Matchings, Hamilton cycles and cycle packings in uniform hypergraphs

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It is well known that every bipartite graph with vertex classes of size n whose minimum degree is at least $n/2$ contains a perfect matching. We prove an analogue of this result for uniform hypergraphs. We also provide an analogue of Dirac's theorem on Hamilton cycles for 3-uniform hypergraphs: We say that a 3-uniform hypergraph has a Hamilton cycle if there is a cyclic ordering of its vertices such that every pair of consecutive vertices lies in a hyperedge which consists of three consecutive vertices. We prove that for every $\varepsilon > 0$ there is an n_0 such that every 3-uniform hypergraph of order $n \geq n_0$ whose minimum degree is at least $n/4 + \varepsilon n$ contains a Hamilton cycle. Our bounds on the minimum degree are essentially best possible.

Keywords: matchings, Hamilton cycles, packings, uniform hypergraphs

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1 Matchings in uniform hypergraphs

The so called 'marriage theorem' of Hall provides a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph. For hypergraphs there is no analogue of this result—up to now only partial results are known. For example, Conforti et al. [CCKV96] extended Hall's theorem to so-called balanced hypergraphs and Haxell [Hax95] extended Hall's theorem to a sufficient condition for the existence of a hypergraph matching which contains a given set of vertices. Moreover, there are many results about the existence of almost perfect matchings in hypergraphs which are pseudo-random in some sense. Most of these are based on an approach due to Rödl (see e.g. [AS92] for an introduction to the topic or Vu [Vu00] for more recent results). For random uniform hypergraphs, the threshold for a perfect matching is still not known. There are several partial results, see e.g. Kim [Kim03].

A simple corollary of Hall's theorem for graphs states that every bipartite graph with vertex classes A and B of size n whose minimum degree is at least $n/2$ contains a perfect matching. This can also be easily proved directly by considering a matching of maximum size. In [KOb] we proved an analogue of this result for r-uniform hypergraphs. So instead of two vertex classes and a set of edges joining them (as in the graph case), we now have r vertex classes and a set of (unordered) r-tuples, each of whose vertices lies in a different vertex class. A natural way to define the minimum degree of an r-uniform r-partite hypergraph $\mathcal{H}$ is the following. Given $r-1$ distinct vertices $x_1, \dots, x_{r-1}$ of $\mathcal{H}$, the *neighbourhood* $N_{r-1}(x_1, \dots, x_{r-1})$ *of* $x_1, \dots, x_{r-1}$ in $\mathcal{H}$ is the set of all those vertices x which form a hyperedge together with $x_1, \dots, x_{r-1}$. The *minimum degree* $\delta'_{r-1}(\mathcal{H})$ is defined to be the minimum $|N_{r-1}(x_1, \dots, x_{r-1})|$ over all tuples $x_1, \dots, x_{r-1}$ of vertices lying in different vertex classes of $\mathcal{H}$ (Fig. 1). This is often also called the minimum co-degree.

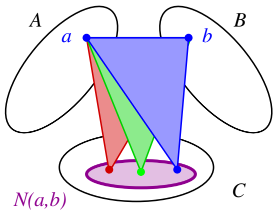

Fig. 1: The neighbourhood of a, b in a 3-uniform 3-partite hypergraph

Theorem 1 *Suppose that $\mathcal{H}$ is an r-uniform r-partite hypergraph with vertex classes of size $n \geq 1000$ which satisfies $\delta'_{r-1}(\mathcal{H}) \geq n/2 + \sqrt{2n \log n}$. Then $\mathcal{H}$ has a perfect matching.*

Theorem 1 is best possible up to the error term $\sqrt{2n \log n}$. The proof relies on a probablistic argument based on the number of perfect matchings in a bipartite graph with given degrees (the latter is given by Brégman's proof [Bré73] of the Minc conjecture on the permanent of a 0-1 matrix).

Surprisingly, a simple argument already shows that a significantly smaller minimum degree guarantees a matching which covers *almost all* vertices of $\mathcal{H}$:

Theorem 2 *Suppose that $\mathcal{H}$ is an r-uniform r-partite hypergraph with vertex classes of size n which satisfies $\delta'_{r-1}(\mathcal{H}) \geq n/r$. Then $\mathcal{H}$ has a matching which covers all but at most $r-2$ vertices in each vertex class of $\mathcal{H}$.*

Again, the bound on the minimum degree in Theorem 2 is essentially best possible: if we reduce it by εn, then we cannot even guarantee a matching which covers all but εn vertices in each vertex class. Both Theorems 1 and 2 can be used to prove analoguous results about matchings in r-uniform hypergraphs which are not required to be r-partite.

