Off-Center Reflections: Caustics and Chaos

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CONTENTS

We study the possible link between the dynamics of a certain family of circle maps and the caustics of their iterates. The maps are defined by *off-center reflections* in a mirrored circle; they can also be regarded as perturbed rotations. Some of our experimental observations can be justified rigorously: for example, a lower bound is given for the number of cusps and the mode-locking behavior are studied. Symplectic topology is a particularly useful tool in this study.

1. INTRODUCTION

We study a particular one-parameter family of circle maps, called *off-center reflections*, first introduced in [Yau 1993, problem 21] (definitions are given in Section 2). We explore the possible link between the dynamics of this family of circle maps and their *caustics*. We observe and prove several phenomena: for example, within a certain generic range of the parameter r, the caustics of odd iterates have exactly four cusps, whereas for even iterates the caustic is a curve tangent to the circle at exactly four points. Other partial results are given in the hope of stimulating further investigations.

An off-center reflection has several interesting analytic forms. It is a Blaschke product restricted to the circle. It has an infinite series expression in the parameter, highlighting its character as a perturbation of a rotation on the circle. Starting with [Arnold 1961], a standard type of perturbations has attracted much interests in mathematics and physics communities [Bak et al. 1988; Ding and Hemmer 1988; Zheng 1991]. This standard type is exactly a reduction of the series of the off-center reflection.

In the family of off-center reflections, as the parameter r goes from 0 to 1, we go from the antipodal map to the doubling map on the circle. This provides a nice particular example of a deformation leading from simple dynamics to chaos, with the unexpected phenomenon of "half-bifurcations" along

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the way (see Figure 9 on page 298). We hope that some notion of stability about the cusp points and tangent points on the caustics might emerge from further study of this family of circle maps.

Section 2 gives the definition and some analytic properties of the map. Section 3 studies the caustics of the map and of its iterates. Our results concerning the caustics of odd iterates are more conclusive than those for even iterates. Following a symplectic and contact geometry interpretation developed by Arnold [1994], we discover the generating function for the corresponding Lagrangian embedding. As a result, the classical four-vertex theorem is applicable. The method fails for even iterates, but explicit computations still provide reasonable support for certain predictions.

Section 4 presents our main heuristic observations, with illustrations, and the theoretical support for them. For example, we have the partial result that the caustic is stable when $r \leq \frac{1}{3}$. The more tedious computations are segregated into Section 4B for detailed reading.

Section 5 studies the phenomenon of mode-locking for this family of circle maps, and gives an estimate for the width of the resonance zone. This is an attempt to understand the iterates of the map. This family extends a class of examples studied by Arnold and others, which exhibits the same behavior. The mode-locking of the off-center reflections and its "complex conjugates" are totally different. Moreover, $r = \frac{1}{3}$ is the first value of the parameter where this behavior undergoes a structural change. This is probably not simply a coincidence with the bifurcation values of cusps.

2. OFF-CENTER REFLECTION

An *off-center reflection* is a map $\mathbb{S}^1 \to \mathbb{S}^1$ defined as follows: Pick a point, say $(r, 0)$ in the interior of the unit disk D^2. For any point in $\partial D^2 = \mathbb{S}^1$, with angle coordinate φ, say, emit a ray from $(r, 0)$ to φ. The ray will be reflected at φ with $\mathbb{S}^1$ as the curve of reflection and it will hit $\mathbb{S}^1$ again at $R_r(\varphi)$ on. The map $R_r : \mathbb{S}^1 \to \mathbb{S}^1$ is the off-center reflection. See Figure 1.

We first establish analytic expressions for the map $R_r : \mathbb{S}^1 \to \mathbb{S}^1$, where $0 \leq r < 1$. Write

$$R_r(\varphi) = \varphi + \pi - 2\alpha \bmod 2\pi,$$

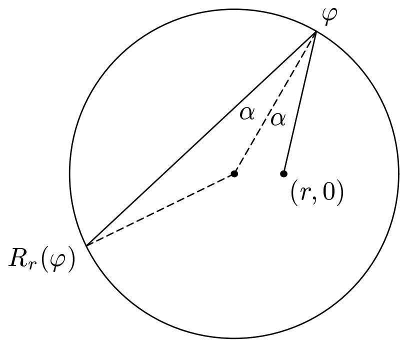


FIGURE 1. The map $\varphi \mapsto R_r(\varphi)$ is the off-center reflection with center $(r, 0)$.

with

$$\alpha = \alpha(\varphi) := \operatorname{Arg}(\cos\varphi - r + i\sin\varphi) - \varphi.$$

Here Arg is the principal argument, taking values in $(-\pi, \pi]$. Since the incidence angle α is an odd function of φ, it has a Fourier sine series expansion, say $\alpha = \sum a_k(r)\sin(k\varphi)$; to compute its coefficients, we notice that

$$\frac{d\alpha}{dr} = \frac{\sin\varphi}{(\cos\varphi - r)^2 + \sin^2\varphi}$$

also has a sine expansion, with coefficients

$$\begin{aligned}\frac{\partial a_k}{\partial r} &= \frac{2}{\pi}\int_0^\pi \frac{\sin\varphi\sin(k\varphi)}{1-2r\cos\varphi+r^2}\,d\varphi \\ &= \frac{1}{\pi}\int_0^\pi \left(\frac{\cos(k-1)\varphi}{1-2r\cos\varphi+r^2} - \frac{\cos(k+1)\varphi}{1-2r\cos\varphi+r^2}\right)d\varphi \\ &= \frac{1}{\pi}\left(\frac{\pi r^{k-1}}{1-r^2} - \frac{\pi r^{k+1}}{1-r^2}\right) = r^{k-1}.\end{aligned}$$

Integrating we conclude that $a_k = r^k/k$, and we can write

$$R_r(\varphi) = \varphi + \pi - 2\sum_{k=1}^{\infty}\frac{r^k}{k}\sin(k\varphi) \bmod 2\pi. \tag{2–1}$$

This formula (without the modulo 2π) is exactly the lifting of R_r to a function from $\mathbb{R}$ to $\mathbb{R}$ taking 0 to π. We will often omit the mod 2π when the context is clear.

By playing with the argument of a complex number, we get another expression for the map R_r, as the restriction to the unit circle of the map

$$z \mapsto -z^2\,\frac{1 - rz}{z - r} \tag{2–2}$$

on the unit disk punctured at $(r, 0)$. Thus the off-center reflection is a special case of a Blaschke product. Therefore, this function $R_r(\varphi)$ is harmonic when (r, φ) are treated as the polar coordinates, since it is the argument of the analytic function

$$-z(1-z)^2.$$

By changing a sign in the formula for R_r, we have another map $\bar{R}_r$ defined by

$$\bar{R}_r : \varphi \mapsto \varphi + \pi + 2\sum_{k=1}^{\infty} \frac{r^k}{k} \sin(k\varphi) \bmod 2\pi.$$

Geometrically, $\bar{R}_r(\varphi)$ is the reflection of $R_r(\varphi)$ in the diameter joining φ to $\varphi + \pi$. The map $\bar{R}_r$ can be extended to the unit disk, via

$$z \mapsto -\frac{z-r}{1-rz}.$$

Therefore, it defines a map in $\mathrm{PSL}(2, \mathbb{R})$, the isometry group of the hyperbolic disk. The dynamics of R_r and this "conjugated" map $\bar{R}_r$ are completely different. See [Herman 1979] and Section 5B.

For sufficiently small r, R_r behaves very similarly to R_0, the antipodal map. When $r < \frac{1}{3}$, in fact, R_r is in the component of R_0 in the group of orientation preserving diffeomorphisms of $\mathbb{S}^1$. However, $R_{1/3}$ is only a homeomorphism on $\mathbb{S}^1$ and R_r a degree 1 map when $\frac{1}{3} < r < 1$. This can be easily seen from the derivatives of R_r, whose expressions will be useful later:

$$R'_r(\varphi) = \frac{1 - 4r\cos\varphi + 3r^2}{1 - 2r\cos\varphi + r^2},$$

$$R''_r(\varphi) = \frac{2r(1-r^2)\sin\varphi}{(1 - 2r\cos\varphi + r^2)^2},$$

$$R'''_r(\varphi) = \frac{2r(1-r^2)\big((1+r^2)\cos\varphi - 2r(1+\sin^2\varphi)\big)}{(1 - 2r\cos\varphi + r^2)^3}.$$

Section 4B will give more information about the fixed point and other special points of R_r. More dynamical properties such as periodic cycles and whether they are attracting are discussed in [Au 1999].

