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Tiling a Rectangle with Polyominoes

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A polycube in dimension d is a finite union of unit d-cubes whose vertices are on knots of the lattice $\mathbb{Z}^d$. We show that, for each family of polycubes E, there exists a finite set F of bricks (parallelepiped rectangles) such that the bricks which can be tiled by E are exactly the bricks which can be tiled by F. Consequently, if we know the set F, then we have an algorithm to decide in polynomial time if a brick is tilable or not by the tiles of E.

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1 Introduction

A *polycube in dimension d* (or more simply a *polycube*) is a finite -not necessarily connected- union of unit cubes whose vertices are on nodes of the lattice $\mathbb{Z}^d$. Given a family (finite or not) of polycubes E, a polycube P is *tilable by E* if there exists a finite sub-family E_1 of E such that P is a union of translated copies of disjoint interiors of elements of E_1. An important problem about polyominoes tiling (polycubes in dimension 2) concerns their capacity to tile some rectangles (see works of Klarner [7], Golomb [5,6], Reid [9], Marshall [8], Dahlke [4],...). Especially, Klarner has defined the notion of polyomino *order* as the smallest number of congruent copies of a polyomino which form a rectangle. The question whether there exists a polyomino of odd order $n \geq 5$ is totally open and still remains one of the great discoveries to be made on the polyominoes theory. To understand the difficulty of that problem, we can remind ourselves that the order (seen as a function) is not algorithmically calculable [1]. The main subject of this paper is to show that, for each family of polycubes E, there exists a finite set F of bricks (parallelepiped rectangles) such that the bricks that can be tiled by E are exactly the bricks that can be tiled by F. In other words, the set of bricks which can be tiled by a fixed family of polycubes is generated by a finite set of bricks called the *prime bricks*. The proof of this theorem cannot be constructive. In fact, given a family E of polycubes, the problem whether "there exists a rectangle which can be tiled by E" is algorithmically undecidable [1]. Thus, a theorem of finite basis type is a bit surprising, although the determination of the cardinality of this basis remains undecidable.

2 Algebraic Relaxation

We are going to deal with algebraically simpler objects : for that, we put weights on the cubes of the polycube (we identify a polycube by the set of cubes which compose it). This brings us to introduce the following formal definition : Let $\mathbb{A}$ be a unitary ring, an $\mathbb{A}$-*weighted* polycube is a map with finite support

P from the set of cubes of the lattice $\mathbb{Z}^d$ into the ring $\mathbb{A}$. For each cube c, the number $P(c)$ is called *the weight of P in c*. The space $P_{\mathbb{A}}$ of the $\mathbb{A}$-weighted polycubes has a natural structure of free $\mathbb{A}$-module and cubes of weight 1 clearly form a basis of $P_{\mathbb{A}}$. Thus, we have a canonical injection of the set of polycubes in the $\mathbb{A}$-module of $\mathbb{A}$-weighted polycubes. We can henceforth introduce the notion of $\mathbb{A}$-tiling in the following way : An $\mathbb{A}$-weighted polycube P will be said $\mathbb{A}$-*tilable* by a family of $\mathbb{A}$-weighted polycubes (called *the tiles*) if and only if P is an $\mathbb{A}$-linear combination of translated copies of elements of this family. Consequently, if a polycube P is tilable by a family of tiles E, then P is necessarily $\mathbb{A}$-tilable by the family E. This provides interesting necessary conditions for the tilability by a family of tiles for a polycube P (see [3] for a development of this idea).

We naturally identify the unit cube $(a_1,a_2,...,a_d)+[0,1]^d$ of $\mathbb{R}^d$ to the vector $(a_1,a_2,...,a_d)$. Moreover, for $a=(a_1,...,a_d)\in\mathbb{Z}^d$, we put by convention $X^a = X_1^{\frac{a_1+|a_1|}{2}}...X_d^{\frac{a_d+|a_d|}{2}}Y_1^{\frac{|a_1|-a_1}{2}}...Y_d^{\frac{|a_d|-a_d}{2}}$ (this does not constitute all the monomials in $X_1,...,X_d,Y_1,...,Y_d$ but only monomials which have not simultaneously some X_i and some Y_i).

