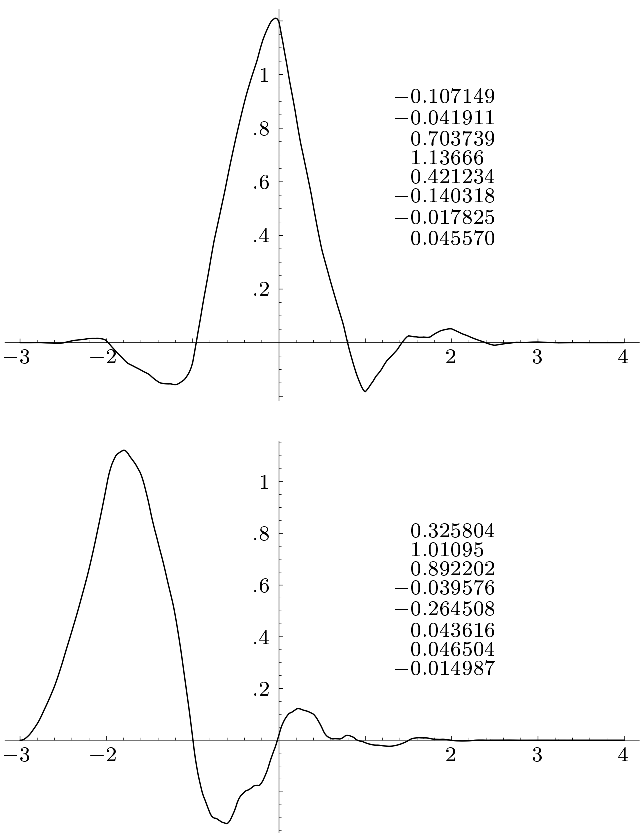
On the Number of Daubechies Scaling Functions and a Conjecture of Chyzak et al.

Yang Wang

Using a result on Riesz factorizations, we show that there are at most 2^{N-1} and at least $2^{\lfloor N/2 \rfloor}$ distinct Daubechies scaling functions with support in $[1-N, N]$.



Two of the four distinct Daubechies scaling functions for $N = 4$. The numbers indicate the coefficients $h_{-3}, \dots, h_4$. The other two scaling functions can be obtained by reversing the coefficients h_k.

We define a *Daubechies scaling function* to be a function $\varphi \in L^2(\mathbb{R})$, with support in $[1-N, N]$ and satisfying the *dilation equation*

$$\varphi(x) = \sum_{k \in \mathbb{Z}} h_k \varphi(2x-k),$$

where N is a positive integer and the sequence $\{h_k\}$, known as the scaling function's *filter sequence*, satisfies the conditions $h_k = 0$ for $k \notin [1-N, N]$ and

$$\left.\begin{aligned} &\sum_{k=1-N}^{N} h_k = 2, \\ &\sum_{k=1-N}^{N} h_k h_{k-2l} = 2\delta_{0,l}, \qquad \text{for } l = 0, \dots, N-1, \\ &\sum_{k=1-N}^{N} (-1)^k h_{1-k} k^l = 0, \quad \text{for } l = 0, \dots, N-1. \end{aligned}\right\} \tag{1}$$

These conditions are motivated by the use of scaling functions and filter sequences in wavelet analysis [Daubechies and Lagarias 1991; 1992]; see also [Chyzak et al. 2001] in this issue, where the equations above are summarized (Section 2) and where it is conjectured (page 75) that there are at most 2^{N-1} solutions to the system (1).

Here we prove that there are at most 2^{N-1} *real* solutions to the system (1), this being the case of consequence in wavelet analysis.

Theorem 1. *For a fixed $N > 0$, the system* (1) *in $h_{1-N}, \dots, h_N$ has at most 2^{N-1} real solutions. In other words, there are at most 2^{N-1} distinct Daubechies scaling functions with support in $[1-N, N]$.*

Research supported in part by NSF grant DMS-97-06793.

