)\Lambda(N/K, \bar{\chi}, s),$$

where the root number $W(\chi)$ is a constant of absolute value 1 [Martinet 1977].

Artin's conjecture says that for every irreducible character $\chi \neq 1$, the Artin L-function $L(N/K, \chi, s)$ is everywhere holomorphic. In particular, the quotient ζ_N/ζ_K should be entire, as a consequence of the Aramata–Brauer Theorem [Murty and Murty 1997]. Now if we restrict our attention to the order of the zero $n_\chi(s_0)$ at some $s_0 \in \mathbb{C}$ of the Artin L-functions, a few results were proved in this direction; see [Stark 1974] for example. By analogy with the conjecture on the simplicity of the zeros of the Riemann zeta function, the main question is to know whether for $\mathrm{Re}\, s_0 > 0$ we have $n_\chi(s_0) \leq 1$ if χ is absolutely irreducible and $K = \mathbb{Q}$.

3. QUATERNION EXTENSIONS

In this section we describe how to compute quaternion fields and give some properties of their associated Artin L-functions.

Definition 3.1. A quaternion extension of $\mathbb{Q}$ is a normal extension N of $\mathbb{Q}$ with Galois group G isomorphic to the quaternion group H_8 of order 8.

The quaternion group H_8 can be written $H_8 = \langle \sigma, \tau \rangle$ with relations $\sigma^4 = 1$, $\tau^2 = \sigma^2$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$. It possesses a unique irreducible character χ of degree 2; one has $\chi(1) = 2$, $\chi(\sigma^2) = -2$ and $\chi(s) = 0$ for $s \neq 1, \sigma^2$.

The field N contains three quadratic subfields k_1, k_2, k_3 with discriminants d_1, d_2, d_3 and a biquadratic subfield K with discriminant $d_1 d_2 d_3$. The theorem below allows us to know under what condition a quadratic field $k = \mathbb{Q}(\sqrt{m})$ can be embedded in a quaternion field N. For a general formulation, see [Witt 1936].

Theorem 3.2. *Let m be a squarefree integer. In order that $k = \mathbb{Q}(\sqrt{m})$ should be a quadratic subfield of some quaternion field N, it is necessary and* *sufficient that m be positive and not congruent to* $-1 \bmod 8$.

By a theorem of Gauss (see [Serre 1970] for a proof), the preceding condition on m holds if and only if $m = p^2 + r^2 + s^2$ where p, r, s are integers. Let $K' = \mathbb{Q}(\sqrt{m}, i)$ with $i^2 = -1$ and let N' be a quartic cyclic extension of K' such that $N'/\mathbb{Q}$ is Galois. Put $< s >= \mathrm{Gal}(K'/k)$, $< \tau >= \mathrm{Gal}(K'/\mathbb{Q}(i))$, and lift them to elements $\bar{s}$, $\bar{\tau}$ in $\mathrm{Gal}(N'/\mathbb{Q})$. By cohomological considerations, we have the following proposition related to the construction of quaternion fields N [Damey and Payan 1970]:

Proposition 3.3
citedam. *$N \subset N'$ if and only if $N'/\mathbb{Q}(i)$ is a quaternion extension and $\bar{s}\bar{\tau} = \bar{\tau}\bar{s}$.*

Now one can write $N' = K'(\sqrt[4]{\alpha})$ where $\alpha \in K' \setminus k^2$, thus one can compute explicitly N' by the following theorem:

Theorem 3.4. *The extension $N'/\mathbb{Q}(i)$ satisfies the conditions of Proposition 3.3 if and only if α can be written*

$$\alpha = m(r + is)^2 \frac{p + \sqrt{m}\,\lambda}{p - \sqrt{m}\,\bar{\lambda}},$$

with

$$\lambda \in \mathbb{Q}(\sqrt{-m}), (r^2 + s^2)\lambda\bar{\lambda} \notin K'^{*^2}.$$

From this, we deduce easily N by computing the fixed subfields of N' by any lifting of s, $\bar{s} \in G' = \mathrm{Gal}(N'/\mathbb{Q})$ of order 2. Since $G' = \mathbb{Z}/2\mathbb{Z} \times H_8$, there are 3 automorphisms in G' of order 2, but only two of them can be a lifting of s and the third one has a square root in G'. Therefore one can compute easily the two quaternion subfields of N'. In the last section we shall give a table of many totally real and imaginary quaternion extensions with their quadratic subfields.