For each $\mathbb{A}$-weighted polycube P, the polynomial :
$Q_P = \sum_{(a_1,...,a_d)\in\mathbb{Z}^d} P((a_1,...,a_d))X^{(a_1,...,a_d)}$ of $\mathbb{A}[X_1,...,X_d,Y_1,...,Y_d]$ is the *polynomial of P*.

For example, the polycube P in dimension 2 made of the unit squares $(0,1)$, $(0,0)$, $(1,0)$, $(2,0)$ has for polynomial $Q_P = X_2+1+X_1+X_1^2$. Similarly, the polycube $b=\{(x_1,...,x_d)\in\mathbb{R}^d$ such as $0\le x_i\le b_i\}$ is called the *brick* of size $b_1,...,b_d$ and denoted by $b_1\times...\times b_d$. Its polynomial is $Q_b=\prod_{i=1}^{d}\frac{(X_i^{b_i}-1)}{X_i-1}$.

Lemma 2.1 *The space $P_{\mathbb{A}}$ is isomorphic to $\mathbb{A}[X_1,...,X_d,Y_1,...,Y_d]/\langle(X_1Y_1-1),...,(X_dY_d-1)\rangle_{\mathbb{A}}$ where $\langle(X_1Y_1-1),...,(X_dY_d-1)\rangle_{\mathbb{A}}$ denotes the ideal of $\mathbb{A}[X_1,...,X_d,Y_1,...,Y_d]$ generated by*

$$(X_1Y_1-1),...,(X_dY_d-1).$$

Proof. There exists a unique linear map f from $\mathbb{A}[X_1,...,X_d,Y_1,...,Y_d]$ to $P_{\mathbb{A}}$ such that $f\left(X_1^{a_1}Y_1^{b_1}...X_d^{a_d}Y_d^{b_d}\right)$ is the unit cube $(a_1-b_1,...,a_d-b_d)$ of weight 1 (since monomials form a basis of $\mathbb{A}[X_1,...,X_d,Y_1,...,Y_d]$).

f is surjective because, for each $\mathbb{A}$-weighted polycube P, we have $f(Q_P)=P$. We can conclude by proving that $\ker(f)=\langle(X_1Y_1-1),...,(X_dY_d-1)\rangle_{\mathbb{A}}$. As we can write (in proceeding by successive divisions by $(X_1Y_1-1),...,(X_dY_d-1)$) every polynomial Q in the form $Q=R+\sum_{i=1}^{d}Q_i(X_iY_i-1)$ where R belongs to the $\mathbb{A}$-module generated by the family $\{X^a;a\in\mathbb{Z}^d\}$, we see that $f(Q)$ is the empty polycube, denoted by 0 (it is the polycube P of weight 0 on all the cubes in the space) if and only if $f(R)=0$ (because $f(Q_i(X_iY_i-1))=0$ by linearity and by definition of f). Moreover, it is clear that $f(R)=0\Leftrightarrow R=0$ and that $R=0\Leftrightarrow Q\in\langle(X_1Y_1-1),...,(X_dY_d-1)\rangle_{\mathbb{A}}$, so $\ker(f)=\langle(X_1Y_1-1),...,(X_dY_d-1)\rangle_{\mathbb{A}}$. □

We denote by L the linear map from $P_{\mathbb{A}}$ into $\mathbb{A}[X_1,...,X_d,Y_1,...,Y_d]/\langle(X_1Y_1-1),...,(X_dY_d-1)\rangle_{\mathbb{A}}$ which associates to P the polynomial Q_P and by L' the linear quotient map : $f/\langle(X_1Y_1-1),...,(X_dY_d-1)\rangle_{\mathbb{A}}$ from $\mathbb{A}[X_1,...,X_d,Y_1,...,Y_d]/\langle(X_1Y_1-1),...,(X_dY_d-1)\rangle_{\mathbb{A}}$ into $P_{\mathbb{A}}$.

We can notice immediately that L and L' are reciprocal maps.

