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Queue Layouts of Graph Products and Powers[†]

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A *k-queue layout* of a graph G consists of a linear order σ of $V(G)$, and a partition of $E(G)$ into k sets, each of which contains no two edges that are nested in σ. This paper studies queue layouts of graph products and powers.

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1 Introduction

Let G be a graph. (All graphs considered are finite, simple and undirected.) The vertex and edge sets of G are denoted by $V(G)$ and $E(G)$, respectively. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *density* of G is $\eta(G) := |E(G)|/|V(G)|$.

A *vertex ordering* of G is a bijection $\sigma : V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$. In a vertex ordering σ of G, let $L_\sigma(e)$ and $R_\sigma(e)$ denote the endpoints of each edge $e \in E(G)$ such that $\sigma(L_\sigma(e)) < \sigma(R_\sigma(e))$. Where the vertex ordering σ is clear from the context, we will abbreviate $L_\sigma(e)$ and $R_\sigma(e)$ by L_e and R_e, respectively. For edges e and f of G with no endpoint in common, there are the following three possible relations with respect to σ, as illustrated in Figure 1:

(a) e and f *nest* if $\sigma(L_e) < \sigma(L_f) < \sigma(R_f) < \sigma(R_e)$,

(b) e and f *cross* if $\sigma(L_e) < \sigma(L_f) < \sigma(R_e) < \sigma(R_f)$,

(c) e and f are *disjoint* if $\sigma(L_e) < \sigma(R_e) < \sigma(L_f) < \sigma(R_f)$.

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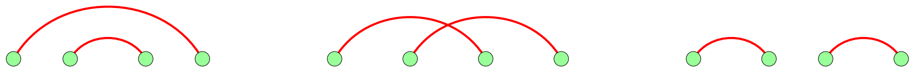


Fig. 1: Relationships between pairs of edges with no common endpoint in a vertex ordering.

A *queue* in σ is a set of edges $Q \subseteq E(G)$ such that no two edges in Q are nested. Observe that when traversing σ from left to right, the left and right endpoints of the edges in a queue are reached in first-in-first-out order—hence the name 'queue'. Observe that $Q \subseteq E(G)$ is a queue if and only if for all edges $e, f \in Q$,

$$\begin{aligned} &\sigma(L_e) \leq \sigma(L_f) \text{ and } \sigma(R_e) \leq \sigma(R_f) \ , \\ \text{or } &\sigma(L_f) \leq \sigma(L_e) \text{ and } \sigma(R_f) \leq \sigma(R_e) \ . \end{aligned} \tag{1}$$

A k-*queue layout* of G is a pair

$$(\sigma, \{Q_1, Q_2, \dots, Q_k\})$$

where σ is a vertex ordering of G, and $\{Q_1, Q_2, \dots, Q_k\}$ is a partition of $E(G)$, such that each Q_i is a queue in σ. The *queue-number* of a graph G, denoted by $\mathsf{qn}(G)$, is the minimum k such that there is a k-queue layout of G.

Queue layouts were introduced by Heath et al. [15, 19]. Applications of queue layouts include sorting permutations [12, 20, 22, 24, 27], parallel process scheduling [3], matrix computations [23], and graph drawing [4, 6]. Other aspects of queue layouts have been studied in the literature [7, 9, 10, 13, 25, 26]. Queue layouts of directed graphs [5, 11, 17, 18] and posets [16] have also been investigated.

Table 1 describes the best known upper bounds on the queue-number of various classes of graphs. Planar graphs are an interesting class of graphs for which it is not known whether the queue-number is bounded (see [6, 23]).

This paper studies queue layouts of graph products and graph powers. To prove optimality we use the following lower bound by Heath and Rosenberg [19]. See Pemmaraju [23] and Dujmović and Wood [9] for slightly more exact lower bounds.