3. CAUSTICS

3A. Caustic of the Off-center Reflection

Two classical examples of caustics are the locus of focal points with respect to a point on a surface and the focal curve of a convex plane curve, which gave rise to the famous Geometric Theorem (Conjecture) of Jacobi and the four-vertex theorem. There are many interesting at-least-four results related to caustics; see [Arnold 1994; 1996; Tabachnikov 1990; 1995]. The caustic of an off-center reflection provides another one. The conjugate locus of a point on a flat flying disc is, at degenerate situation, the caustic of the off-center reflection. Bruce and other [Bruce et al. 1982; Bruce and Giblin 1984; 1985; Giblin and Kingston 1986] have analyzed the singularities of the caustics produced by a point light source when it is reflected in a codimension 1 "mirror" in $\mathbb{R}^2$ and $\mathbb{R}^3$. Their emphasis though is on the "source genericity": whether the caustics could be made generic by moving the source. See also [Bruce et al. 1981].

For a circle map $f : \mathbb{S}^1 \to \mathbb{S}^1$, the family of lines joining φ to $f(\varphi)$ is

$$\begin{aligned} &F(\varphi, x, y) \\ &\quad = (\sin f\varphi - \sin\varphi)(x - \cos\varphi) - (\cos f\varphi - \cos\varphi)(y - \sin\varphi) \\ &\quad = (\sin f\varphi - \sin\varphi)x - (\cos f\varphi - \cos\varphi)y - \sin(f\varphi - \varphi), \end{aligned}$$

where $f\varphi = f(\varphi)$ and so on. The *caustic* of the map f is defined to be the envelope of these lines. Thus, it is given by the equations

$$\frac{\partial F}{\partial\varphi}(\varphi, x, y) = 0 = F(\varphi, x, y),$$

that is,

$$\begin{pmatrix} \sin f\varphi - \sin\varphi & -\cos f\varphi + \cos\varphi \\ f'\varphi\cos f\varphi - \cos\varphi & f'\varphi\sin f\varphi - \sin\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(f\varphi - \varphi) \\ (f'\varphi - 1)\cos(f\varphi - \varphi). \end{pmatrix}$$

Solving for x, y, we obtain a parameterization of the caustic:

$$\begin{aligned} x(\varphi) &= \frac{f'\varphi\cos\varphi + \cos f\varphi}{1 + f'\varphi}, \\ y(\varphi) &= \frac{f'\varphi\sin\varphi + \sin f\varphi}{1 + f'\varphi}. \end{aligned} \tag{3–1}$$

The tangent direction of the caustic, which is degenerate at the cusps, is given by the following formulas:

$$x'(\varphi) = \frac{f''\varphi(\cos\varphi - \cos f\varphi) \\ -f'\varphi(1+f'\varphi)(\sin\varphi + \sin f\varphi)}{(1+f'\varphi)^2},$$
$$y'(\varphi) = \frac{f''\varphi(\sin\varphi - \sin f\varphi) \\ +f'\varphi(1+f'\varphi)(\cos\varphi + \cos f\varphi)}{(1+f'\varphi)^2}, \tag{3–2}$$

The caustic (3–1) of the off-center reflection may run to infinity since $1+R_r'(\varphi)$ may be equal to zero. In fact, this is so if and only if $r \geq \frac{1}{2}$. It would be more appropriate to define the caustic as the envelope of the geodesic normal field on the sphere. After stereographic projection, it does not matter whether the caustic is defined on the plane or the sphere, as the local properties remain unchanged (Darboux's Theorem of symplectic structure). As we will see, the local properties of the caustic of R_r can be understood by direct computation.

Theorem 3.1. *For all $0 < r < 1$, there are exactly four cusp points on the caustic of R_r. Two of them correspond to the R_r-orbit $\{0, \pi\}$, of period two.*

Proof. The derivatives of x and y can be expressed in terms of r and φ, namely:

$$x'(\varphi) = \frac{6r^2(-\cos\varphi + r\cos(2\varphi))(r-\cos\varphi)\sin\varphi}{(-1-2r^2+3r\cos\varphi)^2}$$

$$y'(\varphi) = \frac{6r^2(-1+2r\cos\varphi)(r-\cos\varphi)\sin^2\varphi}{(-1-2r^2+3r\cos\varphi)^2}.$$

The common solutions for $x'(\varphi) = 0 = y'(\varphi)$ are $\varphi = 0, \pi$ and two values of φ with $\cos\varphi = r$. Clearly, 0 and π are zeros of x' of first order and of y' of second order, thus, these are semicubical cusps. If $\cos\varphi = r$, after further differentiation and evaluation at the point, one has

$$x''(\varphi) = -12r^3, \qquad y''(\varphi) = \frac{6r^2(2r^2-1)}{\sqrt{1-r^2}},$$

$$x'''(\varphi) = \frac{12(5r^4+r^2)}{\sqrt{1-r^2}}, \quad y'''(\varphi) = \frac{-6r^3(10r^2-3)}{1-r^2}.$$

Thus,

$$x''y''' - x'''y'' = 72r^4/(1-r^2) \neq 0,$$

and there are also semicubical cusps at those values of φ with $\cos\varphi = r$. □

Figure 2 shows caustics of R_r for $r < \frac{1}{2}$ and $r > \frac{1}{2}$. Since the second one runs to infinity, it is drawn with a "compressed" scale, where a circle of radius greater than one represents the point of infinity and the caustic has a self-intersection there.

3B. A Symplectic Reformulation

The explicit computations in the previous section give an exact count of the number of cusp points, but they are difficult to extend to the study of the iterated map. In this section, we use a symplectic approach that helps overcome this difficulty. We follow the terminology of [Arnold 1994].

Denote the coordinates of the unit cotangent bundle $ST^*(\mathbb{R}^2)$ by (p_x, p_y, x, y), where $(x, y) \in \mathbb{R}^2$ and $p_x^2 + p_y^2 = 1$. This bundle is a contact 3-manifold with the contact 1-form $p_x\,dx + p_y\,dy$; the cotangent manifold $T^*(\mathbb{R}^2)$ is symplectic with the symplectic 2-form $d\,(p_x\,dx + p_y\,dy)$.

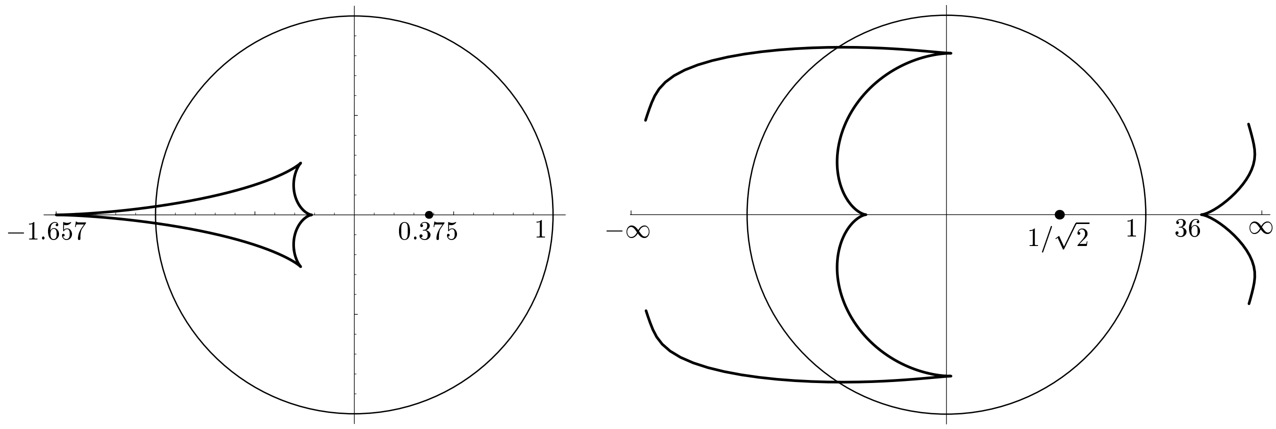


FIGURE 2. Sample caustics of R_r (left, $r = 0.375$; right, $r = 1/\sqrt{2}$).

The unit vector (p_x, p_y) in the direction from φ to $R_r(\varphi)$ is given by

$$p_x = \cos(\varphi + \pi - \alpha), \quad p_y = \sin(\varphi + \pi - \alpha).$$

Then the formula

$$\begin{pmatrix} \varphi \\ S \end{pmatrix} \mapsto \begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\varphi+\pi-\alpha) \\ \sin(\varphi+\pi-\alpha) \\ \cos\varphi + S\cos(\varphi+\pi-\alpha) \\ \sin\varphi + S\sin(\varphi+\pi-\alpha) \end{pmatrix}$$

defines a map $L : \mathbb{S}^1 \times \mathbb{R}^1 \longrightarrow T^*(\mathbb{R}^2)$, which may be thought of as a flow (in the parameter S) of unit speed in the direction of the reflection lines, starting with the round circle $S = 0$. This is a case of what Arnold [1994] called *Legendrian collapsing*.