Lemma 2.2 *Let $\mathbb{A}$ be a unitary ring and E a set of $\mathbb{A}$-weighted polycubes. An $\mathbb{A}$-weighted polycube P is $\mathbb{A}$-tilable by E if and only if $Q_P\in\langle Q_{P'}$ with $P'\in E, X_1Y_1-1,...,X_dY_d-1\rangle_{\mathbb{A}}$.*

Proof. By definition, an $\mathbb{A}$-weighted polycube P is $\mathbb{A}$-tilable by E if and only if there exists for all $1 \leq i \leq t$, $\lambda_i \in \mathbb{A}$, $P^i \in E$ and $a^i = \left(a^i_1, ..., a^i_d\right) \in \mathbb{Z}^d$ such that $P = \sum_{i=1}^{t} \lambda_i P^i_{\left(a^i_1,...,a^i_d\right)}$ where $P^i_{\left(a^i_1,...,a^i_d\right)}$ denotes the translated of P^i by the vector $\left(a^i_1, ..., a^i_d\right)$. By the isomorphism L, this is equivalent to $Q_P = \sum_{i=1}^{t} \lambda_i X^{a^i} Q_{P^i}$ in $\mathbb{A}[X_1, ..., X_d, Y_1, ..., Y_d] / \langle (X_1Y_1 - 1), ..., (X_dY_d - 1) \rangle_{\mathbb{A}}$ and so P is $\mathbb{A}$-tilable by E if and only if $Q_P \in \langle Q_{P'} \text{ with } P' \in E, X_1Y_1 - 1, ..., X_dY_d - 1 \rangle_{\mathbb{A}}$. □

In the following, the set $\langle Q_{P'} \text{ with } P' \in E, X_1Y_1 - 1, ..., X_dY_d - 1 \rangle_{\mathbb{A}}$ is denoted by $I(E)$. This is what we call the *ideal of E*.

3 Existence of Finite Generated Families

Firstly (until the theorem 3.4), we are going to present a short proof of a fundamental result due to Barnes [2].

In order to do that, we are going to define new notations. A *color c*, is a d-tuple $(c_1, ..., c_d)$ where c_i belongs to $\mathbb{N} \setminus \{1\}$. We say that a color $(c_1, ..., c_n)$ *conceals* a brick $B_1 \times ... \times B_d$ if there exists an i such that c_i divides B_i.

Let c be a color, we denote by e^{ic} the d-tuple $(x_1, ..., x_d)$ where $x_i = \begin{cases} 1 & \text{if } c_i = 0 \\ e^{\frac{2i\pi}{c_i}} & \text{otherwise} \end{cases}$.

Lemma 3.1 *For every brick B (embedded in $\mathbb{C}$) and for every color c, $Q_B(\alpha(c))$ is null if and only c conceals B.*

Proof. $Q_B = \prod_{i=1}^{d} \frac{(X_i^{B_i} - 1)}{X_i - 1} = 0$ if and only if there exists an i such that the factor $\frac{X_i^{B_i} - 1}{X_i - 1}$ in Q_B vanishes during the evaluation. Thus, $c_i \mid B_i$. □

Let B be a brick $B_1 \times ... \times B_d$ and let $\{b^1_1 \times ... \times b^1_d, ..., b^m_1 \times ... \times b^m_d\}$ be a set E of bricks. We denote by m_i the least common multiple of $b^1_i, ..., b^m_i$. A color $(c_1, ..., c_d)$ is $(m_1, ..., m_d)$*-primal* if every positive c_i is a prime power which divides m_i. The set C_E will be the set of the $(m_1, ..., m_d)$-primal colors which conceal all bricks in E.

Lemma 3.2 *If a brick B is $\mathbb{C}$-tilable by E, then $\forall c \in C_E$, c conceals B.*

Proof. It is clear that $Q_B = \sum_{b \in E} P_b Q_b$ where P_b is a polynomial. Now, let c be an $(m_1, ..., m_d)$-primal color which conceals all bricks in E, then $\alpha(c)$ is a zero of every Q_b. So, $\alpha(c)$ is a zero of Q_B and by lemma 3.1, d conceals B. □

In the sequel, we denote by $v_p(n)$ the *p-valuation* of n, that is to say the greatest integer k such that $p^k \mid n$.