1058-6458/2001 $0.50 per page

The proof uses a lemma on Riesz factorizations. Let $G(z)$ be a Laurent polynomial, that is,

$$G(z)=\sum_{k\in\mathbb{Z}} c_k z^k,$$

where finitely many $c_k \neq 0$. We consider the question: How may real Laurent polynomials $f(z)$ are there such that $f(z)f(z^{-1}) = G(z)$ (Riesz factorization)? Obviously if $f(z)$ is a solution then so is $\pm z^m f(z)$ for any $m \in \mathbb{Z}$. Call two Laurent polynomials $f(z)$ and $g(z)$ *equivalent* if $g(z) = z^m f(z)$ or $g(z) = -z^m f(z)$ for some $m \in \mathbb{Z}$. So our question concerns the number of inequivalent solutions.

Not every Laurent polynomial $G(z)$ has a Riesz factorization $G(z) = f(z)f(z^{-1})$ for some real Laurent polynomial $f(z)$. If it does we call $G(z)$ *Riesz factorizable.* It is well known [Daubechies 1992] that $G(z)$ is Riesz factorizable if and only if $G(z) = G(z^{-1})$ and $G(z) \geq 0$ on the unit circle $|z| = 1$.

Lemma 2. *Let $G(z) = \sum_{k=-M}^{M} c_k z^k$ be real and Riesz factorizable. Then the number of inequivalent real Laurent polynomials $f(z)$ satisfying*

$$f(z)f(z^{-1}) = G(z)$$

is at most 2^{r+s}, where $2r$ and $4s$ denote the number of real and complex roots of $G(z)$ (counting multiplicity) not on the unit circle. In particular, it is at most 2^M.

Proof. For any two roots z_1 and z_2 of $G(z)$ write $z_1 \sim z_2$ if z_2 is one of z_1, z_1^{-1}, $\bar{z}_1$, or $\bar{z}_1^{-1}$. It follows from $G(z) = f(z)f(z^{-1})$ that if z_* is a root of G then so is every $w \sim z_*$ and with the same multiplicity. We partition the roots of G not on the unit circle into equivalent classes of the relation $\sim$, and label them (counting multiplicity)

$$\mathcal{R}_1,\dots,\mathcal{R}_r,\mathcal{C}_1,\dots,\mathcal{C}_s,$$

where each $\mathcal{R}_i$ and $\mathcal{C}_j$ contain real and complex roots of G not on the unit circle, respectively. Clearly $|\mathcal{R}_i| = 2$ and $|\mathcal{C}_j| = 4$. Let $\mathcal{U}$ denote the roots of G that are on the unit circle.

Observe that up to equivalence a Riesz factorization $G(z) = f(z)f(z^{-1})$ is completely determined by the roots of $f(z)$. Furthermore, if z_0 is a root of $f(z)$ with $|z_0| = 1$ then so is $\bar{z}_0 = z_0^{-1}$. It follows that z_0 must also be a root of $f(z^{-1})$. Hence all roots in $\mathcal{U}$ have even multiplicities and they split evenly between $f(z)$ and $f(z^{-1})$. This fact implies that the Riesz factorization $G(z) = f(z)f(z^{-1})$ is determined completely by the roots of $f(z)$ that do not lie on the unit circle.

To count the number of different factors $f(z)$, note that if $z_i \in \mathcal{R}_i$ is a root of $f(z)$ then the other element z_i^{-1} in $\mathcal{R}_i$ must be a root of $f(z^{-1})$. Similarly if $z_j \in \mathcal{C}_j$ is a root of $f(z)$ then so is $\bar{z}_j$, while the other two elements in $\mathcal{C}_j$ will be roots of $f(z^{-1})$. So there are two ways to select roots for $f(z)$ from each of $\mathcal{R}_i$ and $\mathcal{C}_j$. The number of different factors $f(z)$ such that $G(z) = f(z)f(z^{-1})$ is therefore at most 2^{r+s}. Finally, $G(z)$ has at most $2M$ roots. Hence $2^{r+s} \leq 2^M$. □