Let $p : T^*(\mathbb{R}^2) \to \mathbb{R}^2$ be the canonical projection. The Jacobian $J(p \circ L)$ of $p \circ L$ is

$$\det \begin{pmatrix} \cos(\varphi+\pi-\alpha) & -\sin\varphi - S(1-\alpha')\sin(\varphi+\pi-\alpha) \\ \sin(\varphi+\pi-\alpha) & \cos\varphi + S(1-\alpha')\cos(\varphi+\pi-\alpha) \end{pmatrix},$$

so

$$\begin{aligned} J(p \circ L) &= \cos(\varphi+\pi-\alpha)\cos\varphi + \sin(\varphi+\pi-\alpha)\sin\varphi \\ &\qquad + S(1-\alpha') \\ &= -\cos\alpha + S(1-\alpha'). \end{aligned}$$

So the equation for the *critical curve* on $\mathbb{S}^1 \times \mathbb{R}^1$ is

$$S = \frac{\cos\alpha}{1-\alpha'}.$$

A translation of the definition of the caustic into symplectic terms (or direct calculation for the specific map at hand) shows that:

Proposition 3.2. *The critical curve, when mapped to the (x, y)-plane, agrees with the caustic of R_r.*

A simple calculation also shows that

$$p_x\,dx + p_y\,dy = \sin\alpha\,d\varphi + dS,$$

so the image of L is a Lagrange cylinder in $T^*(\mathbb{R}^2)$. Since α is an odd function of φ, we have

$$\int_{\mathbb{S}^1} \sin\alpha\,d\varphi = 0,$$

so $p(L)$ is an exact Lagrange cylinder. Define a function $S(\varphi)$ on the circle by

$$S(\varphi) = -\int_0^{\varphi} \sin\alpha\,d\varphi;$$

this gives a section in the Lagrange cylinder $p(L)$. It is easy to see that $-S(\varphi)$ is increasing for $0 \le \varphi \le \pi$ and decreasing when $\pi \le \varphi \le 2\pi$. Therefore, the curve C given by

$$\begin{aligned} x &= \cos\varphi + S(\varphi)\cos(\varphi + \pi - \alpha) \\ y &= \sin\varphi + S(\varphi)\sin(\varphi + \pi - \alpha) \end{aligned}$$

is quite likely to be a convex plane curve. We show that this is the case when $r \le \frac{1}{2}$.

First we note that C has a continuous normal field $(\cos(\varphi+\pi-\alpha),\ \sin(\varphi+\pi-\alpha))$. It is easy to compute

$$x'^2 + y'^2 = 4\sin^2\alpha + (\cos\alpha - S(1-\alpha'))^2.$$

Therefore, $x'^2 + y'^2 = 0$ is possible only when $\varphi = \pi$. But the number of zeros of $x'^2 + y'^2$ should be even (geometrically, because C is co-oriented). Thus $x'^2 + y'^2 > 0$ all the time and C is a smooth simple closed curve. The curvature of C is

$$\kappa = \frac{1-\alpha'}{\sqrt{x'^2 + y'^2}},$$

which is nonnegative when $r \le \frac{1}{2}$, so C is convex in this case. Now the family of reflection lines of R_r is identical to the family of normal lines of this convex curve C. Therefore, the caustic of R_r has at least 4 cusp points [Arnold 1994].

The function $S(\varphi)$ should be thought of as the *generating function* of the circle map R_r. The curve C is related to the *orthotomic* of such a reflection. We will study such generating functions for general circle maps in a forthcoming work.

3C. Iterations of Reflections

For an integer n, we denote the n-th iterate of R_r by $R_r^n = R_r \circ R_r^{n-1} : \mathbb{S}^1 \to \mathbb{S}^1$. Equations (3–1) and (3–2), with $f = R_r^n$, give a parametrization of the caustic of R_r^n and its tangent.

The cusps on caustics of R_r^n are more intriging and complicated than those of R_r. There are fundamental differences between the caustics depending on whether n is odd or even. To see this difference, we may consider the trivial case $r = 0$. When n is odd, R_0^n is the antipodal map and its caustic is a point (and so has a cusp), whereas for n even R_0^n is the identity map, whose caustic, being defined by the family of tangents, is the circle itself (and so is smooth). It is to be expected that this contrast remains while r is close to 0.

Theorem 3.3. *For small enough $r > 0$, the caustic of R_r^{2m+1} has at least 4 cusp points.*

Proof. To some extent, the symplectic method in Section 3B may be adopted for R_r^{2m+1}. It can be proved by induction on m that

$$R_r^{2m+1}(\varphi) = \varphi + \pi - 2\tilde{\alpha}_m(\varphi)$$

for some odd function $\tilde{\alpha}_m(\varphi)$. For example,

$$\begin{aligned}\tilde{\alpha}_1(\varphi) &= \alpha(\varphi) + \alpha\big(\varphi + \pi - 2\alpha(\varphi)\big)\\ &\quad + \alpha\big(\varphi - 2\alpha(\varphi) - 2\alpha(\varphi + \pi - 2\alpha(\varphi))\big).\end{aligned}$$

Therefore, we still have an exact Lagrangian cylinder and a sectional curve defined by

$$\tilde{S}(\varphi) = -\int_0^{\varphi} \sin\tilde{\alpha}_m \, d\varphi.$$

For small r, it is a convex curve and hence there are at least 4 cusp points on the caustic. □

Remark. The argument fails for even iterates of R_r, because the analog of $\tilde{\alpha}$ is not an odd function. Furthermore, it is not clear about how small the range of r should be. Yet, from experimental observation, there are at least four cusps for any $r > 0$ and there are exactly four for $0 < r < \frac{1}{3}$. For more, please see the discussion after proposition 4.3.

4. EXPERIMENTS AND OBSERVATIONS

To get more accurate information about the caustics of iterates, we have to rely on lengthy calculations. Our investigation is indeed partly theoretical and partly experimental. We will first describe some interesting properties with illustrations. The technical details of justification are left to the interested reader in Section 4B.

4A. Observations

We first put forward a conjecture about the exact picture of the caustic when R_r^n is still a diffeomorphism. Then we look at the bifurcation process of the structure of the caustics when r varies. Finally, we compare the caustics for different n. We will soon see that the 2-cycles of R_r play a special role (Propositions 4.2 and 4.4). The 2-cycles are $\{0, \pi\}$ and $\{\pm\varphi_c\}$ where $\varphi_c \in (0, \pi)$ and $\cos\varphi_c = (1 - \sqrt{1+8r^2})/(4r)$. We will often refer to this notation.

Conjecture 4.1. *For $0 < r \le \frac{1}{3}$, the caustic of R_r^{2m+1} is a C^∞ curve with exactly four cusp singularities, with two of them occurring at $\varphi = 0, \pi$. On the other hand, the caustic of R_r^{2m} is a differentiable curve; C^∞ everywhere except at exactly the four 2-periodic points of R_r, where the caustic is tangent to the unit circle.*

The conjecture is illustrated in Figure 3.

Proposition 4.2. *Any caustic curve of R_r^{2m} is tangent to the unit circle at any point φ satisfying $R_r^{2m}(\varphi) = \varphi$. In particular, this includes the points 0, π, and $\pm\varphi_c$. If $r \le \frac{1}{3}$, these are the only four points where the caustic meets the circle.*

Proof. By substitution of such φ into equations (3–1) with $f = R_r^n$, we have $x(\varphi) = \cos\varphi$ and $y(\varphi) = \sin\varphi$. Applying Lemma 4.5 to such φ's, the assertion about the tangential property of the caustics of even iterates of R_r follows easily. Next,

$$(R_r^n)'(\varphi)\cos\varphi - \cos R_r^n(\varphi) = 0$$
$$(R_r^n)'(\varphi)\sin\varphi - \sin R_r^n(\varphi) = 0.$$

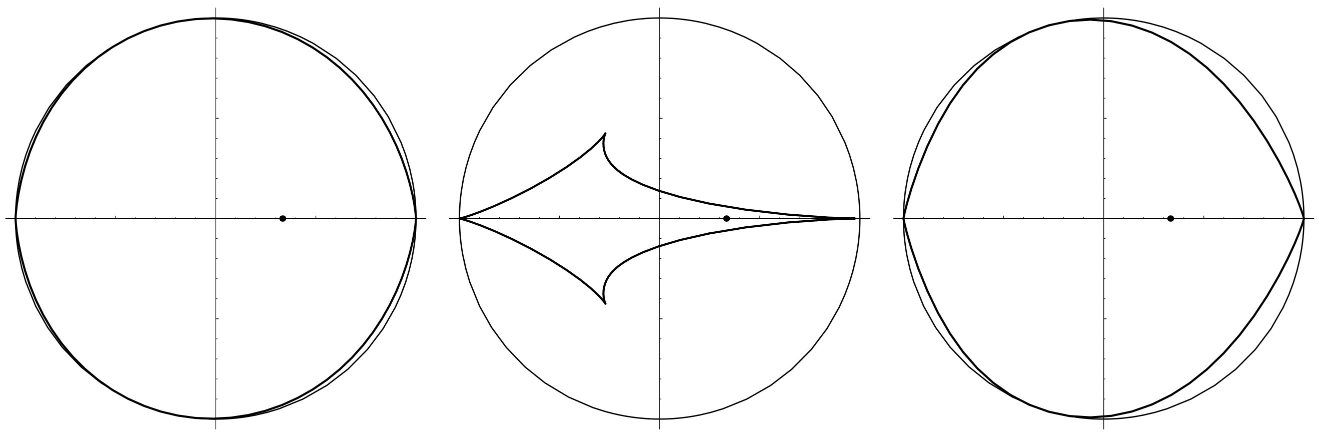


FIGURE 3. Caustics of iterates R_r^n, for $r = \frac{1}{3}$. From left to right, $n = 2, 3, 4$.

since $x^2+y^2=1$. This leads to $R_r^n(\varphi)=\varphi$. If $r\le\frac{1}{3}$, by Lemma 4.8, n must be even and φ is one of the four 2-periodic points. $\square$

We have already seen from symplectic topology that caustics of odd iterates, R_r^{2m+1}, always have at least four cusps for sufficiently small $r>0$. Moreover, for all R_r, two of the cusps occur at $\varphi=0,\pi$. Now, we may extend this result to odd iterates of R_r with isolated exceptional values of r. It turns out these exceptional values only occur in $r>\frac{1}{3}$.

