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Packing Three-Vertex Paths in a Subcubic Graph

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In our paper we consider the P_3-packing problem in subcubic graphs of different connectivity, improving earlier results of Kelmans and Mubayi (5). We show that there exists a P_3-packing of at least $\lceil 3n/4 \rceil$ vertices in any connected subcubic graph of order $n > 5$ and minimum vertex degree $\delta \geq 2$, and that this bound is tight. The proof is constructive and implied by a linear-time algorithm. We use this result to show that any 2-connected cubic graph of order $n > 8$ has a P_3-packing of at least $\lceil 7n/9 \rceil$ vertices.

Keywords: three-vertex paths, subcubic graphs, path packing

1 Introduction

Generalized matching problems have been studied in a wide variety of contexts (1; 2; 4). One of the possible generalizations is the problem of finding the maximum number of vertex-disjoint copies of some fixed graph H in a graph G (*maximum H-packing*), and herein we study lower bounds on the size of the maximum P_3-packing in certain classes of cubic and subcubic graphs (P_3 denotes a path of order 3), a problem first discussed by Akiyama and Kano in 1985 (1).

In 2004 Kelmans and Mubayi (5) showed that any cubic graph of order n must have a P_3-packing of at least $\lceil 3n/4 \rceil$ vertices (the presented 20-page proof is constructive and implied by a quadratic-time algorithm). In Subsection 2.1 we show that a more general result holds, namely that any connected graph of order $n \neq 5$, with vertices of degree 2 and 3 only, has a P_3-packing of at least $\lceil 3n/4 \rceil$ vertices. The proof immediately implies a linear-time algorithm for finding such a packing. This bound is shown to be tight. We then briefly remark on general subcubic graphs, for which we show a tight bound of $\lceil 3n/5 \rceil$, provided $n > 2$. In Subsection 2.2 we use these results to show that any 2-connected cubic graph of order $n > 8$ has a P_3-packing of at least $\lceil 7n/9 \rceil$ vertices.

2 Bounds on the size of a P_3-packing for subcubic graphs

2.1 Packing P_3 in $(2,3)$-regular graphs

Let us recall that a graph is *subcubic* if all its vertex degrees are at most three. Next, we will call a graph $(2,3)$-*regular* if it has vertices of degree 2 and 3 only. In order to prove that a connected $(2,3)$-regular

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graph of order $n \neq 5$ has a P_3-packing of at least $\lceil 3n/4 \rceil$ paths, we will show that such a graph must have a spanning tree admitting such a packing. A modified version of the well known DFS approach may be applied to find such a tree.

Definition 1 *A graph is said to be* pendant-P_k-free *if it contains no path of $k+1$ vertices such that one of the end vertices of the path is of degree 1 in G, the other is of degree 3, while all other vertices of the path are of degree 2. In particular, a graph is* pendant-P_2-free *if none of its vertices of degree 2 is adjacent to both a vertex of degree 1 and a vertex of degree 3.*

Lemma 2 *If a subcubic tree T of order $n \notin \{1, 2, 5\}$ is pendant-P_2-free, then there exists a P_3-packing in T of at least $\lceil n/4 \rceil$ paths.*

Proof: The proof proceeds by induction on the number of vertices in the graph. If tree T has either 3 or 4 vertices or T is a path of length $n \geq 6$ then the thesis holds. It now suffices to show that if tree T fulfills the assumption and $n \geq 6$, we can find a possibly disconnected subgraph $S \subseteq T$ of order at most 4 such that S has a P_3-packing of one path, while $T \setminus S$ is a tree fulfilling the assumptions of the lemma.

First, consider the case when T has a pendant path of length k for some $k \geq 4$ (i.e. a path of length k connected to a vertex of degree 3). We will show by contradiction that the subgraph S may be chosen as either the 3-vertex path L_3, or the 4-vertex path L_4 at the end of the pendant k-path. Suppose that neither case is possible. The graph $T \setminus L_3$ is a tree of at least 3 vertices, so therefore it must either have 5 vertices, or have a pendant P_2. In the first case we immediately have that $T \setminus L_4$ has 4 vertices, and since any connected graph of 4 vertices fulfills the assumption of the lemma, we reach a contradiction. In the second case, it transpires that $k = 5$. Graph $T \setminus L_4$ does not have pendant P_2 and has at least 3 vertices, so it does not fulfill the assumptions of the lemma only when it has 5 vertices. It can easily be shown from here that since T does not have pendant P_2, T can only be the path P_9, a contradiction with $k = 5$. This completes the proof for the case when T has a pendant path of length at least 4.