Remark. The number of Riesz factorizations of $G(z)$ is exactly 2^{r+s} if all roots of G not on the unit circle are distinct. Otherwise it is strictly less. The exact number is not hard to compute, following the proof of the lemma. Let $\mathcal{R}'_1,\dots\mathcal{R}'_{r'}$ and $\mathcal{C}'_1,\dots,\mathcal{C}'_{s'}$ be the distinct equivalent classes of $\mathcal{R}_1,\dots\mathcal{R}_r$ and $\mathcal{C}_1,\dots,\mathcal{C}_s$, respectively. Let m_i and n_j denote the multiplicity of the roots of $\mathcal{R}'_i$ and $\mathcal{C}'_j$, respectively. Then the number of inequivalent Riesz factorizations of $G(z)$ is

$$(m_1{+}1)\cdots(m_{r'}{+}1)(n_1{+}1)\cdots(n_{s'}{+}1). \tag{2}$$

Proof of Theorem 1. Suppose that the sequence of real numbers $\{h_k : 1{-}N \leq k \leq N\}$ satisfies (1). Let

$$H(z)=\frac{1}{2}\sum_{k=1-N}^{N} h_k z^k.$$

Recall (from [Daubechies 1992], for example) that the third set of equations in (1) is equivalent to

$$H(z)=\Big(\frac{1+z}{2}\Big)^N f(z) \tag{3}$$

for some real Laurent polynomial $f(z)$, whereas the middle set of equations is equivalent to

$$H(z)H(z^{-1})+H(-z)H(-z^{-1})=1. \tag{4}$$

Furthermore $H(z)$ satisfies (4) if and only if

$$f(e^{i\theta})f(e^{-i\theta})=\sum_{k=0}^{N-1}\binom{N{+}k{-}1}{k}(1-\cos\theta)^k,$$

which is equivalent to

$$f(z)f(z^{-1})=\sum_{k=0}^{N-1}\binom{N{+}k{-}1}{k}\Big(\frac{2-z-z^{-1}}{2}\Big)^k. \tag{5}$$

Expanding the right hand side of (5) yields

$$G(z) = \sum_{k=1-N}^{N-1} c_k z^k$$

for some c_k. By Lemma 2 there are at most 2^{N-1} inequivalent solutions $f(z)$ satisfying (5).

We now need only show that any two equivalent solutions $f_1(z)$ and $f_2(z)$ of (3) and (5) must be identical. First, it follows from $H(1) = 1$ that $f_1(1) = f_2(1) = 1$. Hence $f_2(z) = z^m f_1(z)$ for some $m \in \mathbb{Z}$. Next, any solution $f(z)$ to (3) and (5) must have the form

$$f(z) = \sum_{k=1-N}^{0} c_k z^k$$

with $c_{1-N} \neq 0$ by (5). Hence $m = 0$, and so $f_1(z) = f_2(z)$. $\square$

Remark. This also shows that there are at least $2^{\lfloor N/2 \rfloor}$ distinct Daubechies scaling functions with support in $[1-N, N]$, where $\lfloor N/2 \rfloor$ denotes the largest integer not exceeding $N/2$. This is because the function in (5) cannot have zeros on the unit disk.

ACKNOWLEDGEMENT

The author is greatly indebted to the Editor, Silvio Levy, for his encouragement and help.

REFERENCES

[Chyzak et al. 2001] F. Chyzak, P. Paule, O. Scherzer, A. Schoisswohl, and B. Zimmermann, "The construction of orthonormal wavelets using symbolic methods and a matrix analytical approach for wavelets on the interval", *Experiment. Math.* **10**:1 (2001), 67–86.

[Daubechies 1992] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics **61**, SIAM, Philadelphia, 1992.

[Daubechies and Lagarias 1991] I. Daubechies and J. C. Lagarias, "Two-scale difference equations, I: Existence and global regularity of solutions", *SIAM J. Math. Anal.* **22**:5 (1991), 1388–1410.

[Daubechies and Lagarias 1992] I. Daubechies and J. C. Lagarias, "Two-scale difference equations, II: Local regularity, infinite products of matrices and fractals", *SIAM J. Math. Anal.* **23**:4 (1992), 1031–1079.

Yang Wang, School of Mathematics, Georgia Institute of Mathematics, Atlanta, GA 30332, United States (wang@math.gatech.edu, http://www.math.gatech.edu/~wang)

Received July 25, 2000; accepted August 17, 2000