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Performance Evaluation of Demodulation Methods: a Combinatorial Approach

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This paper provides a combinatorial approach for analyzing the performance of demodulation methods used in GSM. We also show how to obtain combinatorially a nice specialization of an important performance evaluation formula, using its connection with a classical bijection of Knuth between pairs of Young tableaux and $\{0,1\}$-matrices.

Keywords: Young Tableaux, Bijective Combinatorics, Algebraic Combinatorics, Signal Modulation, Signal Processing, Mobile Communications

1 Introduction

Modulation, i.e. transforming a numeric signal into a wave form, is a technique of main interest in a number of ingeneering domains (computer networks, mobile communications, satellite transmissions, ...) as well as the important subject of studies in signal processing (cf Chapter 5 of [13]). One of the most important problems in this area is to be able to evaluate the performance characteristics of the optimum receivers associated with a given modulation method, which reduce to the computation of various probabilities of errors (see again [13]).

The demodulation decision of an important class of modulation protocols, where both the signal itself and the modulation reference (i.e. a fixed digital sequence) are modulated and transmitted, needs to take into account several noisy informations (the transmitted signal, the transmitted reference, but also copies of these two signals). It appears that the probability of errors appearing in such contexts involve very often to compute the following type of probability:

$$P(U < V) = P\left(U = \sum_{i=1}^{N} |u_i|^2 \ < \ V = \sum_{i=1}^{N} |v_i|^2 \right) , \tag{1}$$

where the u_i and v_i's stand for independent centered complex Gaussian random variables with variances denoted $E[\,|u_i|^2\,] = \chi_i$ and $E[\,|v_i|^2\,] = \delta_i$ for every $i \in [1,N]$ (see also Section 3.1).

The problem of computing explicitely this probability was studied by several researchers from signal processing (cf [2, 9, 13, 14]). The most interesting result in this direction was obtained by Barett (cf [2]) who proved that the probability defined by (1) is equal to

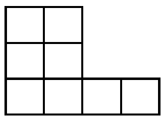
$$P(U < V) = \sum_{k=1}^{N} \left(\prod_{j \neq k} \frac{1}{1 - \delta_k^{-1}\delta_j} \prod_{j=1}^{N} \frac{1}{1 + \delta_k^{-1}\chi_j} \right) . \tag{2}$$

This last formula allows in fact a purely combinatorial description in terms of Young tableaux (cf Section 3.2), which provides the first algorithmically efficient and numerically stable practical method for computing the probability $P(U < V)$ (cf [5, 6]). We continue here the combinatorial study of Barett's formula by connecting it with a very classical bijection of Knuth (cf [7, 10]) between pairs of Young tableaux of conjugated shapes and $\{0,1\}$-matrices. These considerations allowed us in particular to study combinatorially an important specialization of formula (2) (cf Section 5). Note finally that no proofs will be given here. The complete version of this paper will be published elsewhere.

2 Background

2.1 Partitions and Young tableaux

A *partition* is a finite nondecreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of positive integers. The number m of elements of λ is called the *length* of the partition λ. One can represent each such partition λ by a *Ferrers diagram* of *shape* λ, that is to say by a diagram of $\lambda_1 + \ldots + \lambda_m$ boxes whose i-th row contains exactly λ_i boxes for every $1 \leq i \leq n$. The Ferrers diagram associated to $\lambda = (2,2,4)$ is for instance given below.



The *conjugated* partition $\tilde{\lambda}$ of a given partition λ is then just the partition obtained by reading the heights of the columns of the Ferrers diagram associated with λ. One has here for instance $\tilde{\lambda} = (1,1,3,3)$ when $\lambda = (2,2,4)$ as it can be seen on the previous picture.

When λ is a partition whose Ferrers diagram is contained into the square (N^N) with N rows of length N, one can also define the *complementary* partition $\overline{\lambda}$ of λ which is the conjugate of the partition ν whose Ferrers diagram is the complement (read from bottom to top) of the Ferrers diagram of λ in the square (N^N). For instance, for $N = 6$ and $\lambda = (1,1,2,3)$, we have $\nu = (3,4,5,5,6,6)$ and $\overline{\lambda} = (2,4,5,6,6,6)$ as it can be checked on Figure 1.