Proposition 4.3. *For $0<r<\frac{1}{3}$ and for generic $\frac{1}{3}<r<1$, the caustic of R_r^{2m+1} always has cusps at $0,\pi$.*

Proof. For $\varphi_a=0,\pi$, one has $R_r(\varphi_a)=\varphi_a+\pi \bmod 2\pi$, so we may apply Lemma 4.5 to check whether there are cusps. We also have, $R_r^2(\varphi_a)=\varphi_a$ and $R_r''(\varphi_a)=0$, so we can simplify using Lemma 4.6 and obtain $(R_r^{2m+1})''(\varphi_a)=0$ and

$$\begin{aligned}(R_r^{n+2})'''(\varphi_a) &= R_r'(R_r(\varphi_a))(R_r^n)'(\varphi_a)R_r'''(\varphi_a)\\ &\quad + R_r'(\varphi_a)^3\big((R_r^n)'(\varphi_a)R_r'''(R_r(\varphi_a))\\ &\qquad + R_r'(R_r(\varphi_a))^3(R_r^n)'''(\varphi_a)\big).\end{aligned}$$

We temporarily define, for positive integers n,

$$A_n=-(R_r^n)'+(R_r^n)'^3+2(R_r^n)'''.$$

So it suffices to check that $A_{2m+1}(\varphi_a)\neq 0$. We will proceed by induction on m. First, by direct computation,

$$A_1(0)=\frac{24r^2}{(1-r)^2}>0,\qquad A_1(\pi)=\frac{24r^2}{(1+r)^2}>0.$$

Then, it can be shown that

$$\begin{aligned}&A_{n+2}(\varphi_a)\\ &\quad=\left(\frac{1-9r^2}{1-r^2}\right)^3 A_n(\varphi_a)+(R_r^n)'(\varphi_a)\\ &\qquad\times\big(2R_r'''(\varphi_a+\pi)R_r'(\varphi_a)^3+2R_r'(\varphi_a+\pi)R_r'''(\varphi_a)\\ &\qquad\quad -R_r'(\varphi_a)R_r'(\varphi_a+\pi)+R_r'(\varphi_a)^3R_r'(\varphi_a+\pi)^3\big)\\ &\quad=\left(\frac{1-9r^2}{1-r^2}\right)^3 A_n(\varphi_a)+A_2(\varphi_a)\cdot(R_r^n)'(\varphi_a).\end{aligned}$$

In particular,

$$\begin{aligned}A_{2m+1}(0) &= \left(\frac{1-9r^2}{1-r^2}\right)^3 A_{2m-1}(0)\\ &\qquad + A_2(0)\left(\frac{1-3r}{1-r}\right)^m\left(\frac{1+3r}{1+r}\right)^{m-1},\end{aligned}$$

$$\begin{aligned}A_{2m+1}(\pi) &= \left(\frac{1-9r^2}{1-r^2}\right)^3 A_{2m-1}(\pi)\\ &\qquad + A_2(\pi)\left(\frac{1-3r}{1-r}\right)^{m-1}\left(\frac{1+3r}{1+r}\right)^{m},\end{aligned}$$

where

$$\begin{aligned}A_2(0) &= \frac{48r^2(1+r)}{(1-r^2)^3}(1-3r+13r^2-15r^3),\\ A_2(\pi) &= \frac{48r^2(1+r)}{(1-r^2)^3}(1+3r+13r^2+15r^3).\end{aligned}$$

It can be easily computed that $A_2(0)>0$ for $0<r<\frac{1}{3}$ and $A_2(\pi)>0$ for all $r>0$. Thus,

$$A_{2m+1}(\varphi_a)\ge\left(\frac{1-9r^2}{1-r^2}\right)^3 A_{2m-1}(\varphi_a)>0.$$

For $r>\frac{1}{3}$, A_{2m+1} may have zeros. We can only conclude from above that A_{2m+1} is a rational function in r with denominator being a power of $(1-r^2)$. Therefore, it has only isolated zeros and cusps at $0,\pi$ occur for $r>\frac{1}{3}$ generically. $\square$

We observe that at cusp points, (3–2) gives a set of "homogeneous" equations which has zero determinant. Thus, except at a couple of φ's, it is sufficient to solve only one equation, say, $x'(\varphi)=0$. It is likely that this equation has exactly four solutions for $0<r<\frac{1}{3}$. However, it is still hard to solve explicitly especially for high iterates of R_r. Figure 4 shows the functions $(R_r^3)''/\big((R_r^3)'(1+(R_r^3)')\big)$ and $(\sin\varphi+\sin R_r^3(\varphi))/(\cos\varphi-\cos R_r^3(\varphi))$ for $r=0.1$ and $r=0.33$. Exactly four cusp solutions appear in each of them. We do not have a proof of this graphical fact. Perhaps it may be proved by detailed curve sketching argument and comparison of R_r^{2m-1} and R_r^{2m+1} using the known properties of R_r^2 given in Lemma 4.8.

This situation may be compared with the "Jacobi conjecture" promoted by Arnold [1994; 1996]. As mentioned in the beginning of Section 3A, the caustic of R_r agrees with the locus of conjugate points of $(r,0)$ on a flat flying disk. Although the loci of higher order conjugate points are not the same as the caustics of odd iterates of R_r, the geometric nature of their common contact indicates that two problems — whether there are exactly four cusps on the loci of higher order conjugate points and whether there are exactly four cusps of the caustics of odd iterate of R_r — might be related. In both

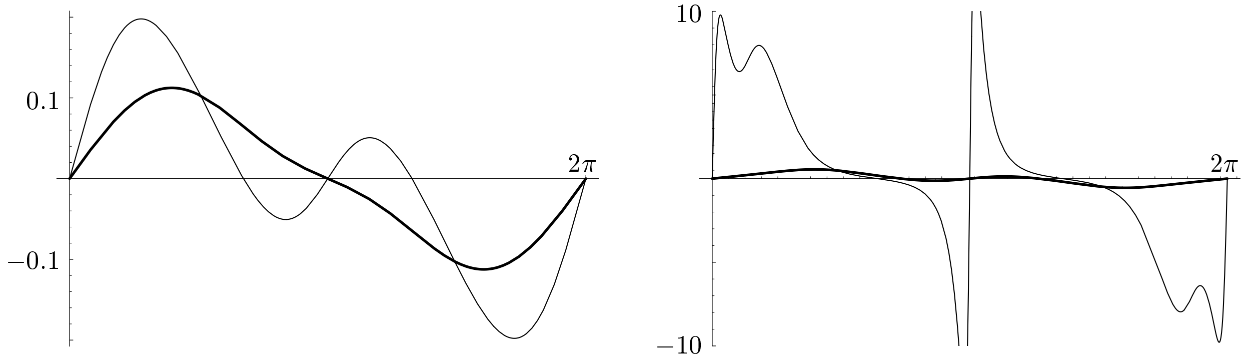


FIGURE 4. Graphs of $(R_r^3)''/\big((R_r^3)'(1+(R_r^3)')\big)$ (left) and $(\sin\varphi+\sin R_r^3(\varphi))/(\cos\varphi-\cos R_r^3(\varphi))$ (right), for $r=0.1$ (thin lines) and $r=0.33$ (thick lines).

cases, since we do not have exact nice formulas for the loci of higher order conjugate points and the caustics of odd iterates of R_r, it would be very difficult to have an exact count of cusps. Of course, Conjecture 4.1 deals with a very special situation. From graphical evidence, it is tempting to think that a proof should not be out of reach.