Lemma 1 ([19]) *Every graph G has queue-number $\mathsf{qn}(G) > \eta(G)/2$.*

This paper is organised as follows. In Section 2 we introduce the concepts of strict queue layout and strict queue-number. Many of the upper bounds on the queue-number that are presented in later sections will be expressed as functions of the strict queue-number. In Section 3 we prove bounds on the queue-number of the power of a graph in terms of the queue-number of the underlying graph. In Section 4 we define the graph products that will be studied in later sections. In Section 5 we study the queue-number of the cartesian product of graphs. Finally in Section 6 we study the queue-number of the direct and strong products of graphs.

‡ Dujmović and Wood [8] gave a simple proof of this result.

Tab. 1: Upper bounds on the queue-number.

graph family	queue-number	reference
n vertices	$\lfloor \frac{n}{2} \rfloor$	Heath and Rosenberg [19]
m edges	$\boldsymbol{e}\sqrt{m}$	Dujmović and Wood [9]
tree-width w	$3^w \cdot 6^{(4^w-3w-1)/9} - 1$	Dujmović *et al.* [6]
tree-width w, max. degree Δ	$36\Delta w$	Wood [29]
path-width p	p	Dujmović *et al.* [6]
band-width b	$\lceil \frac{b}{2} \rceil$	Heath and Rosenberg [19]
track-number t	$t-1$	Dujmović *et al.* [6]
2-trees	3	Rengarajan and Veni Madhavan [25]‡
k-ary butterfly	$\lfloor \frac{k}{2} \rfloor$+1	Hasunuma [14]
d-ary de Bruijn	d	Hasunuma [14]
Halin	3	Ganley [13]
X-trees	2	Heath and Rosenberg [19]
outerplanar	2	Heath *et al.* [15]
arched levelled planar	1	Heath *et al.* [15]
trees	1	Heath and Rosenberg [19]

2 Strict Queue Layouts

Let σ be a vertex ordering of a graph G. We say an edge e is *inside* a distinct edge f, and e and f *overlap*, if

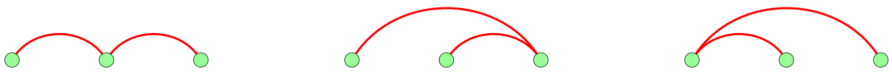
$$\sigma(L_f) \leq \sigma(L_e) < \sigma(R_e) \leq \sigma(R_f) \ .$$

A set of edges $Q \subseteq E(G)$ is a *strict queue* in σ if no edge in Q is inside another edge in Q. Alternatively, Q is a *strict queue* in σ if

$$\begin{aligned} \sigma(L_e) < \sigma(L_f) \ \text{ and } \ \sigma(R_e) < \sigma(R_f) \ , \\ \text{or } \sigma(L_f) < \sigma(L_e) \ \text{ and } \ \sigma(R_f) < \sigma(R_e) \ . \end{aligned} \tag{2}$$

Note that Equation (2) is obtained from Equation (1) by replacing "$\leq$" by "$<$".

Hence a strict queue is a set of edges, no two of which are nested or overlapping, as illustrated in Figure 2. Note that edges forming a 'butterfly' can be in a single strict queue.



(a) butterfly (b) left overlap (c) right overlap

Fig. 2: Relationships between pairs of edges with a common endpoint in a vertex ordering.

A *strict k-queue layout* of G is a pair $(\sigma, \{Q_1, Q_2, \dots, Q_k\})$ where σ is a vertex ordering of G, and $\{Q_1, Q_2, \dots, Q_k\}$ is a partition of $E(G)$, such that each Q_i is a strict queue in σ. We sometimes write

$\mathsf{queue}(e) = i$ for each edge $e \in Q_i$. The *strict-queue-number* of a graph G, denoted by $\mathsf{sqn}(G)$, is the minimum k such that there is a strict k-queue layout of G.