Let A be a totally ordered alphabet. A *tabloid* of shape λ over A is then a filling of the boxes of a Ferrers diagram of shape λ with letters of A. A tabloid is called a *Young tableau* when its rows and its columns consist respectively of increasing and strictly increasing sequences of letters of A. One can see for instance below a Young tableau of shape $(2,2,4)$ over $A = \{a_1 < \ldots < a_5\}$.

a_3	a_5		
a_2	a_2		
a_1	a_1	a_1	a_4

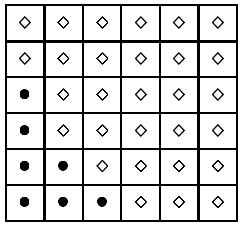


Fig. 1: Two complementary Ferrers diagrams.

2.2 Knuth's bijection

Knuth's bijection is a famous one-to-one correspondance between $\{0,1\}$-matrices and pairs of Young tableaux of conjugated shapes (cf [10]). It is based on the *column insertion* process which is a classical combinatorial construction that we will first present. Let therefore A be a totally ordered alphabet. The fundamental step of the column insertion process associates with a letter $a \in A$ and a Young tableau T over A a new Young tableau $T(a)$ over A defined as follows.

1. If a is strictly larger than all the entries of the first column of T, the tableau $T(a)$ is obtained by putting a in a new box at the top of the first column of T.
2. If it is not the case, one can consider the smallest entry b of the first column of T which is greater than or equal to a. The tableau $T(a)$ is then obtained by replacing b by a and by applying recursively our insertion scheme, starting now by trying to insert b in the second column of T. Our process continues until a replaced entry can go at the top of the next column or until it becomes the only entry of a new column.

One can easily check that $T(a)$ is always a Young tableau. Moreover our process can be reverted if one knows which new box it created. Let now $w = a_1 \dots a_N$ be a word over A. The result of the column insertion process applied to w is then the Young tableau obtained by column inserting successively $a_1, \dots, a_N$ as described above, starting from the empty Young tableau.

Let now M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\})$. Knuth's bijection associates then to M a pair (P,Q) of Young tableaux with conjugated shapes over the alphabet $[1,N]$ which is constructed as described below.

1. Construct first the 2-row array A_N which is equal to the sequence of the N^2 pairs (i,j) of $[1,N] \times [1,N]$ taken in the lexicographic order, i.e.

$$A_N = \left(\begin{array}{cccccccccc} 1 & \dots & 1 & 2 & \dots & 2 & \dots\dots & N & \dots & N \\ 1 & \dots & N & 1 & \dots & N & \dots\dots & 1 & \dots & N \end{array} \right) .$$

2. Select in this array all the entries corresponding to the 1's of M in order to get an array

$$\mathcal{A}(M) = \left(\begin{array}{cccc} u_1 & u_2 & \dots\dots & u_r \\ v_1 & v_2 & \dots\dots & v_r \end{array} \right) .$$

3. Form the word $w_1(M) = v_1 \dots v_r$ obtained by reading from left to right the second entries of $\mathcal{A}(M)$. The column insertion process applied to $w_1(M)$ gives the Young tableau P.

4. Form finally the second Young tableau Q by placing for every $i \in [1,r]$ the i-th element u_i of the first row of $\mathcal{A}(M)$ in the box which is conjugated to the i-th box created during the column insertion process that lead to P.

Since the Young tableau Q encodes the order in which P is constructed in a column insertion process, one can clearly reconstruct the array $\mathcal{A}(M)$ (and hence M) from the pair (P,Q).

Example 2.1 *Let us consider the matrix*

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} .$$

Then the arrays $\mathcal{A}_3$ and $\mathcal{A}(M)$ are respectively equal to

$$\mathcal{A}_3 = \begin{pmatrix} 1 & 1 & \boxed{1} & \boxed{2} & 2 & 2 & 3 & \boxed{3} & \boxed{3} \\ 1 & 2 & \boxed{3} & \boxed{1} & 2 & 3 & 1 & \boxed{2} & \boxed{3} \end{pmatrix} \quad \textit{and} \quad \mathcal{A}(M) = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 3 & 1 & 2 & 3 \end{pmatrix}$$

where we boxed in $\mathcal{A}_3$ the entries corresponding to the 1's of M. Thus $w_1(M) = (3,1,2,3)$. Knuth's bijection associates then with M the following pair (P,Q) of conjugated Young tableaux:

$$(P,Q) = \left(\begin{array}{|c|c} \hline 3 & \\ \hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \; , \; \begin{array}{|c|c|c|} \hline 2 & \multicolumn{2}{c}{} \\ \hline 1 & 3 & 3 \\ \hline \end{array} \right) .$$

3 Performance analysis of demodulation protocols

3.1 Demodulation with diversity

Our initial motivation for studying Barett's formula came from mobile communications. The probability $P(U < V)$ given by formula (1) appears indeed naturally in the performance analysis of demodulation methods based on diversity which are standard in such a context. In order to motivate more strongly our paper, we first present in details this last situation.

