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Removing Even Crossings

Michael J. Pelsmajer[1], Marcus Schaefer[2] and Daniel Štefankovič[2]

[1] *Department of Applied Mathematics, Illinois Institute of Technology, Chicago, Illinois 60616*
[2] *Department of Computer Science, DePaul University, Chicago, Illinois 60604*
[3] *Computer Science Department, University of Rochester, Rochester, NY 14627-0226*

An edge in a drawing of a graph is called *even* if it intersects every other edge of the graph an even number of times. Pach and Tóth proved that a graph can always be redrawn such that its even edges are not involved in any intersections. We give a new, and significantly simpler, proof of a slightly stronger statement. We show two applications of this strengthened result: an easy proof of a theorem of Hanani and Tutte (not using Kuratowski's theorem), and the result that the odd crossing number of a graph equals the crossing number of the graph for values of at most 3. We begin with a disarmingly simple proof of a weak (but standard) version of the theorem by Hanani and Tutte.

Keywords: Hanani's theorem, Tutte's theorem, even crossings, crossing number, odd crossing number, independent odd crossing number

1 The Hanani and Tutte Theorem

In 1970 Tutte published his paper "Toward a Theory of Crossing Numbers" (12) containing the following beautiful theorem.

> In any planar drawing of a non-planar graph there are two non-adjacent edges that cross an odd number of times. In other words: if a graph can be drawn such that every pair of non-adjacent edges intersects an even number of times, then the graph is planar.

Tutte acknowledges earlier proofs of the same result, including the paper "Über wesentlich unplättbare Kurven im drei-dimensionalen Raume" (4) published in 1934 by Chaim Chojnacki (who later changed his name to Haim Hanani). While there is general agreement that the result itself is "remarkable" (9; 2), and "nice" (1), the same cannot be said for its proofs. Both Hanani and Tutte took the same general approach using Kuratowski's theorem: If the graph is non-planar it contains a subdivision of $K_{3,3}$ or K_5, so they only have to show that any drawing of *these* graphs contains two non-adjacent edges that cross an odd number of times. Hanani opts for a more topological approach, while Tutte develops an algebraic theory of crossing numbers.

We want to present a very intuitive and entirely geometric proof of the result which, furthermore, does not use Kuratowski's theorem. We begin by proving a slightly weaker result.

Let us call an edge in a drawing *even* if it intersects every other edge an even number of times.

Theorem 1.1 (Hanani and Tutte, weak version) *If G can be drawn in the plane such that all its edges are even, then G is planar.*

Proof We may assume that G is connected, since components may be redrawn arbitrarily far apart. Fix a plane drawing D of G in which every pair of edges intersects an even number of times. We prove the result by induction on the number of edges in G. To make the inductive step work, we keep track of the *rotation* of each vertex, that is, the cyclic order in which edges leave the vertex in the drawing. The mapping from the vertices of G to their rotations is called the *rotation system* of D. We will prove the following stronger statement:

> If D is a drawing of a multigraph G such that any pair of edges intersects an even number of times in D, then G is planar and can be drawn without changing the rotation system.

We begin with the inductive step: if there are at least two vertices in G, then there is an even edge $e = uv$. Pull v towards u as shown in the left part of Figure 1.

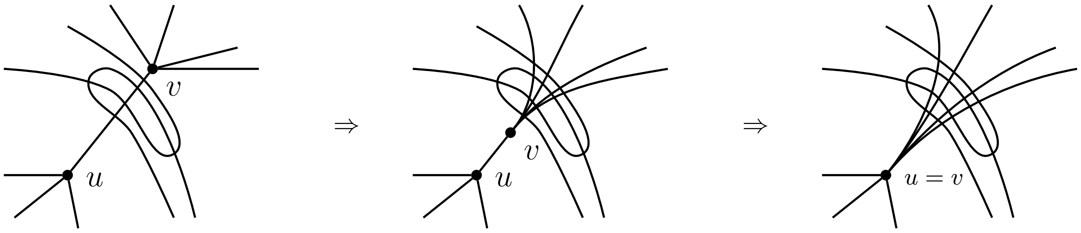

Fig. 1: Pulling an endpoint (left) and contracting the edge (right).