Heath and Rosenberg [19] proved that a fixed vertex ordering of a graph G admits a k-queue layout of G if and only if it has no $(k+1)$-edge rainbow, where a *rainbow* is a set of pairwise nested edges, as illustrated in Figure 3(a). Consider the analogous problem for strict queues: assign the edges of a graph G to the minimum number of strict queues given a fixed vertex ordering σ of G. As illustrated in Figure 3(b), a *weak rainbow* in σ is a set of edges R such that for every pair of edges $e, f \in R$, e is inside f or f is inside e.

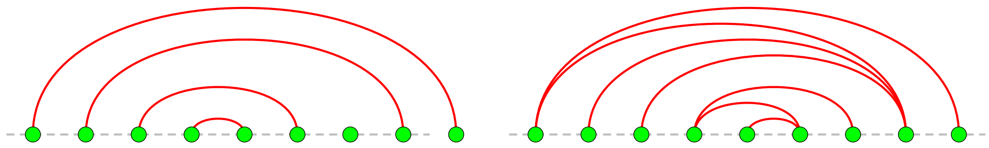


Fig. 3: (a) rainbow, (b) weak rainbow

Lemma 2 *A vertex ordering of a graph G admits a strict k-queue layout of G if and only if it has no $(k+1)$-edge weak rainbow.*

Proof: A strict k-queue layout has no $(k+1)$-edge weak rainbow since each edge of a weak rainbow must be in a distinct strict queue. Conversely, suppose we have a vertex ordering with no $(k+1)$-edge weak rainbow. For every edge $e \in E(G)$, let $\mathsf{queue}(e)$ be one plus the maximum number of edges in a weak rainbow consisting of edges that are inside e. If e is inside f then $\mathsf{queue}(e) < \mathsf{queue}(f)$. Hence we have a valid strict queue assignment. The number of strict queues is at most k. $\square$

A *linear forest* is a graph in which every component is a path. The *linear arboricity* of a graph G, denoted by $\mathsf{la}(G)$, is the minimum integer k such that $E(G)$ can be partitioned in k linear forests; see [1, 2, 30, 31]. We have the following lower bounds on $\mathsf{sqn}(G)$.

Lemma 3 *The strict queue-number of every graph G satisfies:*

(a) $\mathsf{sqn}(G) \geq \mathsf{la}(G) > \eta(G)$,

(b) $\mathsf{sqn}(G) \geq \mathsf{la}(G) \geq \Delta(G)/2$, *and*

(c) $\mathsf{sqn}(G) \geq \delta(G)$.

Proof: Say Q is a strict queue in a vertex ordering σ of G. Every 2-edge path (u, v, w) in Q has $\sigma(u) < \sigma(v) < \sigma(w)$ (or $\sigma(w) < \sigma(v) < \sigma(u)$). Thus no vertex is incident to three edges in Q, and Q induces a linear forest. Hence $\mathsf{la}(G) \leq \mathsf{sqn}(G)$.

Since a linear forest in G has at most $|V(G)| - 1$ edges, $\mathsf{la}(G) \geq |E(G)|/(|V(G)| - 1) > \eta(G)$. This proves (a). At most two edges incident to each vertex are a linear forest. Thus $\mathsf{la}(G) \geq \Delta(G)/2$. This proves (b).

In every vertex ordering of G, every edge incident to the first vertex is in a distinct strict queue. Hence $\mathsf{sqn}(G) \geq \delta(G)$. This proves (c). $\square$

Obviously a proper edge $(\Delta(G)+1)$-colouring [28] can be combined with a $\mathsf{qn}(G)$-queue layout to obtain a strict queue layout.

Lemma 4 *Every graph G has strict queue-number* $\mathsf{sqn}(G) \leq (\Delta(G)+1) \cdot \mathsf{qn}(G)$. $\square$

3 Graph Powers

Let G be a graph, and let $d \in \mathbb{Z}^+$. The d-th *power* of G, denoted by G^d, is the graph with vertex set $V(G^d) = V(G)$, where $vw \in E(G^d)$ if and only if the distance between v and w in G is at most d. The following general result is similar to a theorem of Dujmović and Wood [10].