We consider a model where one transmits a data $b \in \{-1,+1\}$ on a noisy channel. A reference r (corresponding always to the data $b = 1$) is also sent on the noisy channel at the same time than b. We assume that we receive N pairs $(x_i(b), r_i)_{1 \leq i \leq N} \in (\mathbb{C} \times \mathbb{C})^N$ of datas (the $x_i(b)$'s) and references (the r_i's) † that have the following form

$$\begin{cases} x_i(b) &= a_i\, b + \nu_i & \text{for every } 1 \leq i \leq N, \\ r_i &= a_i \sqrt{\beta_i} + \nu'_i & \text{for every } 1 \leq i \leq N, \end{cases}$$

† This situation corresponds to spatial diversity, i.e. when more than one antenna is available, but also to multipath reflexion contexts. These two types of situations typically occur in mobile communications.

where $a_i \in \mathbb{C}$ is a complex number that models the channel fading associated to $x_i(b)$ ‡, where $\beta_i \in \mathbb{R}^+$ is a positive real number that represents the excess of signal to noise ratio (SNR) which is available for the reference r_i and where $\nu_i \in \mathbb{C}$ and $\nu'_i \in \mathbb{C}$ denote finally two independant complex white Gaussian noises. We also assume that every a_i is a complex random variable distributed according to a Gaussian density of variance α_i for every $i \in [1,N]$.

According to these assumptions, all observables of our model, i.e. the pairs $(x_i(b), r_i)_{1 \leq i \leq N}$, are complex Gaussian random variables. We finally also assume that these N observables are N independant random variables of $\mathbb{C}^2$. Under these hypotheses, one can then prove ([4]) that

$$\log\left(\frac{P(b=+1|X)}{P(b=-1|X)}\right) = \sum_{i=1}^{N} \frac{4\,\alpha_i\sqrt{\beta_i}}{1+\alpha_i\,(\beta_i+1)}\,(x_i(b)|r_i) \tag{3}$$

with $X = (x_i(b), r_i)_{1 \leq i \leq N}$ and where $(\star|\star)$ denotes the Hermitian scalar product. The demodulation decision is based on the associated Bayesian criterium. One indeed decides that b was equal to 1 (resp. to -1) when the right hand side of Formula (3) is positive (resp. negative).

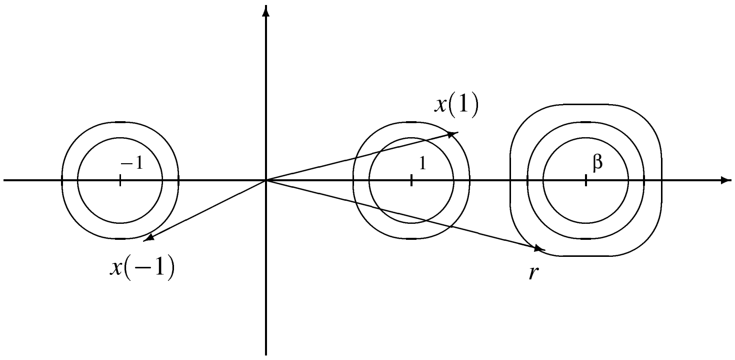


Fig. 2: Two possible noisy bits $x(1)$ and $x(-1)$ and a noisy reference r in the case $N = 1$.

Intuitively this means that one decides that the data $b = 1$ was sent when the $x_i(b)$'s are more or less globally in the same direction than the r_i's. Figure 2 illustrates the case $N = 1$ and one can see that a noisy reference r has a positive (resp. negative) Hermitian scalar product with a noisy data x when x corresponds to a small pertubation of 1 (resp. -1).