Now we restrict our attention to the Artin L-function $L(s, \chi)$ associated to the unique character χ of degree 2 of H_8. If we write $L(s, \chi)$ in terms of Dedekind zeta functions, we have:

Proposition 3.5. *Let K be the quartic subfield of N, we have:*

$$\zeta_N(s) = \zeta_K(s)L(s, \chi)^2 = \zeta_K(s)L(N/K, \chi', s),$$

where χ' is the nontrivial character associated to the quadratic extension N/K.

From the preceding identity, we deduce that $L(s, \chi)^2$ is an entire function. Since $L(s, \chi)$ is meromorphic then $L(s, \chi)$ is entire too.

In Theorems 3.6 and 3.7, we give an explicit computation of $W(\chi)$ for tamely ramified extensions (those such that 2 is not ramified in $N/\mathbb{Q}$). We start by defining an invariant U_N of the quaternion extension N, by setting it to $+1$ if the ring of integers O_N of N is a free $\mathbb{Z}[G]$-module, and to -1 otherwise. The Fröhlich theorem gives the general equality:

Theorem 3.6 [Fröhlich 1972]. $W(\chi) = U_N$.

Set

$$\varepsilon = \begin{cases} +1 & \text{if } N \text{ is real,} \\ -1 & \text{if } N \text{ is imaginary.} \end{cases}$$

In [Martinet 1971], one can find an explicit criterion to know whether O_N is a free $\mathbb{Z}[G]$-module or not:

Theorem 3.7. *O_N is a free $\mathbb{Z}[G]$-module if and only if*

$$\varepsilon \prod_{p \mid d_N} p \equiv \frac{1 + d_1 + d_2 + d_3}{4} \bmod 4.$$

A look at the functional equation of $L(s, \chi)$ shows:

Theorem 3.8. *If $W(\chi) = +1$ then n_χ is even,*
If $W(\chi) = -1$ then n_χ is odd.

and the conjecture on n_χ can be expressed in the following way:

Conjecture 3.9. *If $W(\chi) = +1$ then $n_\chi = 0$,*
If $W(\chi) = -1$ then $n_\chi = 1$.

4. COMPUTATION OF n_χ

In this section we give an explicit method to compute n_χ and verify numerically Conjecture 3.9 in many cases (see Section 6). For that purpose, we use Weil's explicit formula [1972], as reformulated by K. Barner [1981] for ease of computation. One can adapt this formula to $L(N/K, \chi', s)$ and then evaluate the sum on the zeros of the Artin L-function $L(s, \chi)$ in the explicit formula.

Theorem 4.1. *Let F satisfy $F(0) = 1$ and the following conditions:*

(A) *F is even, continuous and continuously differentiable everywhere except at a finite number of* *points a_i, where $F(x)$ and $F'(x)$ have only a discontinuity of the first kind, such that* $F(a_i) = \frac{1}{2}\big(F(a_i+0) + F(a_i-0)\big)$.

(B) *There exists a number $b > 0$ such that $F(x)$ and $F'(x)$ are $O(e^{-(\frac{1}{2}+b)|x|})$ as $|x| \to \infty$.*