Now, let us suppose that T has a pendant path L_3 of length exactly 3. If the tree $T \setminus L_3$ does not fulfill the assumption of the lemma, then it has 5 vertices or a pendant P_2. In the first case T has order 8 and can be covered by two paths P_3 (one for L_3, one for $T \setminus L_3$). The second case is illustrated in Fig. 1(b). Taking $S = L_3 \cup K$ (where K is the one-vertex graph induced by the end of the pendant P_2 in $T \setminus L_3$), we ensure that $T \setminus S$ has no pendant P_2. Taking into account that T does not contain pendant P_4, $T \setminus S$ may never have 5 vertices, which completes the proof in this case.

Finally, we consider the case when T has no pendant P_k, for any $k \geq 2$. It must therefore have a vertex of degree 3 adjacent to two vertices of degree 1. Consider the path L_3 induced by three such vertices. The proof follows analogously as in the previous case. The only difference is that this time there are two cases corresponding to the situation when $T \setminus L_3$ has a pendant P_2 (depicted in Fig. 1(c) and (d)), which necessitate a different choice of vertex K for which $S = L_3 \cup K$ fulfills the inductive assumption. □

As a side note, let us recall that Masuyama and Ibaraki (7) showed that the maximum P_i-packing problem in trees can be solved in linear time, for any $i \geq 3$. The idea of their algorithm is to treat a tree T as a rooted tree (T, r) (with an arbitrary vertex r a the root) and to pack i-vertex paths while traversing (T, r) in the bottom-up manner.

Corollary 3 *If graph G of order n has a spanning forest whose trees fulfill the assumptions of Lemma 2, then there exists a P_3-packing in G of at least $\lceil n/4 \rceil$ paths, which can be found in linear time.*

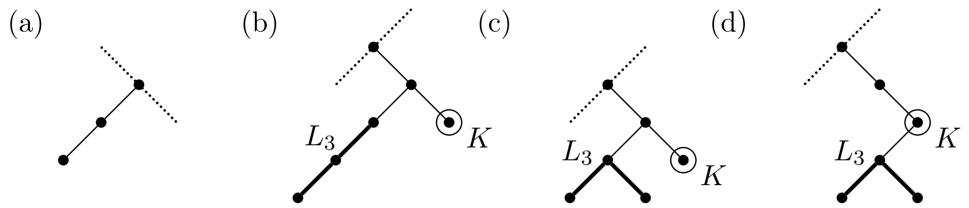

Fig. 1: (a) a tree containing a pendant P_2 path, (b) choice of path L_3 and vertex K for trees with pendant P_3, (c), (d) choice of path L_3 and vertex K for trees without pendant paths longer than 1

Theorem 4 *There exists a P_3-packing of at least $\lceil n/4 \rceil$ paths for any connected $(2,3)$-regular graph of $n > 5$ vertices.*

Proof: Consider a $(2,3)$-regular graph G of order n which fulfills the assumptions of the theorem. Taking into account Corollary 3, it suffices to show an algorithm for constructing a spanning forest in G whose trees fulfill the assumptions of Lemma 2. Such an algorithm is presented below.

1. For each connected component of G successively consider all edges e connecting vertices of degree 3. If removal of the edge from G does not create a new connected component of order 5, remove the edge from G and continue the process. Otherwise mark one of the endpoints of e as a cut vertex and proceed to the next connected component.

2. For each connected component H of G construct a Depth First Search (DFS) spanning tree T. The tree should be rooted following one of the rules below:

   (a) If H has a cut vertex v marked in Step 1, let vertex v be the root of the tree.

   (b) If there exists an induced path (u_1, u_2, u_3) in H such that u_2 and u_3 are adjacent and of degree 2 in H, let u_1 be the root of the tree and let u_2 be the first vertex visited while recursing.

   (c) If neither rule (a) nor rule (b) can be applied, let any vertex of degree 2 in H be the root of the tree.

3. For each connected component H, if the resulting DFS tree T is not pendant-P_2-free, then for each DFS leaf v at the end of a pendant P_2, remove from T the edge incident to v and insert into T any other edge which is incident to v in H. The set of spanning trees obtained in this way (taken over all components of G) is the sought pendant-P_2-free spanning forest.

