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Kernel perfect and critical kernel imperfect digraphs structure

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A kernel N of a digraph D is an independent set of vertices of D such that for every $w \in V(D) - N$ there exists an arc from w to N. If every induced subdigraph of D has a kernel, D is said to be a kernel perfect digraph. Minimal non-kernel perfect digraph are called critical kernel imperfect digraph. If F is a set of arcs of D, a semikernel modulo F, S of D is an independent set of vertices of D such that for every $z \in V(D) - S$ for which there exists an $Sz-$arc of $D - F$, there also exists an $zS-$arc in D. In this talk some structural results concerning critical kernel imperfect and sufficient conditions for a digraph to be a critical kernel imperfect digraph are presented.

Keywords: kernel, semikernel, semikernel modulo F, kernel perfect digraph, critical kernel imperfect digraph

Let D be a digraph; $V(D)$ and $A(D)$ will denote the set of vertices and arcs of D respectively. Let S_1, S_2 be subsets of $V(D)$. The arc u_1u_2 of D will be called an S_1S_2-arc whenever $u_1 \in S_1$ y $u_2 \in S_2$. Let H be a subdigraph of D. If $uv \in A(D) - A(H)$ then uv is called a *pseudodiagonal of* H. $\Gamma^+(u)$, (resp. $\Gamma^-(u)$) is the exneighbourhood (resp. inneighbourhood) of u in D.

A *kernel* N of D is an independent set of vertices such that for every $w \in V(D) - N$ there exists an arc from w to a vertex in N. The concept of kernel was introduced by Von Neumann and Morgenstern (10) as an abstract generalization of their concept of solution for cooperative games. The problem of the existence of a kernel in a given digraph has been studied by several authors, since it is important in the context of Game Theory and Decision Theory, so the main question is: Which structural properties of a graph imply the existence of a kernel?

The classical results (1) are:

1. A symmetric digraph is kernel perfect;
2. A transitive digraph is kernel perfect, and all kernels have the same cardinality (König);
3. A digraph without cycles is kernel perfect, and its kernel is unique (von Neumann);
4. A graph without cycles of odd length is kernel perfect (Richardson)

Many extensions of Richardson's Theorem have have been found. An easy one is:

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Proposition 1 *Let D a digraph such that every cycle of odd length is symmetrical. Then D is kernel perfect.*

Others theorems have been found, in particular the following:

1. *If every cycle of odd length $(x_1, x_2, \ldots x_{2k+1}, x_1)$ has two pseudodiagonals of the type (x_i, x_{i+2}), (x_{i+1}, x_{i+3}) then the digraph is kernel perfect.* (3)

2. *If every cycle of odd length has at least two symmetrical arcs, then the digraph is kernel perfect.* (2)

A directed cycle of length 3 will be called a triangle and a forbidden triangle is a triangle with at most one symmetrical arc. *M-oriented digraphs* have no forbidden triangles. The *covering number* of a digraph D, denoted by $\theta(D)$ is the minimum number of complete subdigraphs of D that partition $V(D)$.

The following are sufficient conditions for a M-oriented digraphs with $\theta(D) \leq 3$ is kernel perfect:

- If each directed cycle $\mathcal{C}$ of length 5 contained in D satisfies at least one of the following properties: (a) $\mathcal{C}$ has two diagonals, (b) $\mathcal{C}$ has three symmetrical arcs.

- If every directed cycle of length 5 has three symmetrical arcs.

- If every directed cycle of length 5 has a symmetrical diagonal.

- If every directed cycle of length 5 has two diagonals.

A *semikernel* S of D is an independent set of vertices such that for every $z \in V(D) - S$ for which there exists an arc from a vertex in S to z, there also exists an arc from z to a vertex in S. Notice that a kernel N of D is a semikernel of D. A digraph D is *kernel perfect* if every non-empty induced subdigraph of D has a kernel. We say that D is a *critical kernel imperfect digraph* if D does not have a kernel but each proper induced subdigraph of D does have at least one .

In (9), Neumann-Lara introduced the concept of a semikernel and, considering the kernel perfect digraphs, obtained sufficient conditions for the existence of a kernel in a digraph in terms of semikernels.

Teorema 2 *(9) Let D be a digraph. If every induced subdigraph of D has a non-empty semikernel then D is kernel perfect.*

This result provides another equivalent definition of a kernel perfect digraph: a digraph is kernel perfect if every non-empty induced subdigraph has a non-empty semikernel.

Theorem 2 allows us to prove in a simpler way Richardson's Theorem (7), which originally had a long and complicated proof: any digraph which does not contain directed cycles of odd length has a kernel; its enough to prove that every bipartite digraph has a semikernel. Theorem 2 also provides tools to give some general sufficient conditions for a digraph to be a kernel perfect digraph and some structural properties on critical kernel imperfect digraphs. Therefore, the concept of a semikernel has been very important in the development of Kernel Theory.

In (5), Galeana-Sánchez introduced the following concept: let F be a set of arcs of D. A set $S \subseteq V(D)$ is called a *semikernel of D modulo F* if S is an independent set such that for every $z \in V(D) - S$ for which there exists an arc from a vertex in S to z of $D - F$, there also exists an $zS-$arc in D. We can observe that a semikernel S is a semikernel modulo F, (for some F).

A digraph D will be called *asymmetrically transitive* whenever $uv, vw \in Asym(D)$ implies $uw \in Asym(D)$, where $Asym(D)$ is the spanning subdigraph of D whose arcs are asymmetrical arcs of D.

In this work the concept of semikernel modulo F is used to obtain new sufficient conditions for the existence of kernels in digraphs; this results are more general than those obtained by using the concept of semikernel and also apply for infinite digraphs.