Then the Mellin transform of F,

$$\Phi(s) = \int_{-\infty}^{+\infty} F(x)e^{(s-\frac{1}{2})x}dx,$$

is holomorphic in every vertical strip $-a \leq \sigma \leq 1+a$ where $0 < a < b$, $a < 1$, and the sum $\sum \Phi(\rho)$ running over the non trivial zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ with $|\gamma| < T$ tends to a limit as T tends to infinity. This limit is given by

$$\begin{aligned}\lim_{T\to+\infty} \sum_{|\gamma|<T} \Phi(\rho) &= \ln \mathfrak{f}_\chi - \sum_{\mathfrak{p},m} \frac{\ln N_{K/\mathbb{Q}}(\mathfrak{p})}{N_{K/\mathbb{Q}}(\mathfrak{p})^{m/2}} \chi'(\mathfrak{p})^m F(m \ln N_{K/\mathbb{Q}}(\mathfrak{p})) \\ &\quad - 2(\ln 2\pi + \gamma + 2\ln 2) - 2\varepsilon J(F) + 2I(F),\end{aligned}$$

where

$$J(F) = \int_0^{+\infty} \frac{F(x)}{2\cosh(x/2)}\, dx,$$

$$I(F) = \int_0^{+\infty} \frac{1-F(x)}{2\sinh(x/2)}\, dx,$$

$\gamma = 0.57721566\ldots$ *is the Euler constant and ε is defined by Theorem* 3.6.

4A. The Conditional Case

Now we assume the Generalized Riemann Hypothesis (GRH) for $L(s,\chi)$ which asserts that all the nontrivial zeros of $L(s,\chi)$ lie on the critical line $\operatorname{Re} s = \frac{1}{2}$. Now we write Theorem 4.1 for Serre's choice $F_y(x) = e^{-yx^2}$ $(y > 0)$. The Mellin transform $\Phi(s)$ of F_y is

$$\Phi_y(s) = \sqrt{\frac{\pi}{y}}\, e^{(s-\frac{1}{2})^2/(4y)},$$

and the Fourier transform $\hat{F}_y$ of F_y is

$$\hat{F}_y(t) = \sqrt{\frac{\pi}{y}}\, e^{-t^2/(4y)}.$$

If we assume the GRH for $L(s,\chi)$, we can write $\Phi_y(\rho) = \hat{F}_y(t)$ where $\rho = \frac{1}{2} + it$. For every $k \geq 1$, we denote by t_k the positive imaginary part of the k-th zero of the Artin L-function $L(s,\chi)$, and n_k its multiplicity. We have the identity

$$\begin{aligned}S(y) &= n_\chi + 2\sum_{k\geq 2}^{+\infty} n_k e^{-\frac{t_k^2}{4y}} \\ &= -\sqrt{\frac{y}{\pi}} \sum_{\mathfrak{p},m} \frac{\ln N_{K/\mathbb{Q}}(\mathfrak{p})}{N_{K/\mathbb{Q}}(\mathfrak{p})^{m/2}} \chi'(\mathfrak{p})^m e^{-y(m\ln N_{K/\mathbb{Q}}(\mathfrak{p}))^2} \\ &\quad + \sqrt{\frac{y}{\pi}} \big(\ln \mathfrak{f}_\chi - 2(\ln 2\pi + \gamma + 2\ln 2) \\ &\qquad - 2\varepsilon J(F_y) + 2I(F_y)\big).\end{aligned}$$

To compute n_χ, one needs:

Proposition 4.2. *Assuming the GRH, we have*

$$n_\chi \leq S(y) \quad \text{and} \quad \lim_{y\to 0} S(y) = n_\chi$$

for all $y > 0$.

The advantage of Serre's choice in Weil's explicit formula is that the series $S(y)$ converges rapidly to n_χ when $y \to 0$. In practice we prove for many quaternion fields that when $W(\chi) = +1$, we have $n_\chi \leq S(y) < 2$ for some $y > 0$ and so $n_\chi = 0$. Similarly for $W(\chi) = -1$, we can prove the inequality $n_\chi \leq S(y) < 3$ for some $y > 0$ and so $n_\chi = 1$. Actually, using Theorem 3.8, Conjecture 3.9 can be stated thus:

Proposition 4.3. *Under GRH, Conjecture* 3.9 *holds if and only if there exists $y > 0$ such that $S(y) < 2$.*

4B. The Unconditional Case

The unconditional bounds of n_χ are less good than the GRH ones in Proposition 4.2 because of the requirement that $\operatorname{Re}\Phi(s) \geq 0$ on the whole critical strip. By using an argument of Odlyzko [Poitou 1977], this last condition holds when we take in Theorem 4.1 the function $G_y(x) = F_y(x)/\cosh(x/2)$ with $F_y(x) = e^{-yx^2}$ $(y > 0)$. Thus we obtain the following bound of n_χ.