Lemma 3.3 *Let $\{b^1_1 \times ... \times b^1_d, ..., b^m_1 \times ... \times b^m_d\}$ be a set E of bricks and $B_1 \times ... \times B_d$ a brick B. If the brick B verifies the following conditions :*

1. *$\forall c \in C_E$, c conceals B.*
2. *$\forall i$, there exists a prime number p_i such that for every prime number $p \neq p_i$, we get $v_p(B_i) \geq \max_k \left(v_p\left(b^k_i\right)\right)$*

Then $B_1 \times ... \times B_d$ is a multiple of one of the bricks of E.

Proof. Let $c = (c_1, ...c_d)$ be the color where $c_i = p_i^{v_{p_i}(B_i)+1}$. By lemma 3.1, it follows that $Q_B(\alpha(c)) \neq 0$. So, there exists a brick b in E such that the evaluation of Q_b in $\alpha(c)$ is not null too (this follows from the first condition of the lemma). So, for every i, c_i does not divide b_i, which implies $v_{p_i}(B_i)+1 > v_{p_i}(b_i)$. Now, by hypothesis, for every $p \neq p_i$, $v_p(B_i) \geq v_p(b_i)$. We conclude that for every i, b_i divides B_i. □

We note $\mathbb{B}_{>k}$ the set of bricks B such that all the edges of B are greater than k. In other words, $\mathbb{B}_{>k} = \{b_1 \times ... \times b_d \text{ with } b_i > k\}$. Now, we can prove the following theorem due to Barnes [2]:

Theorem 3.4 *Let E be a finite set of bricks, then there exists an integer k, which depends uniquely on this set, such that for each brick $B \in \mathbb{B}_{>k}$, B is tilable by E if and only if B is $\mathbb{C}$-tilable by E.*

Proof. Let $\{b_1^1 \times ... \times b_d^1, ..., b_1^m \times ... \times b_d^m\}$ be the bricks in E. We recall that m_i is the least common multiple of $b_i^1, ..., b_i^m$. m_i can be factorized in $m_i = q_{i,1}...q_{i,t_i}$ where each $q_{i,j}$ is a prime power and $q_{i,j} \wedge q_{i,j'} = 1$ for $j \neq j'$. Suppose that B is $\mathbb{C}$-tilable by E. By lemma 3.2, $\forall c \in C_E$, c conceals B. Let δ_i be the greatest common divisor of m_i and B_i. Now, we can find $u_{i,j}$ such that for every i :

$$\frac{m_i}{q_{i,1}} u_{i,1}\delta_i + ... + \frac{m_i}{q_{i,t_i}} u_{i,t_i}\delta_i = B_i$$

If B_i is sufficiently large, all the $u_{i,j}$ can be chosen non-negative and then the non-empty bricks $\frac{m_1}{q_{1,j_1}} u_{1,j_1}\delta_1 \times ... \times \frac{m_d}{q_{d,j_d}} u_{d,j_d}\delta_d$ with $(j_1, ..., j_d)$ belonging to $\{1, ..., t_1\} \times ... \times \{1, ..., t_d\}$ constitutes a tiling of $B_1 \times ... \times B_d$. Then $\forall c \in C_E$, c conceals all this bricks. Indeed, $c_i \mid m_i$ by definition of the $(m_1, ..., m_d)$-primal colors and as c conceals B, there exists an i such that $c_i \mid B_i$. So, for a $j \in \{1, ..., t_i\}$, we get $c_i \mid \delta_i \frac{m_1}{q_{1,j_1}} u_{1,j_1}$ and thus, c conceals the non-empty bricks $\frac{m_1}{q_{1,j_1}} u_{1,j_1}\delta_1 \times ... \times \frac{m_d}{q_{d,j_d}} u_{d,j_d}\delta_d$. Now, this bricks verify the conditions of the lemma 3.3. In order to prove that, we can observe that if p doesn't divide q_{i,j_i} , then $v_p(\frac{m_i}{q_{i,j_i}} u_{i,j_i}\delta_i) \geq v_p(m_i) = \max_k \left(v_p\left(b_i^k\right)\right)$. So, each of this bricks are multiple of one of the bricks of E and we have find a tiling of B. □

Using a trivial bound of the Frobenius' problem, we can remark that $k \leq \max(m_i^2)$.