Since e was an even edge, the edges incident to v remain even. The pulling move will introduce self-intersections in curves that intersect e and are adjacent to v. Since drawings are typically defined not to have self-intersections, we remove them by using the move shown in Figure 2 (although we could preserve self-intersections and instead modify the analysis slightly).

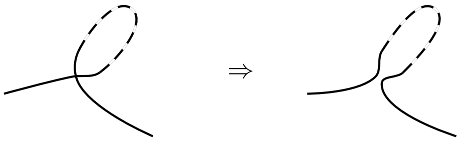

Fig. 2: Removing a self-intersection.

Now that uv no longer has any intersections, we contract it to obtain a new graph G' in which the rotations of u and v are combined appropriately (see the right part of Figure 1).

By the inductive assumption, there is a planar drawing of G' respecting the rotation system. In such a drawing, we can simply split the vertex corresponding to u and v, reintroducing the edge e between them without any intersections. Hence G is planar respecting the rotations of all its vertices.

If G contains only a single vertex, then it might have several loops attached to it. Since all the loops are even edges, it cannot happen that we find edges leaving in order a, b, a, b since this would force an odd number of intersections between a and b. Hence, if we consider the regions enclosed within the two loops in a small enough neighborhood of the vertex, either they are disjoint or one region contains the other. Then it is easy to show that there must be a loop whose ends are consecutive in the rotation system. Removing this edge we obtain a smaller graph G' which, by inductive assumption, can be drawn without intersections and with the same rotation system. We can then reinsert the missing loop at the right location in the rotation system by making it small enough.

In the base case, we simply draw a single vertex with no edges. □

We can restate the result in terms of crossing numbers. The *crossing number* of a drawing of a graph is the total number of crossings of each pair of edges. The *crossing number of* G, $\mathrm{cr}(G)$, is the smallest crossing number of any drawing of G. The *odd crossing number* of a drawing is the number of pairs of edges that cross an odd number of times. The *odd crossing number of* G, $\mathrm{ocr}(G)$, is the smallest odd crossing number of any drawing of G. It follows from the definition that

$$\mathrm{ocr}(G) \leq \mathrm{cr}(G).$$

Theorem 1.1 shows that $\mathrm{ocr}(G) = 0$ implies $\mathrm{cr}(G) = 0$ (that is, G is planar). The original result by Hanani and Tutte draws the same conclusion under the weaker assumption that all pairs of non-adjacent edges intersect an even number of times. This suggests the concept of the *independent odd crossing number*, $\mathrm{iocr}(G)$, as the smallest number of pairs of non-adjacent edges of G that intersect an odd number of times in any drawing of G. The original Hanani and Tutte result (4; 12) can then be stated as follows.

Theorem 1.2 (Hanani and Tutte, strong version) *If* $\mathrm{iocr}(G) = 0$*, then* $\mathrm{cr}(G) = 0$.

We will return to this theorem in the next section.

Remark 1 There have been several proofs of both the weak and the strong version of the Hanani and Tutte theorem.

Let us begin with proofs of the strong version. Two papers in 1976, one by Kleitman (6), the other by Harborth (5) showed that the parity of $\mathrm{iocr}(G)$ is independent of the drawing of G if G is either K_{2j+1} or $K_{2j+1,2j+1}$. Norine (7) supplies a different proof of this result, and observes, that it implies the strong version of the theorem of Hanani and Tutte by an application of Kuratowski's theorem. Székely (11) shows that $\mathrm{iocr}(K_{3,3}) = \mathrm{iocr}(K_5) = 1$ simplifying Tutte's algebraic approach. Again, an application of Kuratowski's theorem yields the strong version of the Hanani and Tutte theorem.