2 Hamilton cycles in 3-uniform hypergraphs

A classical theorem of Dirac states that every graph on n vertices with minimum degree at least $n/2$ contains a Hamilton cycle. If one seeks an analogue of this result for 3-uniform hypergraphs $\mathcal{H}$, then several alternatives suggest themselves. We define the *minimum degree* $\delta(\mathcal{H})$ of $\mathcal{H}$ to be the minimum $|N(x, y)|$ over all pairs of distinct vertices $x, y \in \mathcal{H}$ (where $N(x, y)$ is defined as in the previous section).

We say that a 3-uniform hypergraph $\mathcal{C}$ is a *cycle of order* n if there a exists a cyclic ordering $v_1, \dots, v_n$ of its vertices such that every consecutive pair $v_i v_{i+1}$ lies in a hyperedge of $\mathcal{C}$ and such that every hyperedge of $\mathcal{C}$ consists of 3 consecutive vertices. Thus the cyclic ordering of the vertices of $\mathcal{C}$ induces a cyclic ordering of its hyperedges. A cycle is *tight* if every three consecutive vertices form a hyperedge. A cycle of order n is *loose* if it has the minimum possible number of hyperedges among all cycles on n vertices (Fig. 2). A *Hamilton cycle* of a 3-uniform hypergraph $\mathcal{H}$ is a subhypergraph of $\mathcal{H}$ which is a cycle containing all its vertices.

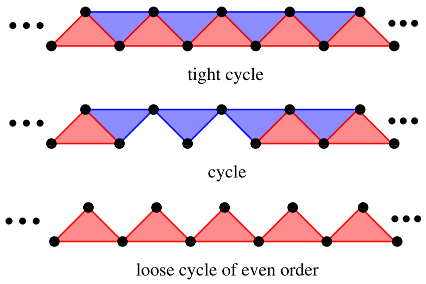

Fig. 2: Segments of cycles in a 3-uniform hypergraph

In [KOa] we proved the following result.

Theorem 3 *For each $\sigma > 0$ there is an integer $n_0 = n_0(\sigma)$ such that every 3-uniform hypergraph $\mathcal{H}$ with $n \geq n_0$ vertices and minimum degree at least $n/4 + \sigma n$ contains a loose Hamilton cycle.*

The bound on the minimum degree in Theorem 3 is best possible up to the error term σn. In fact, if the minimum degree is less than $\lceil n/4 \rceil$, then we cannot even guarantee *any* Hamilton cycle.

Recently, Rödl, Ruciński and Szemerédi [RRS04] proved that if the minimum degree is at least $n/2 + \sigma n$ and n is sufficiently large, then one can even guarantee a tight Hamilton cycle. Their bound improved an earlier one by Katona and Kierstead [KK99] and is best possible up to the error term σn.

The proofs of both our Theorem 3 and the result of Rödl, Ruciński and Szemerédi [RRS04] rely on the Regularity Lemma for 3-uniform hypergraphs due to Frankl and Rödl [FR02]. However, Rödl, Ruciński and Szemerédi make extensive use of the fact that the intersection of the neighbourhoods of any two pairs of vertices is nonempty, which is far from true in our case. For this reason, our argument has a rather different structure. (In fact, if we assume that our hypergraph has minimum degree $n/2 + \sigma n$ and the number of vertices is divisible by four, then our result is much easier to prove). Instead, we prove and use a 'blow up' type result: every 'pseudo-random' hypergraph contains a loose Hamilton cycle. This in turn uses a probabilistic argument based on results about random perfect matchings in pseudo-random graphs [KOc].

3 Perfect cycle packings in 3-uniform hypergraphs

In the case of graphs, there are many results determining the minimum degree which guarantees the existence of spanning substructures other than Hamilton cycles. For instance given two graphs H and G, an *H-packing in G* is a collection of vertex-disjoint copies of H in G. It is *perfect* if all of the vertices of G are covered. Komlós, Sárközy and Szemerédi [KSS01] proved that given a graph H of chromatic number χ, there exists a constant c such that every sufficiently large graph G whose order n is divisible by $|H|$ and whose minimum degree is at least $(1 - 1/\chi)n + c$ has a perfect H-packing. For hypergraphs no analogue of this result exists so far.