In the proof above, we see that for $n \geq 3$, except $A_3(\pi)$, we always have $A_n(0) = 0 = A_n(\pi)$ always at $r = \frac{1}{3}$. There is a possible structural change on the caustic of R_r^n occurring at $r = \frac{1}{3}$ for $n \geq 3$. We observe from experiment that, for the caustic of odd iterates, once $r > \frac{1}{3}$, bifurcation of cusp may occur. Interestingly, from the computed pictures, bifurcation only occurs at the cusp corresponding to $\varphi = 0$ but not others; see Figure 5 for an example. Would the different properties between $A_2(0)$ and $A_2(\pi)$ be part of the reason?

On the other hand, bifurcation into cusps also occurs for even iterates of R_r at $r = \frac{1}{3}$. We begin by examining some Taylor expansions. Since R_r and its iterates are 2π-periodic odd functions, they have particular nice expansions at $\varphi_a = 0, \pi$. This enables us to see the local properties of the caustics more clearly.

Let f be any even iterate of R_r, then $f(\varphi_a) = \varphi_a$. We write $\theta = \varphi - \varphi_a$ and $g(\theta) = f(\varphi) - \varphi_a$ and suppose it has an expansion

$$g(\theta) = \sum_{k=0} a_{2k+1}\theta^{2k+1}.$$

One can inductively work out the coefficients of the expansions of $1 + f'(\varphi) = 1 + g'(\theta)$, $\cos f(\varphi) = \pm\cos g(\theta)$, etc. If P_k, Q_k denote polynomials with $P_k(0,\dots,0) = 0 = Q_k(0,\dots,0)$, one has

$$x(\varphi) = \pm 1 \mp \frac{a_1}{2}\theta^2 + \sum_{k=1} \frac{P_k(a_1,\dots,a_{2k+1})}{(1+a_1)^{2k+1}}\theta^{2k},$$

$$y(\varphi) = \pm\frac{2a_1}{1+a_1}\theta + \sum_{k=1} \frac{Q_k(a_1,\dots,a_{2k+1})}{(1+a_1)^{2k+1}}\theta^{2k+1}.$$

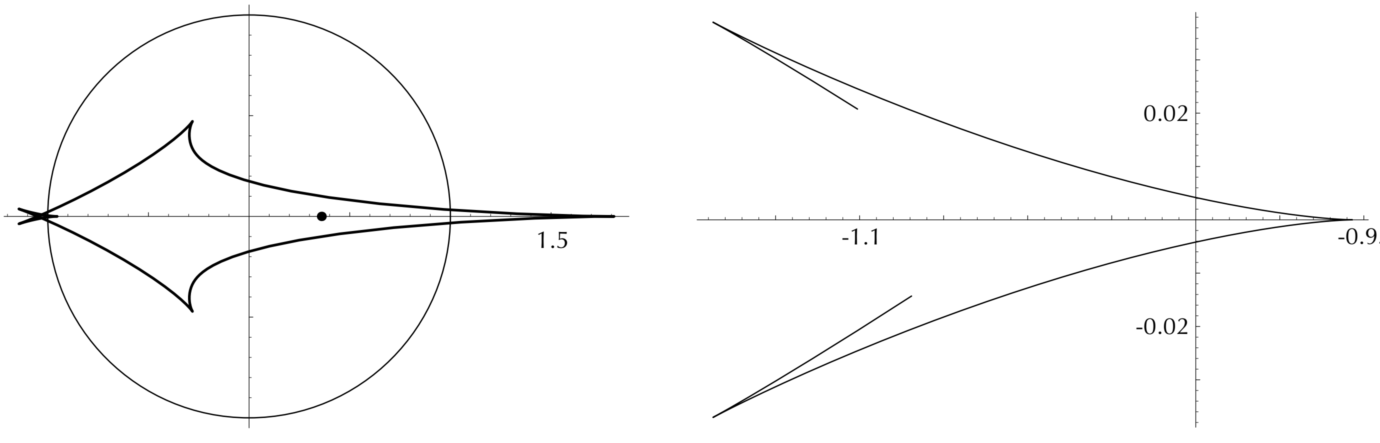


FIGURE 5. Caustic of R_r^n for $r = 0.36$, $n = 3$. On the right, an enlargement of the region $\varphi = 0$.

These expansions help to understand the caustics of R_r^{2m} at $\varphi_a = 0, \pi$. It would be convenient to look at the pictures before we go on.

In Figure 6, we see that cusps are born near $\varphi = 0$ and π. From the enlargement, the caustic bifurcates into $2m$ cusps when r increases across $\frac{1}{3}$, where R_r changes from a diffeomorphism to a degree 1 map. In the expansion of R_r^{2m}, we have

$$a_1 = \left(\frac{1 - 9r^2}{1 - r^2}\right)^m.$$

This allows us to show that $r = \frac{1}{3}$ is where the caustic of R_r^{2m} changes at $\varphi = 0, \pi$. In fact, the caustics of $R_{1/3}^2$ has the following Taylor expansions. At $\varphi_a = 0$,

$$x(\varphi) - x(0) = -\tfrac{27}{4}\theta^4 + O(\theta^6),$$
$$y(\varphi) - y(0) = 18\theta^3 + O(\theta^5),$$

and at $\varphi_a = \pi$,

$$x(\varphi) - x(\pi) = \tfrac{243}{16}\theta^4 + O(\theta^6)$$
$$y(\varphi) - y(\pi) = \tfrac{-81}{2}\theta^3 + O(\theta^5).$$

This shows that the caustic of R_r^2 undergoes a swallowtail bifurcation at $0, \pi$ when $r = \frac{1}{3}$. We may further work out the expansion of $R_{1/3}^{2m}$ as the m-th iterate of $R_{1/3}^2$. Using $a_1 = 0$ and $a_3 \neq 0$ for R_r^2, we have

$$R_{1/3}^{2m}(\varphi) = \varphi_a + \theta^{3^m} U(\theta)$$

for some function U with $U(0) \neq 0$. The bifurcation of the caustics of R_r^{2m} at $\varphi = 0, \pi$ should be of the type $(\theta^{3^m+1}, \theta^{3^m})$ when r passes $\frac{1}{3}$.

We have been considering the behavior of the caustics of R_r^n with parameter r and n fixed. What happens if r is fixed and n is allowed to vary? There is an interesting phenomenon for $r \leq \frac{1}{3}$. Although there is a fundamental difference when n is odd and even, this difference disappears as n goes to infinity. Figure 7 shows how the caustics of R_r^{2m+1} and R_r^{2m}

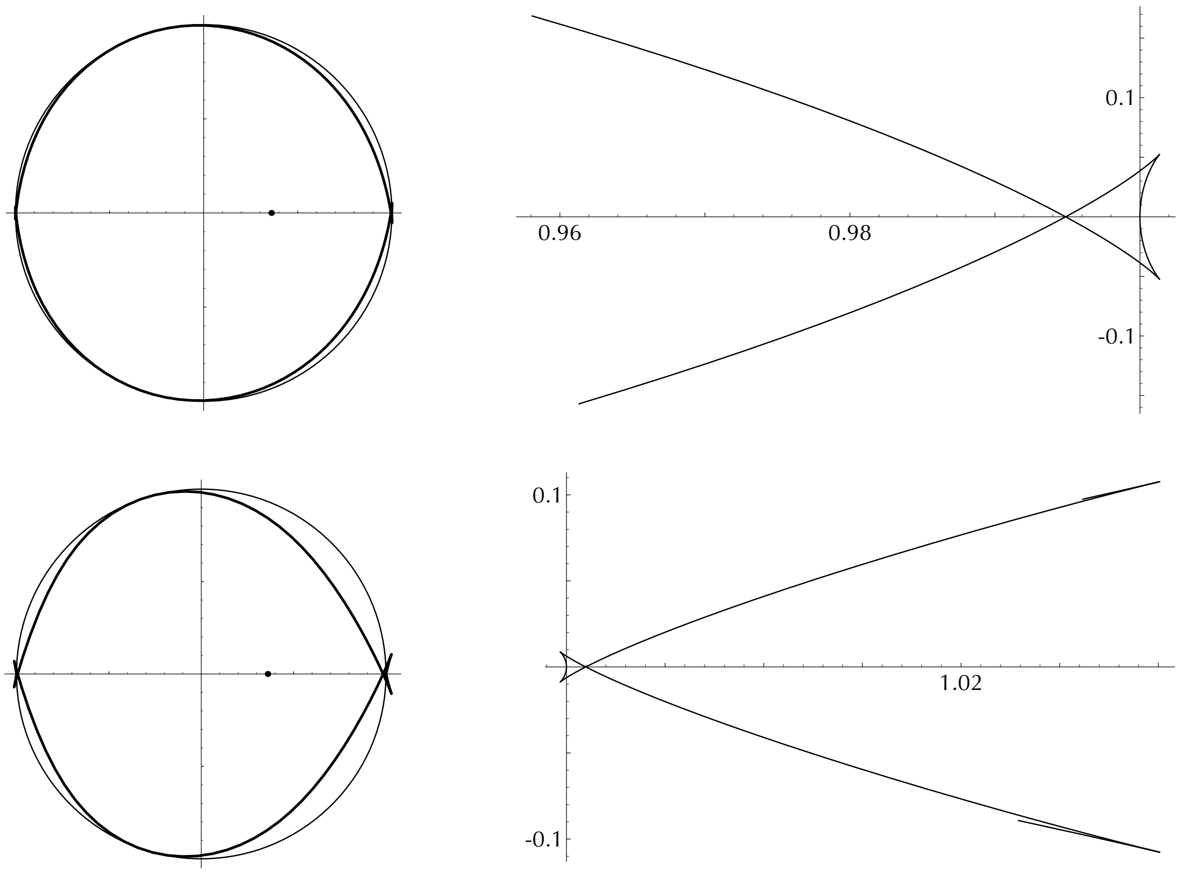


FIGURE 6. Caustics of R_r^n for $r = 0.36$, $n = 2$ (top) and $n = 4$ (bottom). On the right, enlargements of the region $\varphi = 0$.