The bit error probability (BER) of our model is the probability that the data $b = 1$ was decoded in -1, i.e. the probability that one had

$$\sum_{i=1}^{N} \frac{4\,\alpha_i\sqrt{\beta_i}}{1+\alpha_i\,(\beta_i+1)}\,(x_i(1)|r_i) \;< 0\,.$$

‡ Fading is typically the result of the absorbtion of the signal by buildings. Its complex nature comes from the fact that it models both an attenuation (its modulus) and a dephasing (its argument).

Using the parallelogram identity, it is now easy to rewrite this last probability as

$$P\Big(\sum_{i=1}^{N} |u_i|^2 - \sum_{j=1}^{N} |v_i|^2 < 0 \Big)$$

where u_i and v_i denote for every $i \in [1,N]$ the two variables defined by setting

$$u_i = \left(\frac{\alpha_i \sqrt{\beta_i}}{1+\alpha_i\,(\beta_i+1)} \right)^{1/2} (x_i(1)+r_i) \quad \text{and} \quad v_i = \left(\frac{\alpha_i \sqrt{\beta_i}}{1+\alpha_i\,(\beta_i+1)} \right)^{1/2} (x_i(1)-r_i)\ .$$

Our different hypotheses imply then immediately that the u_i's and the v_i's are independant complex Gaussian random variables. Hence the performance analysis of our model relies exactly on Barett's formula (2) as depicted in the introduction of our paper (cf formula (1)).

3.2 The combinatorial version of Barett's formula

Using Barett's formula, one can prove that the probability $P(U < V)$ defined by (1) is equal to

$$P(U<V) = \frac{F(\chi,\delta)}{\prod_{1\le i,j\le N} (\chi_i + \delta_j)} \tag{4}$$

where $F(\chi,\delta)$ denotes the polynomial which is the sum of all the monomials obtained by taking the product of the elements of all square tableaux of shape (N^N) consisting in two Young tableaux of complementary shapes (cf Figure 1 of Section 2.1) that respect the two following constraints:

- **Condition B1**: the first Young tableau is only filled by variables that belong to the ordered alphabet $\delta = \{\,\delta_1 < \ldots < \delta_N\,\}$ and *the length of its first row is equal to N*,
- **Condition B2**: the second Young tableau is only filled by variables that belong to the ordered alphabet $\chi = \{\,\chi_1 < \ldots < \chi_N\,\}$.

A typical example of such a combinatorial structure is given in Figure 3. The first tableau is written here in the usual way. On the other hand, the second tableau is organized differently: its rows (resp. its columns) are placed from top to bottom (resp. from right to left) in the space corresponding to the complement of the first tableau within the square (N^N).

Example 3.1 *For $N = 2$, Barett's formula reduces to*

$$P(U<V) = \frac{\chi_1\chi_2\,(\delta_1^2+\delta_1\,\delta_2+\delta_2^2) + (\chi_1+\chi_2)\,(\delta_1^2\,\delta_2+\delta_1\,\delta_2^2) + \delta_1^2\,\delta_2^2}{(\chi_1+\delta_1)\,(\chi_1+\delta_2)\,(\chi_2+\delta_1)\,(\chi_2+\delta_2)}$$

and one can check that the eight monomials occuring in the numerator of this last expression are exactly the products of the entries of the following eight combinatorial structures:

$$\begin{array}{|c|c|}\hline \chi_1 & \chi_2 \\ \hline \delta_1 & \delta_1 \\ \hline \end{array}\,, \begin{array}{|c|c|}\hline \chi_1 & \chi_2 \\ \hline \delta_1 & \delta_2 \\ \hline \end{array}\,, \begin{array}{|c|c|}\hline \chi_1 & \chi_2 \\ \hline \delta_2 & \delta_2 \\ \hline \end{array}\,, \begin{array}{|c|c|}\hline \delta_2 & \chi_1 \\ \hline \delta_1 & \delta_1 \\ \hline \end{array}\,, \begin{array}{|c|c|}\hline \delta_2 & \chi_2 \\ \hline \delta_1 & \delta_1 \\ \hline \end{array}\,, \begin{array}{|c|c|}\hline \delta_2 & \chi_1 \\ \hline \delta_1 & \delta_2 \\ \hline \end{array}\,, \begin{array}{|c|c|}\hline \delta_2 & \chi_2 \\ \hline \delta_1 & \delta_2 \\ \hline \end{array}\,, \begin{array}{|c|c|}\hline \delta_2 & \delta_2 \\ \hline \delta_1 & \delta_1 \\ \hline \end{array}\,.$$