We write $[E]_{\mathbb{A}}$ the set of all bricks which are $\mathbb{A}$-tilable by E and $[E]$ the set of all bricks which are tilable (in classical sense) by E.

Lemma 3.5 *Let $\mathbb{A}$ be a nœtherian ring and E a set of $\mathbb{A}$-weighted polycubes, then there exists a finite subset F of E such that $[E]_{\mathbb{A}} = [F]_{\mathbb{A}}$.*

Proof. $P \in [E]_{\mathbb{A}}$ if and only if $Q_P \in \langle Q_{P'} \text{ with } P' \in E, X_1Y_1 - 1, ..., X_dY_d - 1\rangle_{\mathbb{A}}$, but as $\mathbb{A}[X_1, ..., X_d, Y_1, ..., Y_d]$ is a nœtherian ring, there exists a finite subset F of E such that $\{Q_{P'} \text{ with } P' \in F, X_1Y_1 - 1, ..., X_dY_d - 1\}$ is a generating family of $\langle Q_{P'} \text{ with } P' \in E, X_1Y_1 - 1, ..., X_dY_d - 1\rangle_{\mathbb{A}}$. Otherwise, for each integer n and $P_1, ..., P_n$ polycubes of E, there exists a polycube P_{n+1} in E such that

$$Q_{P_{n+1}} \notin \langle Q_{P_1}, ..., Q_{P_n}, X_1Y_1 - 1, ..., X_dY_d - 1\rangle_{\mathbb{A}}$$

and so, we can obtain a non stationary increasing sequence of ideals. □

Let E,E' and C be some families of polycubes, E and E' are *two paddles* (resp. $\mathbb{A}$-*paddles* where $\mathbb{A}$ is a unitary ring) *equivalent on* C if and only if $\forall P \in C$, the following statements are logically equivalent :

- P is tilable (resp. $\mathbb{A}$-tilable) by E,

- P is tilable (resp. $\mathbb{A}$-tilable) by E'.

We can now prove the main theorem for the $\mathbb{A}$-tiling. More explicitly :

Corollary 3.6 *Let $\mathbb{A}$ be a nœtherian ring and E a set of $\mathbb{A}$-weighted polycubes, then there exists a finite set F of bricks such that E et F are equivalent $\mathbb{A}$-paddles for the bricks.*

Proof. We have clearly $[[E]_{\mathbb{A}}]_{\mathbb{A}} = [E]_{\mathbb{A}}$ and with the lemma 3.5 applied to $[E]_{\mathbb{A}}$, there exists F such that $[F]_{\mathbb{A}} = [[E]_{\mathbb{A}}]_{\mathbb{A}} = [E]_{\mathbb{A}}$. □

Lemma 3.7 *Let E be a set of bricks, then there exists a finite subset F of E such that [E] = [F].*

Proof. By the lemma 3.5 there exists a finite subset E_1 of E such that $[E]_{\mathbb{C}} = [E_1]_{\mathbb{C}}$.

Moreover, by the theorem 3.4, there exists k such that $[E_1]_{\mathbb{C}} \cap \mathbb{B}_{>k} = [E_1] \cap \mathbb{B}_{>k}$. So, we have :

$$[E_1]_{\mathbb{C}} \cap \mathbb{B}_{>k} = [E_1] \cap \mathbb{B}_{>k} \subset [E] \cap \mathbb{B}_{>k} \subset [E]_{\mathbb{C}} \cap \mathbb{B}_{>k} = [E_1]_{\mathbb{C}} \cap \mathbb{B}_{>k}$$

This involves that $[E_1] \cap \mathbb{B}_{>k} = [E] \cap \mathbb{B}_{>k}$

The elements of $[E]$ which are not in $[E_1]$ are bricks such that the length of at least one of their edges is less than k.

For $1 \leq q \leq k$ and $1 \leq r \leq d$, we put $B_{q,r}$ the set of bricks of $[E]$ such that the length of the r-th edge is q.