change — they tend to the same quadrilateral. The dashed vertical line from φ_c to $-\varphi_c$ in the picture of even or odd caustics is determined by r but not the number of iterates. When n is even, it is where the caustic is tangent to the circle. When n is odd, every caustic is tangent to this vertical line because it is the line joining φ_c and $R_r^{2m+1}(\varphi_c) = -\varphi_c$. The point of tangency occurs exactly at φ_c by definition.

Proposition 4.4. *For $0 < r \le \frac{1}{3}$, as $m \to \infty$, both the caustics of R_r^{2m+1} and R_r^{2m} approach the same quadrilateral defined by the four points 0, π, and $\pm\varphi_c$, which are the only 2-periodic points of R_r.*

Proof. We first show that the caustics of R_r^{2m+1} at the four points tend to the circle as $m \to \infty$. These four φ's are the solution to $R_r^2(\varphi) = \varphi$. Thus, we have $(R_r^{2m+1})'(\varphi) = R_r'(\varphi)^{m+1} \cdot R_r'(R_r(\varphi))^m$.

At $\varphi = 0, \pi$, the coordinates of the caustic are

$$x(\varphi) = \frac{\pm\left((R_r^n)'(\varphi) - 1\right)}{1 + (R_r^n)'(\varphi)}, \qquad y(\varphi) = 0.$$

Clearly,

$$\begin{aligned}(R_r^{2m+1})'(0) &= R_r'(0)^{m+1} R_r'(\pi)^m \\ &= \left(\frac{1-3r}{1-r}\right)^{m+1} \left(\frac{1+3r}{1+r}\right)^m.\end{aligned}$$

Thus, $x(0) \to -1$ as $m \to \infty$. The situation at π is similar.

At the point φ_c with $R_r^n(\varphi_c) = -\varphi_c$, the coordinates of the caustic are

$$x(\varphi_c) = \cos\varphi_c, \qquad y(\varphi_c) = \frac{(-1 + (R_r^n)'(\varphi_c))\sin\varphi_c}{1 + (R_r^n)'(\varphi_c)}.$$

Since R_r is odd and $R_r^2(\varphi_c) = \varphi_c$, it follows that $(R_r^n)'(\varphi_c) = R_r'(\varphi_c)^n$. Moreover, by $4r\cos\varphi_c = 1 - \sqrt{1 + 8r^2}$, one may show that

$$\frac{(-1 + (R_r^n)'(\varphi_c))}{1 + (R_r^n)'(\varphi_c)} \to 1 \qquad \text{as } n \to \infty.$$

Again, these two cusps approach to the unit circle.

Secondly, from Lemmas 4.7 and 4.8, 0 and π are the attracting fixed points of R_r^2 while $\pm\varphi_c$ are repelling. Moreover, the attracting basins for 0 and π are $(-\varphi_c, \varphi_c)$ and $(\varphi_c, 2\pi - \varphi_c)$ respectively. Thus, for any given neighborhood of 0, for sufficiently large m, for any neighboring $\varphi_1, \varphi_2 \in (-\varphi_c, \varphi_c)$, $R_r^{2m}(\varphi_1)$ and $R_r^{2m}(\varphi_2)$ lie in that neighborhood of 0. Hence, the intersection of the lines from φ_j to $R_r^{2m}(\varphi_j)$ lies in a neighborhood of the quadrilateral. The proof for the cases at π and of odd iterates are similar. □

4B. Technical Results

We collect here some technical results needed to justify the observations of Section 4A. They are mostly obtained by direct computation and the reader can skip the proofs.

The first one deals with the existence of cusps at certain special "symmetric" positions.

Lemma 4.5. *Let f denote any iterate of R_r. On the caustic of f, the conditions for the occurrence of a semicubical cusp at φ are*

- *$f'(\varphi) = 0$ and $f''(\varphi) \ne 0$ if $f(\varphi) = \varphi$;*
- *$f''(\varphi) = 0$ and $-f'(\varphi) + f'(\varphi)^3 + 2f'''(\varphi) \ne 0$ if $f(\varphi) = \varphi + \pi$.*

Proof. This is proved by computing the derivatives of (3–1) and (3–2), then evaluating at the particular

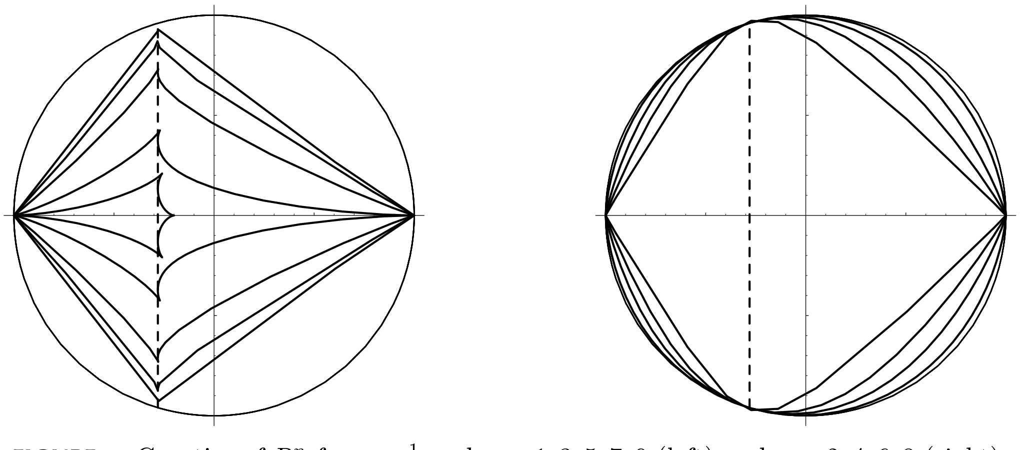


FIGURE 7. Caustics of R_r^n for $r = \frac{1}{3}$ and $n = 1, 3, 5, 7, 9$ (left) and $n = 2, 4, 6, 8$ (right).

values φ_0 or φ_a. The result follows by verifying that $x' = 0 = y'$ and $x''y''' - x'''y'' \neq 0$. □

Remark. Although this lemma is stated for an iterate of R_r, it is actually true for any circle map. Other results in this section also hold in a more general setting, but we will focus on iterates of R_r.

At a point φ with $f(\varphi) = -\varphi$, we always have $x'(\varphi) = 0$. The conditions there are

$$f'(\varphi)(1 + f'(\varphi))\cos\varphi + f''(\varphi)\sin\varphi = 0,$$
$$f'(\varphi)(2 + \cos(2\varphi)) + 6f'(\varphi)^2\cos^2\varphi + (1{+}2\cos(2\varphi))f'(\varphi)^3 - 2f'''(\varphi)\sin^2\varphi \neq 0.$$

Analogously, if $f(\varphi) = \pi - \varphi$, we have $y'(\varphi) = 0$ and conditions

$$f'(\varphi)(1 + f'(\varphi))\sin\varphi - f''(\varphi)\cos\varphi = 0,$$
$$f'(\varphi)(2 - \cos(2\varphi)) + 6f'(\varphi)^2\sin^2\varphi + (1 - 2\cos(2\varphi))f'(\varphi)^3 - 2f'''(\varphi)\cos^2\varphi \neq 0.$$

The next lemma is just the chain rule.

Lemma 4.6. *If $n = p + q$,*

$$\begin{aligned}(R_r^n)'(\varphi) &= (R_r^p)'(R_r^q(\varphi))\,(R_r^q)'(\varphi)\\ &= R_r'(\varphi)R_r'(R_r(\varphi))\cdots R_r'(R_r^{n-1}(\varphi)),\\ (R_r^n)''(\varphi) &= (R_r^p)''(R_r^q(\varphi))\,(R_r^q)'(\varphi)^2\\ &\quad + (R_r^p)'(R_r^q(\varphi))\,(R_r^q)''(\varphi),\\ (R_r^n)'''(\varphi) &= (R_r^p)'''(R_r^q(\varphi))\,(R_r^q)'(\varphi)^3\\ &\quad + (R_r^p)'(R_r^q(\varphi))\,(R_r^q)'''(\varphi)\\ &\quad + 3(R_r^p)''(R_r^q(\varphi))\,(R_r^q)'(\varphi)\,(R_r^q)''(\varphi).\end{aligned}$$

In determining the cusps on the caustic, some orbits in the iteration play a special role. We thus establish the following to handle that.