χ_6	χ_5	χ_4	χ_3	χ_2	χ_1
δ_5	χ_6	χ_5	χ_4	χ_2	χ_1
δ_4	δ_5	δ_6	χ_4	χ_3	χ_2
δ_3	δ_3	δ_5	χ_5	χ_4	χ_3
δ_2	δ_2	δ_3	δ_4	χ_4	χ_3
δ_1	δ_1	δ_2	δ_2	δ_2	δ_3

Fig. 3: A typical example of complementary fillings of a square tableau.

The algorithmic complexity of formula (4) is $O(N^2\,\alpha_N)$ where α_N denotes the number of monomials involved in its numerator or equivalently the number of square tableaux of shape (N^N) filled as in the typical example of Figure 3. Unfortunately one can prove analytically that $\alpha_N = 2^{N^2-1}$ from which it follows that formula (4) is impracticable when N grows. This combinatorial formula is however not useless since it allows to study the specializations of Barett's formula (cf Section 5).

Proposition 3.2 *The number α_N of square tableaux of shape (N^N) filled by two complementary Young tableaux satisfying to conditions* **B1** *and* **B2** *is equal to* $\alpha_N = 2^{N^2-1}$.

4 A bijective proof of Proposition 3.2

This new section is devoted to the construction of a bijective proof that contributes to explain more deeply Proposition 3.2. This bijection will also help us for studying an important specialization of Barett's formula (see Section 5).

4.1 A more general combinatorial structure

Let us first introduce a natural generalization of the combinatorial structures that appeared in Section 3.2, that is to say the set $\mathcal{T}_N$ of all square tableaux of shape (N^N) divided as in this last section into two complementary Young tableaux (but here without any constraint on them) respectively filled by elements of the alphabets δ and χ. The two Young tableaux that form an element of $\mathcal{T}_N$ will again be organized as already depicted in Section 3.2.

As we will see in the sequel, it is in fact possible to construct a bijection between $\mathcal{T}_N$ and the set $\mathcal{M}_{N\times N}(\{0,1\})$ of all square $\{0,1\}$-matrices of size N, which implies that the cardinality of $\mathcal{T}_N$ is equal to 2^{N^2}. It follows then from this last result that $\alpha_N = 2^{N^2-1}$ due to the fact that the number of elements of $\mathcal{T}_N$ whose first tableau has a first row of length N is obviously (use a symmetry with respect to the main diagonal of the square (N^N) and exchange the role of the alphabets χ and δ in order to pass from one case to the other) equal to the number of elements of $\mathcal{T}_N$ whose second tableau has a first row of length N (which means equivalently that the first tableau has a first row of length strictly less than N).

4.2 Description of the bijection

We now present our bijection between $\mathcal{M}_{N\times N}(\{0,1\})$ and $\mathcal{T}_N$. Our construction is based on a slight variation of the well known Knuth's bijection (cf Section 2.2) that has an interesting symmetry property which is fundamental for highlighting Barett's formula in a new way.

Let M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\})$. We apply first Knuth's bijection (as described in Section 2.2) to M in order to get a pair (P,Q) of Young tableaux of conjugated shapes λ_1 and $\lambda_1\tilde{}$. We then associate with Q a new Young tableau $\overline{Q}$ of shape $\overline{\lambda_1}$ (the complementary partition of λ_1 within the square (N^N)) which is defined as follows.

- We denote first by m the length of λ_1. We then decide (by abuse of language) that Q also has columns indexed by integers strictly greater than m which are all empty.
- We can now define a unique tabloid $\overline{Q}$ of shape $\overline{\lambda_1}$ by asking that the i-th column of $\overline{Q}$ consists exactly for every $i\in[1,N]$ of all the letters of the alphabet $\{1,\ldots,N\}$, sorted in increasing order from bottom to top, that do not appear in the $(N-i+1)$-th column of Q.