Careful analysis shows that application of Step 1 and the rules of selecting a DFS root in Step 2 of the algorithm guarantee that the root of the DFS tree is not an end vertex of a pendant P_2 path and does not become one throughout step 3. We will confine ourselves to the proof that applying Step 3 of the algorithm on any DFS spanning tree guarantees that the resultant tree does not have pendant-P_2 containing the DFS-leaves of the tree. Indeed, each pendant P_2 of T at the start of Step 3 either contains the DFS root v, or ends in some DFS leaf u. In the latter case, since vertex u is of degree at least 2 by assumption, there must exist an edge connecting u with some vertex w on the path from v to u in the DFS tree, other than the direct DFS parent of u. It transpires that w is not a DFS leaf and that the subtree of T rooted at w is not a path. Performing the reconnection operation described in Step 3 of the algorithm removes one pendant P_2 path from T without creating any new ones. Furthermore, since operations are performed on leaves only, the essential properties of the DFS tree are not lost throughout the algorithm.

The runtime of all the steps of the presented algorithm is linear and requires little comment. □

The bound given by Theorem 4 is tight even for the class of 2-connected $(2, 3)$-regular graphs. The class of graphs obtained from cyclic ladders by inserting exactly two vertices on every edge, may serve as an example (see Appendix).

It is interesting to consider in what way Lemma 2 may be generalised if no assumptions are made on the form of the considered spanning tree. Using a similar technique as that used in the proof of Lemma 2, it is easy to show the following statement.

Theorem 5 *There exists a P_3-packing of at least $\lceil(3n-6)/5\rceil$ vertices in any subcubic graph of order n.*

The bound given in Theorem 5 is tight even for subcubic trees (see Appendix).

2.2 Packing P_3 in cubic 2-connected graphs

For connected cubic graphs the known lower bound is a direct conclusion from Theorem 4 (since a cubic graph may not have 5 vertices, all cubic graphs have a P_3-packing of at least $\lceil 3n/4\rceil$ vertices). It is not known whether this bound is tight for arbitrarily large values of n; an upper bound of $\lceil 4n/5\rceil$ can be obtained by considering a class of graphs with numerous pendant 5-vertex components (see Appendix).

The effect of the connectivity of a cubic graph on the size of its P_3-packing was first discussed by Akiyama and Kano (1), who posed the following conjecture (which is still open).

Conjecture 6 (1) *Every 3-connected cubic graph of order divisible by three has a perfect P_3-packing.*

For 2-connected cubic graphs Conjecture 6 does not hold (1), and the problem of finding a maximum P_3-packing in a 2-connected bipartite graph is APX-hard. However it is possible to develop an improved lower bound of $\lceil 7n/9\rceil$ vertices on the size of the P_3-matching in a 2-connected cubic graph.

Due to the lack of space we only present the outline of the proofs. The applied techniques are similar to those used in the proof of Theorem 4 and the proof is based on a very detailed analysis of individual cases.

Lemma 7 *If the vertex set of a connected subcubic graph G of order $n > 8$ can be partitioned into sets of 4, 5 and 8 elements, each of which induces a Hamiltonian subgraph of G, then there exists a P_3-packing of at least $\lceil 7n/9\rceil$ vertices in G.*

Proof: Consider a multigraph M obtained from graph G by replacing each subgraph induced by sets of 4, 5 and 8 elements (called an *s-graph*, where $s = 4, 5, 8$) by a vertex in the multigraph (called an *s-vertex*, where $s = 4, 5, 8$). Two s-vertices are adjacent in M iff there is an edge in G connecting two corresponding s-graphs. As noticed in (6) every connected graph G of at least 3 vertices has a partition into spiders (i.e. stars of at least 3 vertices with attached pendant vertices, at most one per each pendant vertex of the star). Since G has at least 9 vertices, multigraph M can be partitioned into spiders or M has exactly two vertices, one of them representing a 5- or 8-vertex cycle in G. This partition of M induces the corresponding partition of graph G into connected components of one of several possible types (of not more than 136 vertices). By the detailed analysis of these components it can be proven that in each such component C there exists a P_3-packing of at least $\lceil 7|V(C)|/9\rceil$ vertices. □

Class of graphs	Connected	2-connected	3-connected
Subcubic	$\lceil(3n-6)/5\rceil$*	-	-
$(2,3)$-regular	$\lceil 3n/4\rceil, n>5$*	$\lceil 3n/4\rceil, n>5$*	-
Cubic	$\lceil 3n/4\rceil$ (see (5))	$\lceil 7n/9\rceil, n>8$	$\lceil 7n/9\rceil, n>8$

Tab. 1: Proven lower bounds on the number of vertices in a maximum P_3-packing in a subcubic graph. Tight bounds are marked with asterisks.