Lemma 4.7. *For $0 \leq r \leq \frac{1}{3}$, R_r^{2m+1} has no fixed point and $R_r^{2m}(\varphi) \neq \varphi + \pi$.*

Proof. First, one can obtain algebraically the four fixed points of R_r^2. The attracting ones are 0, π, while $\pm\varphi_c$ are repelling. Using calculus we get the corresponding attracting basins and conclude that $R_r^{2m}(\varphi)$ converges to 0 or π mod 2π monotonically. By the series expression (2–1) of $R_r(\varphi)$, one can deduce the estimate

$$\big|R_r(\varphi) - \varphi\big| \geq \pi - 2\big|\log(1 - r)\big|.$$

The lemma follows by the convergence of $R_r^{2m}(\varphi)$. □

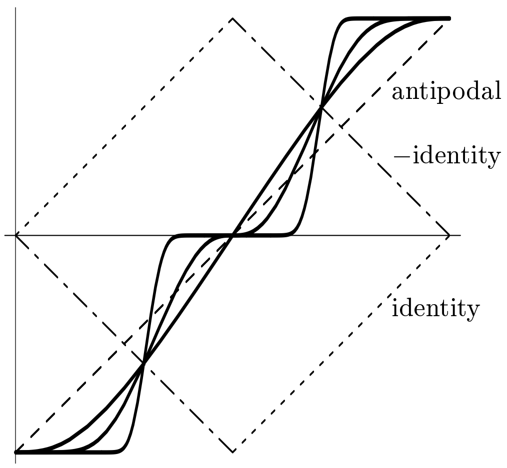
Lemma 4.8. *For $0 < r \leq \frac{1}{3}$, let*

$$\varphi_c = \arccos\left(\frac{1 - \sqrt{1 + 8r^2}}{4r}\right).$$

- *$R_r^n(\varphi) = \varphi$ has solutions if and only if n is even, and they are $0, \pi, \pm\varphi_c$.*
- *$R_r^n(\varphi) = -\varphi$ has solutions $\pm\varphi_c$ if n is odd, and $0, \pi$ if n is even.*
- *$R_r^n(\varphi) = \varphi + \pi$ has solutions if and only if n is odd; two of them are 0 and π.*
- *$R_r^n(\varphi) = \pi - \varphi$ has solutions 0, π when n is odd and two solutions when n is even.*

This follows from the previous lemma and induction.

Lemmas 4.7 and 4.8 are illustrated by Figure 8.



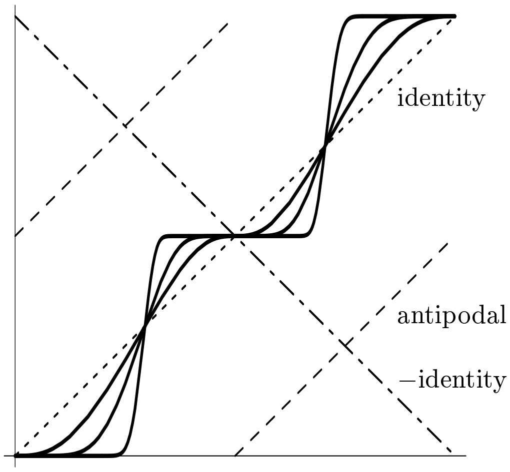


FIGURE 8. Plots of R_r^n for $r = \frac{1}{3}$ and $n = 1, 3, 7$ (left), $r = 2, 6, 8$ (right). The axes correspond to $\varphi = 0$.

In an attempt to understand more about the iterates R_r^n, we computed the *asymptotic orbits* of the two critical points

$$\pm \arccos \frac{1+3r^2}{4r} \tag{4–1}$$

for $\frac{1}{3} \leq r < 1$. The asymptotic orbit of φ is the set $\{R_r^n(\varphi) : N_1 < n < N_2\}$ for large N_1, N_2. Figure 9 shows the corresponding *bifurcation diagram* — that is, the plot of asymptotic orbits against the parameter r — for the critical point corresponding to the $+$ sign in (4–1). The diagram for the other critical points is simply the reflection in a horizontal line.

Thus, as r increases starting from $\frac{1}{3}$, the orbit first undergoes bifurcation at $r = 1/\sqrt{5} \approx 0.447$ where the 2-cycle $\{0, \pi\}$ turns repelling and a 4-cycle occurs. The period doubling continues until $r < 0.62$ and is followed with chaotic behavior when r approaches 1. At $r = 1$, R_r becomes the doubling map.

An interesting feature of Figure 9 is that, in addition to the usual bifurcations, there are "half-bifurcations" or sudden turns (for example, at $r \approx 0.56$). We have noticed that the half-bifurcation at (r, φ) is matched by one at $(r, -\varphi)$ going in the same direction — either both choose the upper branch or both choose the lower branch. This means that the half-bifurcation at (r, φ) is matched by one at (r, φ) in the mirror image diagram (the bifurcation diagram corresponding to the other critical point), going in the *opposite direction*. Thus, if we superimpose Figure 9 with its mirror image (reflection in the horizontal bisector), the result looks like a usual bifurcation diagram.

Should we take these half-bifurcations into consideration when using the Feigenbaum's constant to estimate the limit of period doubling?

Finally, unlike other well-known one-parameter families (logistic, polynomial, cosine), this family R_r does not display the most obvious period-3 window, but a period-8 one at $r \approx 0.68$, though a period 3 window seems to occur at $r \approx 0.86$. According to the Sarkovskii ordering [Devaney 1989, part 1], R_r has periodic orbits of any period for r in the period-3 window. However, for r in the period-8 window, can R_r have a period with odd factors? If not, does it only have periods that are powers of 2?

5. MODE-LOCKING

5A. Background

The study of circle maps is closely related to the study of differential equations on torus (i.e., equations with double periodic coefficients). For such a differential equation, one may consider the Poincaré

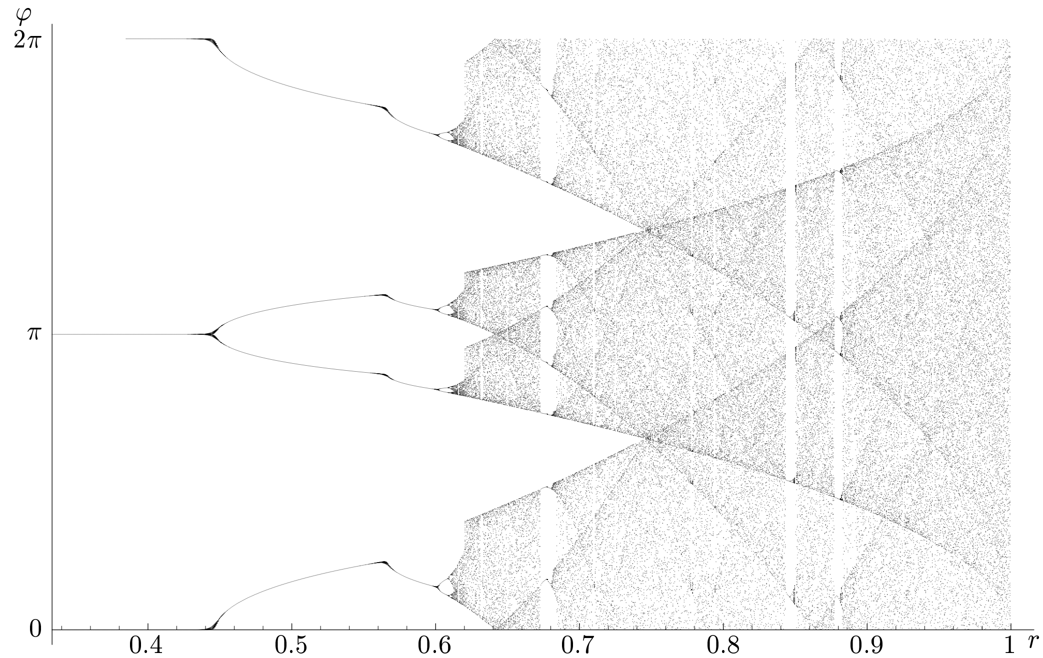


FIGURE 9. Bifurcation diagram of the critical point $\dfrac{1+3r^2}{4r}$.

return map of the flow, which defines a map on a meridian circle of the torus. The stability of the equation is reflected by this circle map.