Theorem 4.4. *For all $y > 0$, we have $n_\chi \leq T(y)$, where*

$$\begin{aligned}T(y) &= \left(2\int_0^{+\infty} \frac{e^{-yx^2}}{\cosh(x/2)}\, dx\right)^{-1} \times \\ &\Big(\ln \mathfrak{f}_\chi - 2\sum_{\mathfrak{p},m} \frac{\ln N_{K/\mathbb{Q}}(\mathfrak{p})}{1+N_{K/\mathbb{Q}}(\mathfrak{p})^m} \chi'(\mathfrak{p})^m e^{-y(m\ln N_{K/\mathbb{Q}}(\mathfrak{p}))^2} \\ &\quad - 2(\ln 2\pi + \gamma + 2\ln 2) - 2\varepsilon J(G_y) + 2I(G_y)\Big).\end{aligned}$$

In practice we check Conjecture 3.9 using this criterion:

Proposition 4.5. *Conjecture* 3.9 *holds if there exists* $y > 0$ *such that* $T(y) < 2$.

To compute $S(y)$ and $T(y)$, we begin by computing the integrals $I(F_y)$, $J(F_y)$, $I(G_y)$ and $J(G_y)$ to a high enough precision, we then compute the series over the prime ideals in the Weil explicit formula by computing $\chi'(\mathfrak{p})$ and $N_{K/\mathbb{Q}}(\mathfrak{p})$ for each prime number p less than some large enough p_0. Actually the number field N is defined by a polynomial $P(x)$; for every prime number p prime to the index of N, the decomposition of the ideal (p) into a product of prime ideals of N is given by the decomposition of $P(x)$ modulo p; see [Cohen 1993]. Since $N/\mathbb{Q}$ is a Galois extension, then one needs to compute only the degree f of the first irreducible polynomial appearing in the decomposition of $P(x)$ modulo p. The computations of $\chi'(\mathfrak{p})$ and $N_{K/\mathbb{Q}}(\mathfrak{p})$ are given in the proposition below:

Proposition 4.6. *Let* $k_1 = \mathbb{Q}(\sqrt{d_1})$, $k_2 = \mathbb{Q}(\sqrt{d_2})$, $k_3 = \mathbb{Q}(\sqrt{d_3})$ *be the quadratic subfields of* N.

- *If* $f = 1$ *then* $N_{K/\mathbb{Q}}(\mathfrak{p}) = p$ *and* $\chi'(\mathfrak{p}) = +1$.
- *If* $f = 4$ *then* $N_{K/\mathbb{Q}}(\mathfrak{p}) = p^2$ *and* $\chi'(\mathfrak{p}) = -1$.
- *If* $f = 2$ *we have two cases*:
  - *If* $(\frac{d_i}{p}) = -1$ *for exactly one* $i \in \{1, 2, 3\}$, *then* $N_{K/\mathbb{Q}}(\mathfrak{p}) = p^2$ *and* $\chi'(\mathfrak{p}) = +1$.
  - *If* $(\frac{d_i}{p}) = +1$ *for exactly one* $i \in \{1, 2, 3\}$, *then* $N_{K/\mathbb{Q}}(\mathfrak{p}) = p$ *and* $\chi'(\mathfrak{p}) = -1$.

Example 4.7. Let $N = \mathbb{Q}(\sqrt{M})$, where

$$M = \frac{5 + \sqrt{5}}{2}\frac{21 + \sqrt{21}}{2}.$$

The quaternion field N could be defined by the polynomial $P(x)$ in example 1 of section 6. One can compute the different terms in $T(y)$ for $y = 0.04$ and show that the sum over the prime ideals is equal to -0.33763, $J(G_y) = 0.89478$ and $I(G_y) = 0.83304$. Thus $T(y) = 0.39377$.