Let G be a 2-connected cubic graph of order $n > 8$. Since G is a 2-connected cubic graph, its vertex set can be partitioned into subsets inducing Hamiltonian subgraphs of order at least 4 (see Lemma 7 (4)). Analogously as in the proof of Lemma 7, we construct the multigraph M with s-vertices corresponding to the subsets of s elements inducing Hamiltonian subgraphs in G. We consider connected components of multigraph M induced by all s-vertices, where $s \in \{4, 5, 8\}$ and the rest of multigraph M, say M'. We attach isolated s-vertices (for $s = 4, 5, 8$) and components of two connected 4-vertices to M'. By Theorem 4 one can prove that M' admits P_3-packing of at least $\lceil 7n/9\rceil$ vertices, and by Lemma 7 we get

Theorem 8 *There exists a P_3-packing of at least $\lceil 7n/9\rceil$ vertices for any 2-connected cubic graph of order $n > 8$.*

A summary of the main results presented in the paper is given in Table 1.

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Appendix

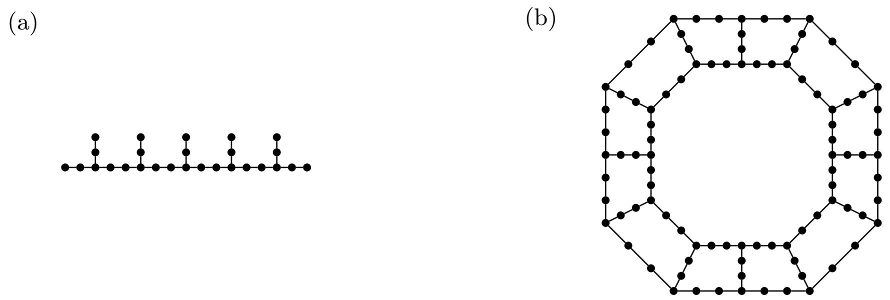

Fig. 2: (a) A tight example for the bound on the number of vertices in the maximum P_3-packing in a subcubic graph ($\lceil(3n-6)/5\rceil$). (b) A tight example for the bound on the number of vertices in the maximum P_3-packing in a 2-connected $(2,3)$-regular graph ($\lceil 3n/4\rceil$).

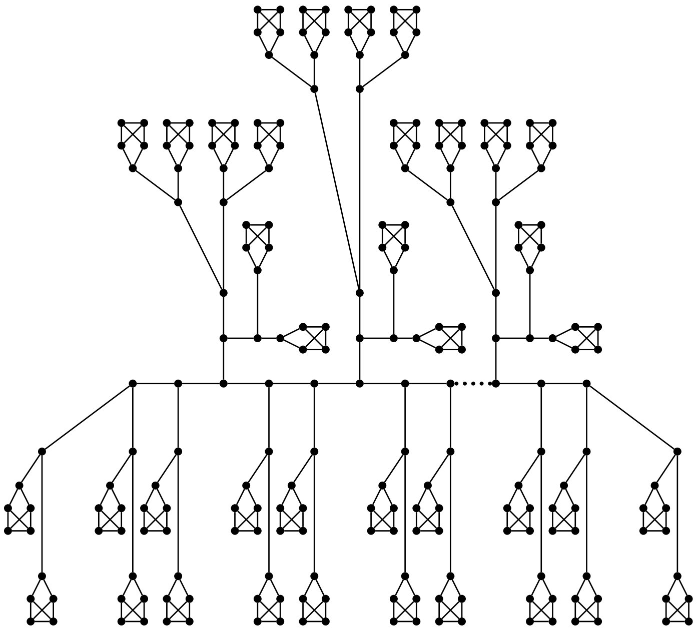

Fig. 3: An example of a class of cubic graphs with a maximum P_3 packing of not more than $4n/5$ vertices.