We conclude by showing that there exists a finite family of polycubes E_2 such that $[E] = [E_1 \cup E_2]$. We are going to proceed by recurrence on d(the dimension of the space).

If $d = 1$, for each q, the set $B_{q,1}$ is constituted by at most one brick and so $\left[E_1 \cup \bigcup_{1 \leq q \leq k} B_{q,1} \right] = [E]$.

Suppose the lemma verified for rank $d-1$. We put :

$$C_{q,r} = \{a_1 \times ... \times a_{r-1} \times a_{r+1} \times ... \times a_d \text{ such that } \\ a_1 \times ... \times a_{r-1} \times q \times a_{r+1} \times ... \times a_d \in B_{q,r}\}$$

So $C_{q,r}$ is a set of bricks of dimension $d-1$ obtained by canonical projection of $B_{q,r}$. By recurrence hypothesis, there exists $D_{q,r}$ a finite set of bricks such that [$D_{q,r}$] = [$C_{q,r}$].

Let

$$E_{q,r} = \{a_1 \times ... \times a_{r-1} \times q \times a_{r+1} \times ... \times a_d \text{ such that } \\ a_1 \times ... \times a_{r-1} \times a_{r+1} \times ... \times a_d \in D_{q,r}\},$$

then $E_{q,r}$ is finite and we clearly have [$E_{q,r}$] = [$B_{q,r}$]. So $E_1 \cup \bigcup_{\substack{1 \leq q \leq k \\ 1 \leq r \leq d}} E_{q,r}$ is finite and $[E_1 \cup \bigcup_{\substack{1 \leq q \leq k \\ 1 \leq r \leq d}} E_{q,r}] = [E]$.

□

Remark 3.8 *The lemma 3.4 works on the same principle as the lemma of Dickson which allows us to obtain the nœtheriality of*

$$\mathbb{C}[X_1, ..., X_n].$$

Theorem 3.9 *Let E be a set of polycubes, then there exists a finite set F of bricks, such that E and F are equivalent paddles on the bricks.*

Proof. We have clearly $[[E]] = [E]$ and $[E]$ is a set of bricks, so by the lemma 3.4, there exists a finite subset F of $[E]$ such that $[F] = [[E]] = [E]$. □

We can observe, in corollary, that for every set E of polycubes, if we make the hypothesis that we know a finite family F of bricks which is an equivalent paddle, we can determine rapidly if a given brick B is tilable by E. Indeed, there are two possibilities :

- Brick B has all its edges of length greater than k (k is supposed determined) and then it suffices to verify that B is $\mathbb{C}$-tilable by E (we can do this in polynomial time using generalized division algorithm [3])
- B is an element of the finite set F.

4 Rectangles which can be Tiled by Small Polyominoes

We propose in this section a survey of the results known on the tilable rectangles by congruent copies of a given edge-connected polyomino. (We cover only the cases where the number of squares of the polyomino is less than 7. The unmentioned polyominoes with less than 7 squares cannot tile any rectangles). Some of this results deserve a citation, the other one are unpublished results and they can be easily verified. We can observe that for some polyominoes the number of rectangles is huge, they are no explanation for this fact. For each polyomino, we present only an "half-base" of rectangles. To obtain a generating base, we have to symmetrize this half-base by adding $b \times a$ if $a \times b$ is in the half-base.