When the conductor $\mathfrak{f}_\chi$ is large, the computation of $S(y)$ and $T(y)$ is slower and this is essentially due to the possible existence of low zeros of the Artin L-function $L(s, \chi)$. Actually when the first zeros of $L(s, \chi)$ distinct from $\frac{1}{2}$ are close to the real axis, one needs to compute $S(y)$ and $T(y)$ for smaller positive values of y in order to be able to bound $S(y)$ and $T(y)$ above by 2 (see Propositions 4.3 and 4.5). An approach to the problem of low zeros of $L(s, \chi)$ in terms of the conductor $\mathfrak{f}_\chi$ is given in the next section.

5. AN UPPER BOUND FOR n_χ AND LOW ZEROS OF $L(s, \chi)$

We now give estimates on the upper bounds of n_χ and the first zero $\rho_\chi = \frac{1}{2} + i\,\beta_\chi$ of $L(s, \chi)$ distinct from $\frac{1}{2}$. For this purpose, we apply Theorem 4.1 to suitable functions with compact supports. If we assume the GRH, then one can prove more precise estimates on n_χ and β_χ. Such improvements have been considered in [Mestre 1986] for L-series of modular forms.

Theorem 5.1. *Under GRH*,

$$n_\chi \ll \frac{\ln \mathfrak{f}_\chi}{\ln\ln \mathfrak{f}_\chi} \quad \text{and} \quad |\beta_\chi| \ll \frac{1}{\ln\ln \mathfrak{f}_\chi}.$$

Proof. We first need an estimate for the sum over the prime ideals of K in Theorem 4.1. Let F be a function with compact support satisfying the hypotheses of Theorem 4.1 and let $F_T(x) = F(\frac{x}{T})$. By using the prime number theorem, one can prove the following estimate:

Lemma 5.2. *The sum over the prime ideals in Theorem* 4.1 *is bounded by the inequality*

$$\left|\sum_{\mathfrak{p},m} \frac{\ln N_{K/\mathbb{Q}}(\mathfrak{p})}{N_{K/\mathbb{Q}}(\mathfrak{p})^{m/2}}\chi'(\mathfrak{p})^m F_T(m \ln N_{K/\mathbb{Q}}(\mathfrak{p}))\right| \le C_0 e^{T/2},$$

with $C_0 > 0$.

We also need an easy lemma:

Lemma 5.3. *Define* F *by*

$$F(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1,\\ 0 & \text{otherwise}.\end{cases}$$

Then F *satisfies the hypotheses of Theorem* 4.1 *and*

$$\hat{F}(u) = \left(\frac{2\sin(u/2)}{u}\right)^2.$$

Now if we put $F_T(x) = F(\frac{x}{T})$ then $\hat{F}_T(u) = T\hat{F}(Tu)$. Applying Weil's explicit formula to F_T and using Lemma 5.2, we obtain the estimate:

$$n_\chi T \le \ln \mathfrak{f}_\chi + C_0 e^{T/2} + 2(I(F_T) + J(F_T)),$$

since $I(F_T)$ and $J(F_T)$ are bounded as T tends to $+\infty$, replacing T by $2\ln\ln\mathfrak{f}_\chi$, we see that

$$n_\chi \ll \frac{\ln\mathfrak{f}_\chi}{\ln\ln\mathfrak{f}_\chi},$$

proving the first inequality in the statement of Theorem 5.1. To prove the theorem's second inequality, we use another even function G with compact support, defined as follows.