Theorem 1 *For every graph G and $d \in \mathbb{Z}^+$,*

$$\mathsf{qn}(G^d) \leq \frac{(2\,\mathsf{sqn}(G))^{d+1}-1}{2\,\mathsf{sqn}(G)-1} - \mathsf{sqn}(G) - 1 \ .$$

Proof: Let σ be the vertex ordering in a strict $\mathsf{sqn}(G)$-queue layout of G. Consider σ to be a vertex ordering of G^d. For every pair of vertices $v, w \in V(G)$ with $\sigma(v) < \sigma(w)$ and at distance $\ell \leq d$, fix a path $P(vw)$ from v to w in G with exactly ℓ edges. Suppose $P(vw) = (x_0, x_1, \dots, x_\ell)$, where $v = x_0$ and $w = x_\ell$. For each $1 \leq i \leq \ell$, let $\mathsf{dir}(x_{i-1}x_i)$ be '$+$' if $\sigma(x_{i-1}) < \sigma(x_i)$, and '$-$' otherwise. Let $f(vw)$ be the vector

$$f(vw) = \Big[\big(\mathsf{queue}(x_{i-1}x_i), \mathsf{dir}(x_{i-1}x_i)\big) : 1 \leq i \leq \ell\Big] \ .$$

Consider two edges $vw, pq \in E(G^d)$ with $f(vw) = f(pq)$. Then $|P(vw)| = |P(pq)|$. Let $P(vw) = (x_0, x_1, \dots, x_\ell)$ and $P(pq) = (y_0, y_1, \dots, y_\ell)$. We have $\mathsf{dir}(x_0x_1) = \mathsf{dir}(y_0y_1)$ and $\mathsf{queue}(x_0x_1) = \mathsf{queue}(y_0y_1)$. Thus $x_0 \neq y_0$. Without loss of generality $\sigma(x_0) < \sigma(y_0)$. By Equation (2), $\sigma(x_1) < \sigma(y_1)$. In general, $\sigma(x_{i-1}) < \sigma(y_{i-1})$ implies $\sigma(x_i) < \sigma(y_i)$, since $\mathsf{queue}(x_{i-1}x_i) = \mathsf{queue}(y_{i-1}y_i)$ and $\mathsf{dir}(x_{i-1}x_i) = \mathsf{dir}(y_{i-1}y_i)$. By induction, $\sigma(x_i) < \sigma(y_i)$ for all $0 \leq i \leq \ell$. In particular, $\sigma(w) < \sigma(q)$. Thus vw and pq can be in the same strict queue. If we partition the edges of G^d by the value of f we obtain a strict queue layout of G^d. The number of queues is

$$\sum_{\ell=1}^{d} (2\,\mathsf{sqn}(G))^\ell \ = \ \frac{(2\,\mathsf{sqn}(G))^{d+1}-1}{2\,\mathsf{sqn}(G)-1} - 1 \ .$$

Observe that for the edges of G we have counted $2\,\mathsf{sqn}(G)$ queues. Of course we need only $\mathsf{sqn}(G)$ queues. Thus the total number of queues is as claimed. $\square$

3.1 Powers of Paths and Cycles

In a vertex ordering σ of a graph G, the *width* of an edge e is $\sigma(R_e) - \sigma(L_e)$. The *bandwidth* of σ is the maximum width of an edge of G. The *bandwidth* of G, denoted by $\mathsf{bw}(G)$, is the minimum bandwidth of a vertex ordering of G. Alternatively, $\mathsf{bw}(G) = \min\{k : G \subseteq P_n^k\}$ for every n-vertex graph G.