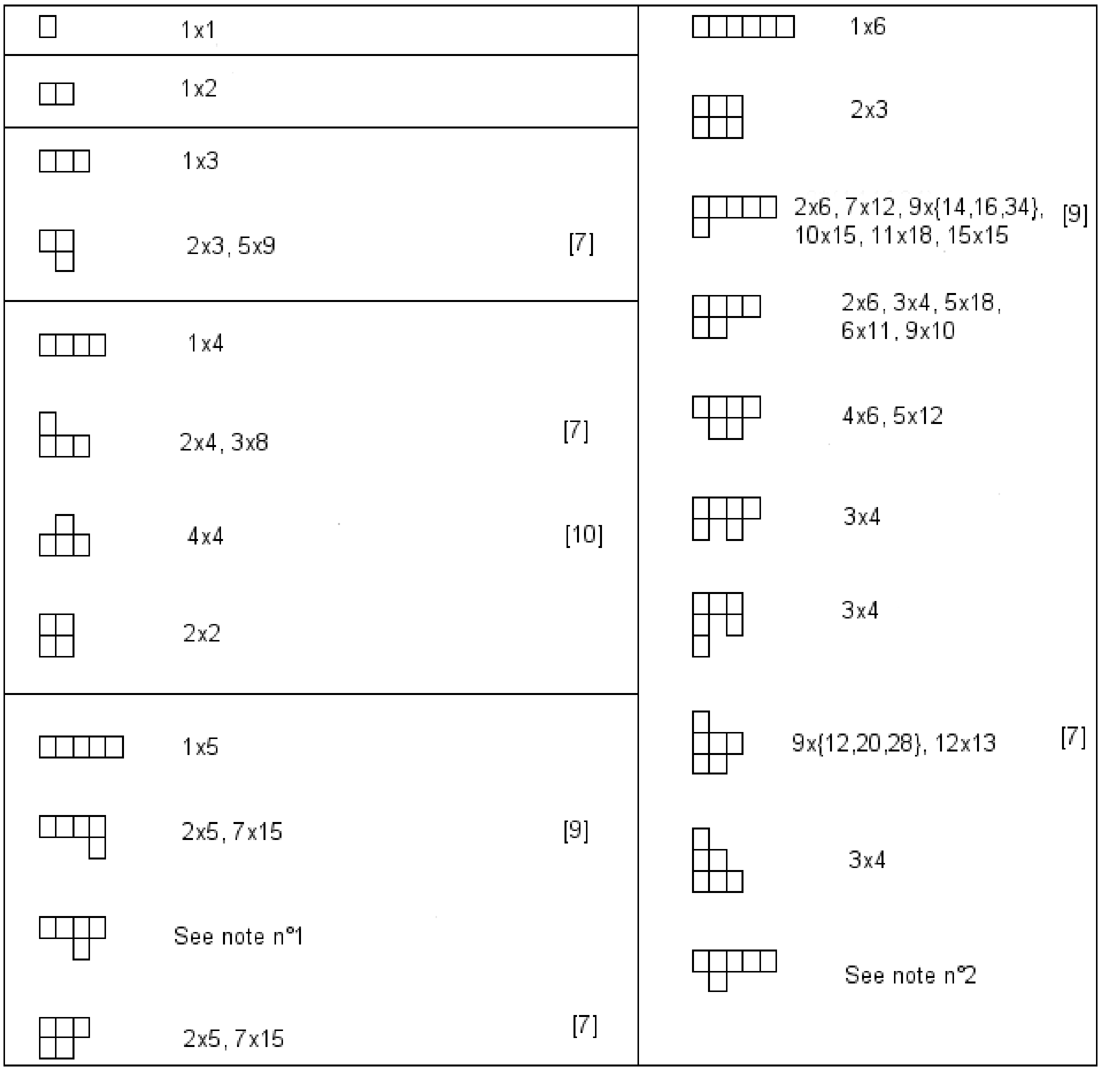

Fig. 1: Rectangles which can be Tiled by Small Polyominoes
Rectangles which can be Tiled by Small Polyominoes.

Note n ° 1 :
5×10
$9 \times \{20, 30, 45, 55\}$
$10 \times \{14, 16, 23, 27\}$
$11 \times \{20, 30, 35, 45\}$
$12 \times \{50, 55, 60, 65, 70, 75, 80, 85, 90, 95\}$
$13 \times \{20, 30, 35, 45\}$
14×15
$15 \times \{15, 16, 17, 19, 21, 22, 23\}$
$17 \times \{20, 25\}$
$18 \times \{25, 35\}, 22 \times 25$

Note n ° 2 :
23×24
$24 \times \{29, 35, 41, 47, 53, 59, 63, 65, 71, 77, 83, 89, 95, 101, 102, 103, 107, 108, 113, 114, 119, 120 \ldots\}$
This polyomino is the *hexomino Y*. It is the smallest case which is still open.

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