Lemma 5.4. *Let*

$$G(x) = \begin{cases} (1-x)\cos(\pi x) + \dfrac{3}{\pi}\sin(\pi x) & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then G satisfies the hypotheses of Theorem 4.1 and

$$\hat{G}(u) = \Big(2 - \frac{u^2}{\pi^2}\Big)\Big(\frac{2\pi}{\pi^2 - u^2}\cos\frac{u}{2}\Big)^2.$$

We now apply once more Weil's explicit formula to $G_T(x) = G(x/T)$ and replace T by $\sqrt{2}\pi/|\beta_\chi|$. We obtain the estimate

$$\begin{aligned} \frac{8}{\pi^2} n_\chi T \ge \ln\mathfrak{f}_\chi - 2(\ln 2\pi + \gamma + 2\ln 2) \\ - 2\varepsilon J(G_T) + 2I(G_T) \\ - \sum_{\mathfrak{p},m} \frac{\ln N_{K/\mathbb{Q}}(\mathfrak{p})}{N_{K/\mathbb{Q}}(\mathfrak{p})^{m/2}} \chi'(\mathfrak{p})^m G_T(m \ln N_{K/\mathbb{Q}}(\mathfrak{p})). \end{aligned}$$

Using Lemma 5.2, the above estimate (1) on n_χ and the fact that the integrals $I(G_T)$ and $J(G_T)$ are bounded as T tends to $+\infty$, we deduce, for some positive constants A and B:

$$\frac{\ln\mathfrak{f}_\chi}{\ln\ln\mathfrak{f}_\chi} AT + Be^{T/2} \ge \ln\mathfrak{f}_\chi,$$

so that

$$T \ge \min\Big(\frac{1}{2A}, 1 - \frac{\ln(2B)}{\ln\ln\mathfrak{f}_\chi}\Big)\ln\ln\mathfrak{f}_\chi.$$

Thus for sufficiently large $\mathfrak{f}_\chi$ we have $T \gg \ln\ln\mathfrak{f}_\chi$, and so

$$|\beta_\chi| \ll \frac{1}{\ln\ln\mathfrak{f}_\chi},$$

concluding the proof of the theorem. □

Corollary 5.5. *If we assume the GRH,*

$$\lim_{\mathfrak{f}_\chi \to +\infty} \rho_\chi = \tfrac{1}{2}.$$

Without assuming the GRH, we have the following estimate for n_χ, which is less good than the one in Theorem 5.1; see [Mestre 1983] for a similar result in the case of elliptic curves.

Theorem 5.6. $n_\chi < \ln\mathfrak{f}_\chi$ *unconditionally.*

Proof. Define the function H_T with compact support by $H_T(x) = F_T(x)/\cosh(x/2)$, where F_T is defined after Lemma 5.3. By using an argument of Odlyzko [Poitou 1977], one can show that the Mellin transform Φ_T of H_T satisfies $\operatorname{Re}\Phi_T(s) \ge 0$ in the critical strip. Thus, when we apply Theorem 4.1 to H_T, we obtain

$$\begin{aligned} n_\chi \Phi_T\big(\tfrac{1}{2}\big) \\ \le \ln\mathfrak{f}_\chi - 2(\ln 2\pi + \gamma + 2\ln 2) \\ - 2\varepsilon J(H_T) + 2I(H_T) \\ - \sum_{\mathfrak{p},m} \frac{\ln N_{K/\mathbb{Q}}(\mathfrak{p})}{N_{K/\mathbb{Q}}(\mathfrak{p})^{m/2}} \chi'(\mathfrak{p})^m H_T(m \ln N_{K/\mathbb{Q}}(\mathfrak{p})). \end{aligned}$$

Since H_T is a decreasing function on $[0, +\infty]$, one can show:

Lemma 5.7.

$$\Big|\sum_{\mathfrak{p},m} \frac{\ln N_{K/\mathbb{Q}}(\mathfrak{p})}{N_{K/\mathbb{Q}}(\mathfrak{p})^{m/2}} \chi'(\mathfrak{p})^m H_T(m \ln N_{K/\mathbb{Q}}(\mathfrak{p}))\Big| \le 4 \sum_{p^m \le e^T} \frac{\ln p}{p^{m/2}} H_T(m \ln p).$$

Thus, by using the inequality before the lemma, we obtain

$$\begin{aligned} n_\chi \Phi_T\big(\tfrac{1}{2}\big) \le \ln\mathfrak{f}_\chi - 2(\ln 4\pi + \gamma) + 2J(H_T) + 2I(H_T) \\ + 4 \sum_{p^m \le e^T} \frac{\ln p}{p^{m/2}} H_T(m \ln p). \end{aligned}$$