An infinite sequence $(x_1, x_2, \ldots)$ of distinct vertices of D_1, such that $x_i x_{i+1} \in A(D_1)$ for each i is called *infinite outward path*.

Teorema 3 *Let D be a (possibly infinite) digraph. Let D_1 be an asymetrically transitive subdigraph of D without infinite outward path, such that every induced subdigraph of D has a non-empty semikernel modulo $A(D_1)$. If D has no induced subdigraph isomorphic to a member of a special family of 14 digraphs, then D is a kernel perfect digraph.*

We will provide an equivalent definition of a kernel perfect digraph for a class of digraphs; If D satisfy:

- There exists $D_1 \subset D$ such that, there is a partial order, $\leq$, in the set of non-empty semikernels of D modulo $A(D_1)$, with a maximal element.
- If S is a non-empty semikernel of D modulo $A(D_1)$, such that $B_S = \{v \in D - S \mid \nexists\, vS-$arc in $D\} \neq \emptyset$ and, if S' is a non-empty semikernel of $D[B_S]$ modulo $A(D_1)$, then $T_S \cup S'$ is non-empty semikernel of D modulo $A(D_1)$ and $T_S \cup S' > S$, where $T_S = \{v \in S \mid \nexists\, vS'-$arc in $D_1\}$.
- If S_0 is maximal with respect to $\leq$, then $S \subset S_0 \cup \{x \in V(D) \mid \exists\, xS_0 -$ arc in $D\}$, for each $S < S_0$

we say that D holds the property $P(\alpha_{D_1}, \leq)$. We say that D satisfy *hereditarily* $P(\alpha_{D_1}, \leq)$ if D holds the property $P(\alpha_{D_1}, \leq)$ and every $H \subset^* D$ holds $P(\alpha_{D_1[V(H)]}, \leq)$, with $\leq$ restricted to $\alpha_{D_1[V(H)]}$. Note that the independent sets of H are also independent in D.

Teorema 4 *Let D be a digraph that satisfy hereditarilly $P(\alpha_{D_1}, \leq)$. D is kernel perfect if every non-empty induced subdigraph has a non-empty semikernel modulo $A(D_1)$.*

Notice that Theorem 3 implies Theorem 2, if we have that D_1 is $Sym(D)$ (the spanning subdigraph of D whose arcs are symmetrical arcs of D). As a consequence of Theorem 3, we obtain a generalization of the following result due to B. Sands, N. Sauer and R. Woodrow (8): *Let D be a digraph whose arcs are colored with two colors. If D contains no monochromatic infinite outward path, then there exists a set S of vertices of D such that no two vertices of S are connected by a monochromatic directed path and for every vertex not in S there is a monochromatic directed path from x to a vertex in S.*

In (6), Galeana-Sánchez and V. Neumann-Lara, using the notions of semikernels, gave sufficient conditions for a digraph to be a kernel perfect digraph. Those conditions generalized those studied by, e.g. Duchet (2). As a example, we have:

Teorema 5 *If every directed cycle C of odd length in D has two pseudodiagonals with consecutive terminal endpoints then D is kernel perfect.*

Galeana-Sánchez and Neumann-Lara also gave some structural properties of critical kernel imperfect digraphs. In particular they proved that every vertex (resp. arc) in a critical kernel imperfect digraph D, is contained in an odd directed cycle containing some ”special pseudodiagonals”.

In this work, we generalize the results of Galeana-Sánchez and Neumann-Lara, using the notions of semikernels modulo $A(D_1)$, where $D_1 \subset D$ and asking for D to hold the property $P(\alpha_{D_1}, \leq)$, (the results of them are obtained if $D_1 = Sym(D)$).

The following theorems let us know some structures of the critical kernel imperfect digraphs:

We say that a cycle $C = (u_0, u_1, \dots u_n)$ in D *alternate* arcs, (resp. vertex), in $A \subset A(D)$, (resp. $B \subset V(D)$), if $u_0u_1, u_2u_3, \dots inA$, (resp. $u_0, u_2, \dots \in B$).

Teorema 6 *Every arc in a critical kernel imperfect digraph D (possibly infinite) holding $P(\alpha_{D_1}, \leq)$ is contained in an odd directed cycle that alternate arcs in $A(D) - A(D_1)$ not containing special pseudodiagonals.*

Remark: Up to now, it is not known if an infinite critical kernel imperfect digraph exists.

Teorema 7 *Every vertex in a critical kernel imperfect digraph D (possibly infinite), holding $P(\alpha_{D_1},\leq)$, which is not a directed cycle of odd length, belongs to at least $\Delta_D(u) + 1$ directed cycle of odd length that alternate arcs in $A(D) - A(D_1)$. ($\Delta_D(u)$=max$\{|\Gamma^-(u)|, |\Gamma^+(u)|\}$).*

In particular, we provide sufficient conditions, as in the following theorems, to assure when a digraph is kernel perfect:

Teorema 8 *Any finite digraph holding $P(\alpha_{D_1}, \leq)$ in which every odd directed cycle that alternate arcs in $A(D) - A(D_1)$, has two pseudodiagonals with consecutive terminal endpoints, is kernel perfect.*

Denote by $\mathcal{V}_{D_1}$, (resp. $\mathcal{F}_{D_1}$), the set of vertices (resp. arcs) of D which do not belong to a directed cycle of odd length that alternate arcs in $A(D) - A(D_1)$.

Teorema 9 *D is kernel perfect digraph iff $D - \mathcal{V}_{D_1}$, (resp. every induced subdigraph H of D such that $A(H) \cap \mathcal{F}_{D_1} = \emptyset$), is a kernel perfect digraph.*

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