The weak version was proved by Pach and Tóth (9), strengthening it in a different direction by allowing the presence of edges that are not even. In Section 2 we will show how to obtain their version of the result using our methods. There also is a proof by Cairns and Nikolayevsky (3, Lemma 3) using homology which shows that the weak version is true on surfaces of any genus.

2 The Pach-Tóth Result and Applications

Pach and Tóth (9, Theorem 1) generalized the weak version of the Hanani and Tutte theorem by showing that one can redraw even edges without crossings (even in the presence of odd edges). Their proof is a nontrivial extension of Tutte's and Hanani's approach of extending Kuratowski's theorem. We show that our inductive approach gives a much simpler proof of the Pach-Tóth result. In fact, it yields the stronger conclusion that we can perform the redrawing without adding pairs of edges that intersect an odd number of times; in particular the odd crossing number does not increase.

Theorem 2.1 *If D is a drawing of G in the plane, and E_0 is the set of even edges in D, then G can be drawn in the plane such that no edge in E_0 is involved in an intersection and there are no new pairs of edges that intersect an odd number of times.*

The proof is in the same spirit as the proof of Theorem 1.1, involving some additional geometric transformations. We mention two applications of our strengthened version of the result by Pach and Tóth. First, we can give an easy proof of Theorem 1.2, which, as far as we know, is the first direct and geometric proof not making use of Kuratowski's theorem. The core idea of the proof is to locate cycles in the graph, make its edges even, and then redraw the cycle without intersections, applying Theorem 2.1. Repeating this process in the right manner will eventually remove all intersections.

The second result is related to the odd crossing number. When applying Theorem 2.1 to draw conclusions about the odd crossing number, we proceed as follows: Draw G to minimize odd crossing number $\mathrm{ocr}(G)$ (call pairs that intersect an odd number of times *odd pairs*, and edges that belong to an odd pair is *odd*; the remaining edges are *even*). Using Theorem 2.1, we can redraw the even edges so they are not involved in any intersections and such that the new drawing still has odd crossing number $\mathrm{ocr}(G)$. Now the even edges form a plane graph G' and each odd edge lies entirely within some face of G'. We can now process the odd edges within each face separately to obtain results on crossing numbers.

Example 2.2 For example, let us reconstruct the argument by Pach and Tóth which shows that $\mathrm{cr}(G) \leq 2\,\mathrm{ocr}(G)^2$. Consider a subgraph H of G drawn in the plane consisting of the (odd) edges within a face of G' and the (even) edges on the boundary of the face. We can redraw the odd edges of H such that each pair intersects at most once; after the redrawing we have at most $\binom{|E(H)|}{2}$ crossings within that face of G'. If we do the same for every face of G' we can conclude that $\mathrm{cr}(G)$ is at most the sum of $\binom{|E(H)|}{2}$ where H ranges over the odd subgraphs in the faces of G'. Furthermore, we know that there are at most $2\,\mathrm{ocr}(G)$ odd edges in total, hence $\mathrm{cr}(G) \leq \binom{2\,\mathrm{ocr}(G)}{2} \leq 2\,\mathrm{ocr}(G)^2$. ■

This approach turns out to be successful when the odd crossing number is small. Recall that by definition $\mathrm{ocr}(G) \leq \mathrm{cr}(G)$, and that by the result of Hanani and Tutte, $\mathrm{ocr}(G) = 0$ implies $\mathrm{cr}(G) = 0$. This suggests that perhaps $\mathrm{ocr}(G) = \mathrm{cr}(G)$ for all graphs G (see (8; 9; 13; 1)). While this conjecture turns out to be false (10), we can show, using our approach, that $\mathrm{ocr}(G) = \mathrm{cr}(G)$ if $\mathrm{ocr}(G)$ is small enough.

Theorem 2.3 *If G is a graph with $\mathrm{ocr}(G) \leq 3$, then $\mathrm{ocr}(G) = \mathrm{cr}(G)$.*

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