One can in fact prove that $\overline{Q}$ is a Young tableau. Hence $\Psi(M)=(P,\overline{Q})$ is a pair of complementary Young tableaux within the square (N^N). To obtain from it an element of $\mathcal{T}_N$, it suffices to associate with each entry i of P (resp. $\overline{Q}$) the letter δ_i (resp. χ_i) of the alphabet δ (resp. χ). We denote then by $\Phi(M)$ the element of $\mathcal{T}_N$ that corresponds in such a way to the initial matrix M. Since the mapping $Q\to\overline{Q}$ is one to one, Ψ is clearly a bijection between $\mathcal{M}_{N\times N}(\{0,1\})$ and pairs of Young tableaux of complementary shapes over the alphabet $[1,N]$ when Φ is a bijection between $\mathcal{M}_{N\times N}(\{0,1\})$ and $\mathcal{T}_N$.

Example 4.1 *Let us continue Example 2.1. Knuth's bijection applied to the matrix M introduced in this example gives here a pair of tableaux (P,Q) of conjugated shapes $\lambda_1=(1,1,2)$ and $\lambda_1\tilde{}=(1,3)$. The shape $\overline{\lambda_1}=(2,3)$, complementary to the shape λ_1 within the square (3^3), provides then the shape of the tableau $\overline{Q}$. Filling in its entries by taking (in the reverse order) the complements in $\{1,2,3\}$ of the entries of the columns of Q, we obtain the tableau*

$$\overline{Q}=\begin{array}{|c|c|c}\cline{1-2} 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\quad .$$

The element $\Phi(M)$ of $\mathcal{T}_3$ associated with M is then the following rewriting of the pair $(P,\overline{Q})$:

$$\Phi(M)\quad=\quad\begin{array}{|c|c|c|}\hline \delta_3 & \chi_2 & \chi_1 \\ \hline \delta_2 & \chi_2 & \chi_1 \\ \hline \delta_1 & \delta_3 & \chi_3 \\ \hline\end{array}\quad .$$

4.3 Symmetry properties of the bijection

This section is devoted to the presentation of a strong symmetry property of the bijection Φ. To this purpose, we give first a new method, described below, for constructing the second Young tableau $\overline{Q}$ associated by Φ with a given $\{0,1\}$-matrix M.

1. Construct the 2-row array B_N which is equal to the sequence of the N^2 pairs (i,j) of $[1,N]\times[1,N]$ taken in the lexicographic order with respect to the second entry, i.e.

$$B_N=\left(\begin{array}{cccccccccc} 1 & \ldots & N & 1 & \ldots & N & \ldots\ldots & 1 & \ldots & N \\ 1 & \ldots & 1 & 2 & \ldots & 2 & \ldots\ldots & N & \ldots & N \end{array}\right).$$

2. Select in this array all the entries corresponding to the 0's of M. We obtain then a word $w_2(M)$ by reading the first component of the selected entries. The result of the column insertion process applied to $w_2(M)$ is a Young tableau Q'.

It appears that the Young tableau Q' obtained in this way is exactly the second Young tableau $\overline{Q}$ constructed by the bijection Ψ, presented in Section 4.2, when applied to the matrix M.

Proposition 4.2 *Let M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\})$, let $\overline{Q}$ be the second Young tableau constructed by the bijection Φ applied to M and let Q' be the Young tableau constructed as above. Then one has $Q' = \overline{Q}$.*

Example 4.3 *This example continues Example 2.1 and Example 4.1. In this case, we have:*

$$B_3 = \begin{pmatrix} \boxed{1} & 2 & \boxed{3} & \boxed{1} & \boxed{2} & 3 & 1 & \boxed{2} & 3 \\ \boxed{1} & 1 & \boxed{1} & \boxed{2} & \boxed{2} & 2 & 3 & \boxed{3} & 3 \end{pmatrix}$$

where we boxed the entries that correspond to the 0*'s of the associated matrix M. Hence* $w_2(M) = (1,3,1,2,2)$. *The column insertion process applied to $w_2(M)$ gives then the Young tableau:*

$$Q' = \begin{array}{|c|c|c} \hline 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} = \overline{Q}\,.$$

5 Some specializations of Barett's formula

This section is now devoted to the obtention, through the bijection constructed in Section 4, of explicit expressions for several specializations of Barett's formula.