Heath and Rosenberg [19] observed that edges whose widths differ by at most one are not nested. Thus $\mathsf{qn}(G) \leq \lceil \mathsf{bw}(G)/2 \rceil$, as mentioned in Table 1. In a vertex ordering, edges with the same width are not nested or overlapping, and thus form a strict queue. The next lemma follows.

Lemma 5 *Every graph G has strict queue-number* $\mathsf{sqn}(G) \leq \mathsf{bw}(G)$. □

We have the following results that give more precise bounds on the queue-number and strict-queue-number of powers of paths and cycles than Theorem 1.

Lemma 6 *The k-th power of a path P_n ($n \geq k+1$) has queue-number* $\mathsf{qn}(P_n^k) = \lceil k/2 \rceil$ *and strict queue-number* $\mathsf{sqn}(P_n^k) = k$

Proof: The bandwidth of a graph G can be thought of as the minimum integer k such that $G \subseteq P_n^k$. Thus the upper bound is nothing more than the result $\mathsf{qn}(G) \leq \lceil \mathsf{bw}(G)/2 \rceil$ of Heath and Rosenberg [19]. The lower bound follows since P_n^k contains a $(k+1)$-clique, which contains $\lceil k/2 \rceil$ pairwise nested edges in any vertex ordering, all of which must be assigned to distinct queues.

The natural vertex-ordering of P_n^k has no $(k+1)$-edge weak rainbow. Thus $\mathsf{sqn}(P_n^k) \leq k$ by Lemma 2. The lower bound follows since P_n^k contains a $(k+1)$-clique, which contains a k-edge weak rainbow in any vertex ordering. □

A graph is *unicyclic* if every connected component has at most one cycle. Heath and Rosenberg [19] proved that any unicyclic graph has a 1-queue layout. In particular, every cycle has a 1-queue layout. More generally,

Lemma 7 *The k-th power of a cycle C_n ($n \geq 2k$) has queue-number* $\frac{k}{2} < \mathsf{qn}(C_n^k) \leq k$, *and strict queue-number* $\mathsf{sqn}(C_n^k) = 2k$.

Proof: Observe that $\delta(C_n^k) = \Delta(C_n^k) = 2k$ and $\eta(C_n^k) = k$. Thus the claimed lower bounds follow from Lemmata 1 and 3. For the upper bounds, say $C_n = (v_1, v_2, \dots, v_n)$. By considering the vertex ordering

$$(v_1, v_n; v_2, v_{n-1}; \dots; v_i, v_{n-i+1}; \dots; v_{\lfloor n/2 \rfloor}, v_{\lceil n/2 \rceil}) \ , \tag{3}$$

we see that $C_n^k \subset P_n^{2k}$. The result follows from Lemma 6. □

4 Graph Products

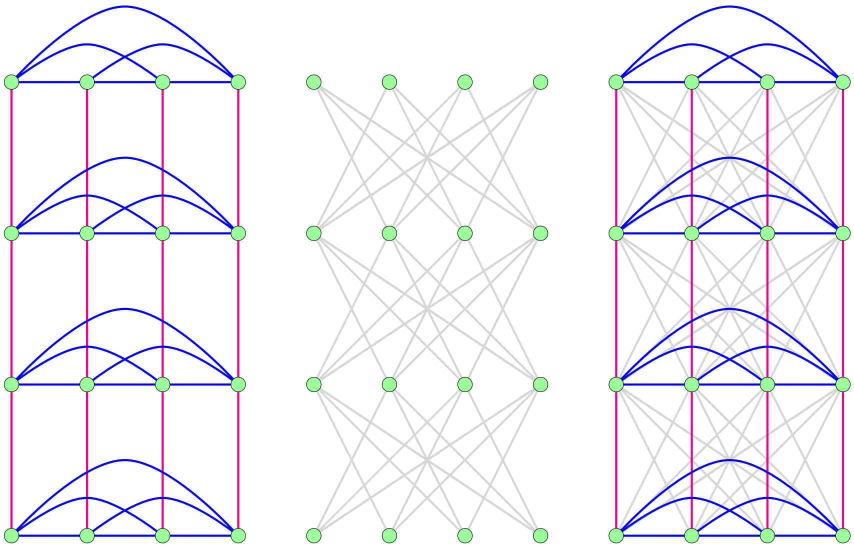
Let G_1 and G_2 be graphs. Below we define a number of graph products whose vertex set is

$$V(G_1) \times V(G_2) = \{(a, v) : a \in V(G_1), v \in V(G_2)\}) \ .$$

We classify a potential edge $(a, v)(b, w)$ as follows:

- *G_1-edge*: $ab \in E(G_1)$ and $v = w$.
- *G_2-edge*: $a = b$ and $vw \in E(G_2)$.
- *direct edge*: $ab \in E(G_1)$ and $vw \in E(G_2)$.

The *cartesian product* $G_1 \square G_2$ consists of the G_1-edges and the G_2-edges. The *direct product* $G_1 \times G_2$ consists of the direct edges. The *strong product* $G_1 \boxtimes G_2$ consists of the G_1-edges, the G_2-edges, and the direct edges. That is, $G_1 \boxtimes G_2 = (G_1 \square G_2) \cup (G_1 \times G_2)$. Note that other names abound for these graph products. Our notation is taken from the survey by Klavžar [21]. Assuming isomorphic graphs are equal, each of the above three products are associative, and for instance, $G_1 \square G_2 \square \cdots \square G_d$ is well-defined. Figure 4 illustrates these three types of graphs products.



(a) $P_4 \square K_4$ (b) $P_4 \times K_4$ (c) $P_4 \boxtimes K_4$

Fig. 4: Examples of graph products: (a) cartesian, (b) direct, (c) strong.

The following lemma is well-known and easily proved.

Lemma 8 *For all graphs G_1 and G_2, the density satisfies*

(a) $\eta(G_1 \square G_2) = \eta(G_1) + \eta(G_2)$,

(b) $\eta(G_1 \times G_2) = 2\eta(G_1) \cdot \eta(G_2)$,

(c) $\eta(G_1 \boxtimes G_2) = 2\eta(G_1) \cdot \eta(G_2) + \eta(G_1) + \eta(G_2)$.

5 The Cartesian Product

We have the following bounds on the queue-number of a cartesian product. In a vertex ordering σ of a graph product, we abbreviate $\sigma((v, a))$ by $\sigma(v, a)$.

Theorem 2 *For all graphs G and H,*

$$\mathsf{qn}(G \square H) \leq \mathsf{sqn}(G) + \mathsf{qn}(H) \ .$$

Furthermore, if for some constant c we have $\mathsf{sqn}(G) \leq c \cdot \eta(G)$ *and* $\mathsf{qn}(H) \leq c \cdot \eta(H)$*, then*

$$\mathsf{qn}(G \,\Box\, H) \;\geq\; \tfrac{1}{2c}\left(\mathsf{sqn}(G) + \mathsf{qn}(H)\right) \;.$$

Proof: First we prove the upper bound. Let σ be the vertex ordering in a strict $\mathsf{sqn}(G)$-queue layout of G. Let π be the vertex ordering in a $\mathsf{qn}(H)$-queue layout of H. Let ϕ be the vertex ordering of $G \,\Box\, H$ in which $\phi(v,a) < \phi(w,b)$ if and only if $\sigma(v) < \sigma(w)$, or $v = w$ and $\pi(a) < \pi(b)$.

For all edges e of G and for all vertices a of H, we have $\phi(L_e,a) < \phi(R_e,a)$. Similarly, for all edges e of H and for all vertices v of G, we have $\phi(v,L_e) < \phi(v,R_e)$.