Now if we put $T = \ln 3$, we obtain

$$\begin{aligned} 1.072 n_\chi &\le \ln\mathfrak{f}_\chi - 6.216 + 0.523 + 4.648 + 0.683 \\ &\le \ln\mathfrak{f}_\chi - 0.362. \end{aligned}$$

And so we find that $n_\chi < \ln\mathfrak{f}_\chi$. □

6. COMPUTATIONS OF n_χ FOR QUATERNION FIELDS

Table 1 gives our computed data. Each box refers to one quaternion field $N/\mathbb{Q}$, giving on the top line a reduced polynomial $P(x)$ ("reduced" meaning that we have written $N = \mathbb{Q}[\theta]$, choosing for θ a minimal primitive vector of the lattice of integers of N for the "twisted" trace form $\operatorname{tr}_{N/\mathbb{Q}}(x\bar{y})$), and on the bottom line other related information. The computations were done using PARI-GP version 2.0.19.

According to [Kwon 1996], the minimum discriminant both in the real and in the imaginary case is

	$P(x)$ and D_N	R/I	quad. subfields	$W(\chi)$	y_0	$S(y_0)$	y	$T(y)$	n_χ
1	$x^8-x^7-34x^6+29x^5+361x^4-305x^3-1090x^2+1345x-395$								
	1340095640625	R	$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{21})$	$+1$	0.04	0.00806	0.04	0.393771	0
2	$x^8+315x^6+34020x^4+1488375x^2+22325625$								
	1340095640625	I	$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{21})$	-1	0.07	1.04505	0.11	1.58039	1
3	$x^8-205x^6+13940x^4-378225x^2+3404025$								
	74220378765625	R	$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{41})$	-1	0.05	1.00067	0.1	1.30413	1
4	$x^8-3x^7+142x^6-115x^5+6641x^4+3055x^3+157938x^2+152941x+2031361$								
	6011850680015625	I	$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{41})$	-1	0.05	1.26425	0.05	2.09134	1
5	$x^8-x^7-178x^6-550x^5+7225x^4+44407x^3+55928x^2-45392x+4096$								
	31172897213027361	R	$\mathbb{Q}(\sqrt{17})$, $\mathbb{Q}(\sqrt{33})$	$+1$	0.04	0.00222	0.04	0.31774	0
6	$x^8-3x^7+106x^6+381x^5+414x^4-8475x^3+44497x^2+151740x+253168$								
	31172897213027361	I	$\mathbb{Q}(\sqrt{17})$, $\mathbb{Q}(\sqrt{33})$	-1	0.04	1.19064	0.04	2.05980	1
7	$x^8-3x^7-475x^6-2386x^5+56669x^4+732202x^3+3280440x^2+5788174x+2396941$								
	12187467896636600569	R	$\mathbb{Q}(\sqrt{37})$, $\mathbb{Q}(\sqrt{41})$	-1	0.03	1.03133	0.03	1.75340	1
8	$x^8-3x^7-847x^6-4250x^5+194805x^4+2321042x^3+4218300x^2-28827252x-48031623$								
	388282220975269366201	R	$\mathbb{Q}(\sqrt{37})$, $\mathbb{Q}(\sqrt{73})$	-1	0.03	1.00010	0.03	1.35751	1
9	$x^8-3x^7+1854x^6+14657x^5+1134753x^4+15385779x^3+370857442x^2+2861780247x+28470071727$								
	3145085989899681866228l	I	$\mathbb{Q}(\sqrt{37})$, $\mathbb{Q}(\sqrt{73})$	-1	0.03	1.84217	0.01	2.83822	1
10	$x^8-3x^7+1042x^6+8233x^5+284219x^4+4899401x^3+42209694x^2+179998937x+404059099$								
	987184899627564646089	I	$\mathbb{Q}(\sqrt{37})$, $\mathbb{Q}(\sqrt{41})$	-1	0.03	1.58551	0.03	2.81849	1
11	$x^8-x^7-866x^6-2686x^5+197617x^4+1072207x^3-8786448x^2-32864208x+159160192$								
	420386522758923179809	R	$\mathbb{Q}(\sqrt{17})$, $\mathbb{Q}(\sqrt{161})$	$+1$	0.03	0.19296	0.03	1.13789	0
12	$x^8-3x^7-1591x^6-7978x^5+718061x^4+8174530x^3-29006964x^2-433628432x+235862473$								
	16964214194699233633081	R	$\mathbb{Q}(\sqrt{37})$, $\mathbb{Q}(\sqrt{137})$	-1	0.03	1.00204	0.03	1.64797	1
13	$x^8-3x^7+3478x^6+27505x^5+4489397x^4+53881703x^3+2972520282x^2+26220344507x+651061429207$								
	1374101349770637924279561	I	$\mathbb{Q}(\sqrt{37})$, $\mathbb{Q}(\sqrt{137})$	-1	0.05	2.24737	0.01	2.88613	1
14	$x^8-12x^6+36x^4-36x^2+9$								
	12230590464	R	$\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$	$+1$	0.05	0.00002	0.08	0.11665	0
15	$x^8+12x^6+36x^4+36x^2+9$								
	12230590464	I	$\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$		0.05	1.000005	0.05	1.05777	≤ 1
16	$x^8-44x^6+308x^4-484x^2+121$								
	29721861554176	R	$\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{11})$	$+1$	0.05	0.01167	0.05	0.36928	0
17	$x^8-76x^6+1748x^4-12996x^2+29241$								
	789298907447296	R	$\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{19})$	$+1$	0.04	0.04449	0.04	0.66149	0