5.1 Matrices involved in the combinatorial version of Barett's formula

Let us denote by $\mathcal{N}_N$ the set of all square matrices M of $\mathcal{M}_{N\times N}(\{0,1\})$ such that the length of the first row of the first Young tableau P associated with M by the bijection Ψ of Section 4.2 is exactly equal to N. Let also $\mu(t)$ stand for the monomial obtained by taking the product of all entries of an arbitrary element t of $\mathcal{T}_N$. According to the results of Section 3.2, the polynomial $F(\chi,\delta)$ which is the denominator of the combinatorial expression (4) of the probability of error (1) can now be expressed as

$$F(\chi,\delta) = \sum_{M\in\mathcal{N}_N} \mu(\Phi(M)) \tag{5}$$

where Φ stands for the second bijection constructed in Section 4.2.

In order to understand better the combinatorial version of Barrett's formula, we will therefore explore the fine structure of $\mathcal{N}_N$. Let again M be a matrix of $\mathcal{M}_{N\times N}(\{0,1\}$. Observe then that the length of the first row of the Young tableau P associated by Ψ with M is exactly the length of the longest decreasing subsequence in $w_1(M)$ according to Greene's theorem (cf [8] or Chapter 3 of [7]) and to the construction of P (cf Section 2.2). Since a decreasing subsequence in $w_1(M)$ corresponds to a strictly increasing subsequence, for the North-East order [§], in the set of the entries of M associated with 1's, we now get the following characterization of $\mathcal{N}_N$.

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§ We define the North-East order $\prec_{NE}$ over $[1,N]\times[1,N]$ by setting $(i,j) \prec_{NE} (k,l)$ iff $i > k$ and $j \geq l$.

Proposition 5.1 *A matrix $M \in \mathcal{M}_{N\times N}(\{0,1\}$ belongs to $\mathcal{N}_N$ if and only if there exists a sequence of 1's of length N in M such that the corresponding entries form a strictly increasing sequence (of length N) for the North-East order.*

Example 5.2 *Let us consider the matrix M' defined by*

$$M' = \left(\begin{array}{ccc} 1 & 0 & \boxed{1} \\ 1 & \boxed{1} & 0 \\ 0 & \boxed{1} & 0 \end{array} \right) .$$

The entries associated with the three 1's of M' boxed on the above picture correspond then to the increasing sequence $(3,2) \prec_{NE} (2,2) \prec_{NE} (1,3)$ for the North-East order. According to Proposition 5.1, M' belongs therefore to $\mathcal{N}_3$, which just means that the length of the first row of the first tableau associated by Ψ to M' is equal to 3 as it can be directly checked.

Let now M be a matrix of $\mathcal{N}_N$. According to Proposition 5.1 and to the definition of the North-East order, there exists a sequence σ of length N of 1's in M such that the corresponding sequence of entries has the form $\sigma' = ((N-k+1, j_k))_{1\leq k\leq N}$ where $(j_k)_{1\leq k\leq N}$ stands for an increasing sequence of integers of $[1,N]$. One can obviously encode such a sequence of 1's by the pseudo-composition [¶] $p(\sigma) = (p_k)_{1\leq k\leq N}$ of N defined by asking p_k to be the number (possibly equal to zero) of 1's of σ that belong to the k-th column of M [‖]. We will now denote by $p(M)$ the greatest (for the lexicographic order on $\mathbb{N}^N$) pseudo-composition that can be associated in such a way with M. The set $\mathcal{N}_N$ can then be partitioned as

$$\mathcal{N}_N = \bigcup_{p\in\mathcal{P}_N} \mathcal{N}_{p,N} \tag{6}$$

where $\mathcal{P}_N$ denotes the set of all pseudo-compositions of length N of N and where $\mathcal{N}_{p,N}$ stands for the set of all matrices $M \in \mathcal{N}_N$ whose associated pseudo-permutation $p(M)$ is equal to p.

Let us now associate with every pseudo-composition $p = (p_1,\ldots,p_N)$ of $\mathcal{P}_N$ the integer $\mu(p)$ defined as the smallest element μ of $[1,N]$ such that $p_1 + \ldots + p_\mu = N$. The following result gives then a fine characterization of the matrices of $\mathcal{N}_{p,N}$.