The following theorem gives a perfect packing result for loose cycles of sufficient length (but bounded when compared to the order of the host hypergraph). It follows easily from Theorem 3 using a random vertex partition argument. We denote a loose cycle on k vertices by $\mathcal{C}_k$.

Theorem 4 *For any $\gamma > 0$ there is an integer $k_0 = k_0(\gamma)$ such that the following holds for all $k \geq k_0$. Suppose that $\mathcal{H}$ is a 3-uniform hypergraph whose number n of vertices is divisible by k and whose minimum degree is at least $n/4 + \gamma n$, then $\mathcal{H}$ contains a perfect $\mathcal{C}_k$-packing.*

As with Theorem 3, the bound on the minimum degree is best possible up to the error term γn. It is also best possible in the sense that the condition that $k \geq k_0(\gamma)$ is needed if k is not divisible by 4.

A modification of the proof of Theorem 3 yields the following result about $\mathcal{C}_4$-packings. (Note the loose cycle $\mathcal{C}_4$ on 4 vertices has 2 hyperedges and these hyperedges share 2 vertices.)

Theorem 5 *For any $\gamma > 0$ there is an integer $n_1 = n_1(\gamma)$ such that every 3-uniform hypergraph $\mathcal{H}$ whose number $n \geq n_1$ of vertices is divisible by 4 and whose minimum degree is at least $n/4 + \gamma n$ contains a perfect $\mathcal{C}_4$-packing.*

Theorem 1 can be used to show that every 3-uniform hypergraph whose order n is sufficiently large and divisible by 3 and whose minimum degree at least $n/2 + 18\sqrt{n \log n}$ contains a perfect matching. This bound is best possible up to the error term $18\sqrt{n \log n}$. In view of this, Theorems 4 and 5 might be surprising at first sight since they imply that as far as the minimum degree is concerned, it is much harder to find a perfect matching in a 3-uniform hypergraph than a perfect $\mathcal{C}_4$-packing or a perfect $\mathcal{C}_k$-packing where k is sufficiently large.

References

[AS92] N. Alon and J.H. Spencer. *The Probabilistic Method*. Wiley-Interscience, New York, 1992.

[Bré73] L.M. Brégman. Some properties of nonnegative matrices and their permanents. *Soviet Mathematics Doklady*, 14:945–949, 1973.

[CCKV96] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković. Perfect matchings in balanced hypergraphs. *Combinatorica*, 16:325–329, 1996.

[FR02] P. Frankl and V. Rödl. Extremal problems on set systems. *Random Struct. Algorithms*, 20:131–164, 2002.

[Hax95] P.E. Haxell. A condition for matchability in hypergraphs. *Graphs and Comb.*, 11:245–248, 1995.

[Kim03] J.H. Kim. Perfect matchings in random uniform hypergraphs. *Random Struct. Algorithms*, 23:111–132, 2003.

[KK99] G.Y. Katona and H.A. Kierstead. Hamiltonian chains in hypergraphs. *J. Graph Theory*, 30:205–212, 1999.

[KOa] D. Kühn and D. Osthus. Loose Hamilton cycles in 3-uniform hypergraphs of high minimum degree. *submitted*.

[KOb] D. Kühn and D. Osthus. Matchings in hypergraphs of large minimum degree. *J. Graph Theory, to appear*.

[KOc] D. Kühn and D. Osthus. Multicoloured Hamilton cycles and perfect matchings in pseudo-random graphs. *submitted*.

[KSS01] J. Komlós, G. N. Sárközy, and E. Szemerédi. Proof of the Alon-Yuster conjecture. *Discrete Math.*, 235:255–269, 2001.

[RRS04] V. Rödl, A. Ruciński, and E. Szemerédi. A Dirac-type theorem for 3-uniform hypergraphs. *preprint*, 2004.

[Vu00] V.H. Vu. New bounds on nearly perfect matchings in hypergraphs: higher codegrees do help. *Random Struct. Algorithms*, 17:29–63, 2000.