TABLE 1 (start). For each quaternion field $N/\mathbb{Q}$, we show a reduced polynomial $P(x)$ (see beginning of Section 6), the discriminant d_N, whether N is real or imaginary, two quadratic subfields $\mathbb{Q}(\sqrt{d_1})$ and $\mathbb{Q}(\sqrt{d_2})$ of N—the third being $\mathbb{Q}(\sqrt{d_1 d_2})$—and the values of $W(\chi)$, y_0, $S(y_0)$ (Proposition 4.3), y, $T(y)$ (Proposition 4.5) and n_χ.

	$P(x)$ and D_N	R/I	quad. subfields	$W(\chi)$	y_0	$S(y_0)$	y	$T(y)$	n_χ
18	$x^8-60x^6+810x^4-1800x^2+900$ 47775744000000	R	$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{6})$		0.07	1.00101	0.07	1.13852	≤ 1
19	$x^8-60x^6+1170x^4-9000x^2+22500$ 47775744000000	R	$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{6})$	+1	0.07	0.09399	0.07	0.61520	0
20	$x^8+60x^6+810x^4+1800x^2+900$ 47775744000000	I	$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{6})$		0.07	1.07405	0.07	1.55366	≤ 1
21	$x^8+60x^6+1170x^4+9000x^2+22500$ 47775744000000	I	$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{6})$		0.08	1.09340	0.07	1.63606	≤ 1
22	$x^8+105x^6+3780x^4+55125x^2+275625$ 343064484000000	I	$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{21})$	+1	0.05	0.54966	0.05	1.53349	0
23	$x^8+205x^6+13940x^4+378225x^2+3404025$ 19000416964000000	I	$\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{41})$		0.05	1.13981	0.03	1.80213	≤ 1

TABLE 1 (continued)

$2^{24}3^6$, attained exactly in the fields 14 and 15; similarly the smallest coincidences between two real or imaginary fields occur for the discriminant $2^{22}5^63^6$, attained exacltly on the four fields 18 to 21. Fields 1 to 13 are tame, the others are not.

ACKNOWLEDGMENTS

I thank J. Martinet for bringing this problem to my attention and for his helpful comments and suggestions. I also profited from discussions with H. Cohen and B. Erez. I also thank J.-P. Serre for his remarks on an earlier version of this paper.

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Received February 14, 2000; accepted in revised form November 4, 2000