Proposition 5.3 *Let $p = (p_1,\ldots,p_N)$ be a pseudo-composition of $\mathcal{P}_N$. Let also $(j_k)_{1\leq k\leq N}$ denote the unique increasing sequence of integers defined by asking every $k \in [1,N]$ to be repeated p_k times. A matrix M belongs then to $\mathcal{N}_{p,N}$ iff it satisfies the two following properties:*

- **Condition C1:** *for every $k \in [1,N]$, the entry of order $(N-k+1, j_k)$ of M is 1;*
- **Condition C2:** *for every $k \in [1,\mu(p)-1]$, the entry of order $(N-(p_1+\ldots+p_k),k)$ of M is 0.*

¶ A pseudo-composition of an integer N is a sequence of positive integers (including 0) whose sum is N.

‖ The sequence $(j_k)_{1\leq k\leq N}$ that characterizes σ' (or equivalently σ) as described above, is indeed the unique increasing sequence of N elements of $[1,N]$ obtained by repeating p_k times each integer $k \in [1,N]$.

Example 5.4 *Let us consider the matrix $M \in \mathcal{M}_{3\times 3}(\{0,1\})$ defined by setting*

$$M = \begin{pmatrix} 0 & 0 & \boxed{1} \\ 0 & \boxed{1} & 1 \\ \boxed{1} & 0 & 0 \end{pmatrix} .$$

The sequences σ_1 and σ_2 of 1's of M given by the associated sequences of entries

$$\sigma_1' = ((3,1) \prec_{NE} (2,2) \prec_{NE} (1,3)) \quad \textit{and} \quad \sigma_2' = ((3,1) \prec_{NE} (2,3) \prec_{NE} (1,3))$$

are the unique sequences of length 3 of 1's in M whose corresponding sequences of entries are strictly increasing for the North-East order. Since $p(\sigma_1) = (1,1,1)$ and $p(\sigma_2) = (1,0,2)$, we therefore get $p(M) = (1,1,1)$. One can also check that Proposition 5.3 holds: we boxed (resp. circled) here the entries of M that are constrained by condition **C1** *(resp.* **C2***) as expected.*

5.2 One of the practically useful specializations

Let us consider the situation where χ_i and δ_i are respectively equal to some fixed values χ and δ for every $i \in [1,N]$. According to relation (5), the polynomial $F(\chi,\delta)$ defined in Section (3.2) reduces then to the two variable polynomial

$$F_1(\chi,\delta) = \sum_{i=N}^{N^2} \alpha_i \chi^{N^2-i} \delta^i \tag{7}$$

where α_i denotes the number of matrices of $\mathcal{N}_N$ with i 1's and $N^2 - i$ 0's (the above expression comes from the fact that $\alpha_i = 0$ for every $0 \le i \le N-1$ since every matrix of $\mathcal{N}_N$ has at least N 1's). It now follows from Relation (6) and from Proposition 5.3 that one has

$$\alpha_i = \sum_{\mu=1}^{N} \sum_{\substack{p \in \mathcal{P}_N \\ \mu(p)=\mu}} \binom{N^2 - (N+\mu-1)}{i-N} \tag{8}$$

since having i 1's in a matrix of $\mathcal{N}_{p,N}$ means placing $i-N$ 1's (N 1's are already constrained by condition **C1**) in the $N^2-(N+\mu(p)-1)$ positions not taken both by the N 1's fixed by Condition **C1** and by the $\mu(p)-1$ 0's fixed by Condition **C2**. Note now that the number of pseudo-compositions p of $\mathcal{P}_N$ such that $\mu(p) = \mu$ is just the number of integer solutions of the equation $i_1 + \ldots + i_\mu = N$ with $i_\mu \ge 1$ or equivalently of the equation $i_1' + \ldots + i_\mu' = N-1$ (without any constraint), which is classically known to be equal to the binomial coefficient of order $(N-1, N-2+\mu)$ (cf [3]). It follows then from relation (8) that one has

$$\alpha_i = \sum_{\mu=1}^{N} \binom{N-2+\mu}{N-1} \binom{N^2-N-\mu+1}{i-N} .$$

Replacing this last value in relation (7), it is now easy to deduce the following simple formula

$$P_1(U < V) = \left(\frac{\delta}{\chi}\right)^N \left(\sum_{\mu=0}^{N-1} \binom{N-1+\mu}{N-1} \left(\frac{\chi}{\chi+\delta}\right)^{N+\mu} \right) \tag{9}$$

for the current specialization of the probability of error (1) that we are presently studying. Note that formula (9) was already obtained in [5] by purely analytic methods.

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