Arnold [1961, § 12] investigated the circle map

$$\varphi \mapsto \varphi + a + \varepsilon \cos \varphi$$

and obtained information on its resonance zone in the (a, ε)-plane. This gave rise to the famous picture of so-called Arnold tongues. Subsequently, numerous studies by physicists and mathematicians [Bak et al. 1988; Ding and Hemmer 1988; Feudel et al. 1995; Jensen et al. 1984; Kaneka 1984; Piña 1986; Zheng 1991] have been published on the perturbation of a rotation

$$\varphi \mapsto \varphi + \Omega - \varepsilon \sin \varphi, \qquad \varepsilon \in [0, 1).$$

The focus is on the phenomenon called mode-locking and the Devil's staircase. Arnold later [1983b] gave a proof of his observation for circle maps of the form

$$\varphi \mapsto \varphi + \Omega + \varepsilon(\text{trigonometric polynomial})$$

as well as analytic reduction of many circle maps [Arnold 1983a, Chapter 3, § 12]. The algebraic nature of the method is also apparent in the problem of particular differential equations. Arnold predicts that a general theorem exists for these equations and general circle maps.

In this section, we will provide further evidence towards Arnold's prediction by showing similar behavior in the off-center reflection. Our off-center reflection is not of the form studied by Arnold, so it may be regarded as another small step towards the general theory.

Consider a two-parameter model of circle maps arising from the off-center reflection map, with parameters $r \in [0, 1)$ and $\Omega \in (-\pi, \pi]$:

$$R_{r,\Omega}(\varphi) = \varphi + \Omega - 2 \sum_{k=1}^{\infty} \frac{r^k}{k} \sin(k\varphi).$$

(We use r instead of ε to be consistent with previous sections.) Unlike the models discussed above, r cannot be factored out. This map can be thought of as an imperfect off-center reflection on the circle, where the reflected angle has a constant deviation from the incident angle. The original off-center reflection corresponds to $\Omega = \pi$. We may not get such a deviation by varying the metric of the circle; it is better understood in terms of symplectic geometry.

For $\varphi_0 \in \mathbb{S}^1$, there is the *rotation number*

$$\omega(R_{r,\Omega}, \varphi_0) = \lim_{n\to\infty} \frac{R^n_{r,\Omega}(\varphi_0) - \varphi_0}{n},$$

where the right-hand side is performed on a lifting of $R_{r,\Omega}$. It is independent of φ_0 if $R_{r,\Omega}$ is diffeomorphic. In such case, one simply denotes $\omega(R_{r,\Omega})$. If $R_{r,\Omega}$ is only a degree 1 map, one has a *rotation interval* instead. These notions are indeed defined for any circle map. Historically, attention has been centred around perturbations of rotations, $\varphi \mapsto \varphi + \Omega + u(\varphi)$. It is natural to ask for the relation between Ω and ω. The physicists usually refer to Ω as internal frequency and ω as resonance frequency. When $\omega = \omega(\Omega)$ is a locally constant function, the situation is called *mode-locking.*

Herman [1977; 1979] studied mode-locking extensively and obtained interesting results, which are applicable to $R_{r,\Omega}$ because it satisfies the property $\mathbf{A}_0$ of Herman.

Theorem 5.1. *For all $\omega_0 \in 2\pi\mathbb{Q}$ and $0 < r \le \frac{1}{3}$, there is an interval $\mathcal{I} = \mathcal{I}_r$ of ω_0 such that for every $\Omega \in \mathcal{I}_r$, the diffeomorphism $R_{r,\Omega}$ has rotation number ω_0.*

The interval $\mathcal{I}$ is called *resonance interval* and its size depends on r (and of course ω_0). Its variance in terms of r defines a picture which looks like a tongue. We will discuss it later. Furthermore, from Herman's study, the off-center reflection model also demonstrates the well-known Devil's staircase.

Theorem 5.2. *For any $0 < r \le \frac{1}{3}$, the function $\Omega \mapsto \omega(R_{r,\Omega})$ is nondecreasing, locally constant at any rational number, and has a Cantor set of discontinuity.*

We have mentioned that if we alter a sign and form the "conjugate" family

$$\bar{R}_{r,\Omega}(\varphi) = \varphi + \Omega + 2 \sum_{k=1}^{\infty} \frac{r^k}{k} \sin(k\varphi),$$

the dynamics is completely different. Actually, $\bar{R}_{r,\Omega}$ can be extended to

$$e^{2\pi i\,\Omega} \frac{z - r}{1 - rz}$$

on the hyperbolic disk, which defines a hyperbolic element in $\mathrm{PSL}(2, \mathbb{R})$. Mode-locking does not occur, i.e., $\omega\left(\bar{R}_{r,\Omega}\right) = 2p\pi/q$ only if $\Omega = p/q$.

5B. Width of the Resonance Zone

Arnold [1983b] discusses the mode-locking situation of a rotation slightly perturbed by a trigonometric polynomial, $g(x)$,

$$f : x \mapsto x + \Omega + \varepsilon g(x).$$

The *resonance zone* is the set $\{(\Omega, \varepsilon) : \Omega \in \mathcal{I}_\varepsilon\}$. Arnold developed a formal calculation to estimate the width of the interval $\mathcal{I}_\varepsilon$ in terms of ε, which gives rises to a picture of the resonance zone. This formal calculation is related to the homological equation of analytical reduction [Arnold 1983a]. If the rotation number is rational, the width of the resonance interval $\mathcal{I}_\varepsilon$ is bounded by a power of ε. The graphical plot of the resonance zone in the $\varepsilon\Omega$-plane form the so-called *Arnold's tongue.*

By a method similar to Arnold's, one may also estimate the width of $\mathcal{I}_r$ for the off-center reflections $R_{r,\Omega}$, $0 \le r \le \frac{1}{3}$. We will show the different behaviors of $R = R_{r,\pi}$ and $\bar{R} = \bar{R}_{r,\pi}$ at the same time.

For simplicity of computation, we first consider the resonance zone containing π. Writing $\Omega = \pi + a$, the second iterates of the maps are

$$\begin{aligned} R^2(x) &= x + 2\pi + 2a - 2\sum_{k=1}^{\infty} \frac{r^k}{k}\sin(kx) \\ &\quad -2\sum_{k=1}^{\infty}\frac{r^k}{k}\sin k\Big(x + \pi + a - 2\sum \frac{r^k}{k}\sin(kx)\Big), \\ \bar{R}^2(x) &= x + 2\pi + 2a + 2\sum_{k=1}^{\infty} \frac{r^k}{k}\sin(kx) \\ &\quad +2\sum_{k=1}^{\infty}\frac{r^k}{k}\sin k\Big(x + \pi + a + 2\sum \frac{r^k}{k}\sin(kx)\Big). \end{aligned}$$

The equations of resonance are $R^2(x) = x + 2\pi$ and $\bar{R}^2(x) = x + 2\pi$. Let $v = a \mp 2\sum (r^k/k)\sin(kx)$, we have

$$0 = v \pm \sum_{k=1}^{\infty}\left(\frac{r^k}{k}\sin(kx) - \frac{(-r)^k}{k}\sin k(x+v)\right),$$

where $v = v_1 r + v_2 r^2 + v_3 r^3 + \cdots$. Note that the solutions of v's for R and $\bar{R}$ do not only differ by a sign. One can see this by the subtle combinations of the signs of the infinite series in their second iterates. Inductively, one may show that for $\bar{R}$, we have

$$v_k = \frac{2}{k}\sin kx,$$

while the values for R are

$$\begin{aligned} v_1 &= -2\sin x, \\ v_2 &= \sin 2x, \\ v_3 &= 2\sin x - \tfrac{8}{3}\sin 3x. \end{aligned}$$

This leads to a r-series for a and its maximum and minimum provide bounds for the resonance zone, namely,

$$a = 2\sin(2x)r^2 + \big(2\sin x - \tfrac{7}{3}\sin 3x\big)r^3 + \cdots$$

for the map R, and $a = 0$ for the map $\bar{R}$. This calculation agrees with our previous remark that mode-locking (near $\omega = \pi$) does not occur for $\bar{R}$. Furthermore:

Theorem 5.3. *The width of* $\mathcal{I}_r$ *is bounded by* Cr^2 *for* $\Omega = \pi$ *and* Cr *for general* Ω.

The computation for the resonance zone at a general $\Omega = 2p\pi/q$ is more complicated. The equation to formally expand is $R^q_{r,a+2p\pi/q}(x) = x + 2p\pi$. The coefficients a_k of

$$a = a_1 r + a_2 r^2 + a_3 r^3 + \cdots$$

provide the estimates of $\mathcal{I}_r$. It turns out that the first term a_1 does not vanish, indeed,

$$qa_1 = 2\sum_{j=0}^{q-1}\sin\left(x + \frac{2jp\pi}{q}\right).$$

This may not be a sharp estimate, yet we can only conclude that the width of $\mathcal{I}_r$ is of order r in general.

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