Consider two G-edges $(L_e,a)(R_e,a)$ and $(L_f,b)(R_f,b)$ of $G \,\Box\, H$, for which e and f are in the same strict queue of G. By Equation (2), without loss of generality, $\sigma(L_e) < \sigma(L_f)$ and $\sigma(R_e) < \sigma(R_f)$. Thus $\phi(L_e,a) < \phi(L_f,b)$ and $\phi(R_e,a) < \phi(R_f,b)$. Hence for each strict queue in G, the corresponding G-edges of $G \,\Box\, H$ form a strict queue in ϕ.

Consider two H-edges $(v,L_e)(v,R_e)$ and $(w,L_f)(w,R_f)$ of $G \,\Box\, H$, for which e and f are in the same queue of H. By Equation (1), without loss of generality, $\pi(L_e) \leq \pi(L_f)$ and $\pi(R_e) \leq \pi(R_f)$. First suppose that $\sigma(v) \leq \sigma(w)$. Then $\phi(v,L_e) \leq \phi(w,L_f)$ and $\phi(v,R_e) \leq \phi(w,R_f)$. Thus $(v,L_e)(v,R_e)$ and $(w,L_f)(w,R_f)$ are not nested in ϕ. Now suppose that $\sigma(w) < \sigma(v)$. Then $\phi(w,L_f) < \phi(w,R_f) < \phi(v,L_e) < \phi(v,R_e)$. Thus $(v,L_e)(v,R_e)$ and $(w,L_f)(w,R_f)$ are disjoint. Thus for each queue in H, the corresponding H-edges of $G \,\Box\, H$ form a queue in ϕ. Therefore ϕ admits a $(\mathsf{sqn}(G)+\mathsf{qn}(H))$-queue layout of $G \,\Box\, H$.

Now we prove the lower bound. By Lemmata 1 and 8(a), $\mathsf{qn}(G \,\Box\, H) > \eta(G \,\Box\, H)/2 = (\eta(G) + \eta(H))/2$. The result follows since $\eta(G) \geq \frac{1}{c}\,\mathsf{sqn}(G)$ and $\eta(H) \geq \frac{1}{c}\,\mathsf{qn}(H)$. □

Theorem 2 has the following immediate corollary.

Corollary 1 *For all graphs* $G_1, G_2, \dots, G_d$*,*

$$\mathsf{qn}(G_1 \,\Box\, G_2 \,\Box\, \cdots \,\Box\, G_d) \;\leq\; \mathsf{qn}(G_1) + \sum_{i=2}^{d} \mathsf{sqn}(G_i) \;.$$

□

5.1 Grids

A *d-dimensional grid* is a graph $P_{n_1} \,\Box\, P_{n_2} \,\Box\, \cdots \,\Box\, P_{n_d}$, for all $n_i \geq 1$. Heath and Rosenberg [19] determined the queue-number of every 2-dimensional grid.

Lemma 9 ([19]) *Every* 2*-dimensional grid has queue-number one.* □

A *generalised d-dimensional grid* is a graph $G = P^k_{n_1} \,\Box\, P^k_{n_2} \,\Box\, \cdots \,\Box\, P^k_{n_d}$, for all $k \geq 1$ and $n_i \geq k+1$. Now P^k_n has $kn - k(k+1)/2$ edges. Thus $\eta(P^k_n) = k - \frac{k(k+1)}{2n}$. By Lemma 8(a),

$$\eta(G) \;=\; \sum_{i=1}^{d}\left(k - \frac{k(k+1)}{2n_i}\right) \;=\; dk - \frac{1}{2}\,k(k+1)\sum_{i=1}^{d}\frac{1}{n_i} \;. \tag{4}$$

Lemma 9 generalises as follows.

Theorem 3 *For all $d \geq 2$, the queue-number of a d-dimensional grid $G = P_{n_1} \,\square\, P_{n_2} \,\square\, \cdots \,\square\, P_{n_d}$ satisfies:*

$$\frac{d}{4} \;\leq\; \frac{1}{2}\left(d - \sum_{i=1}^{d} \frac{1}{n_i}\right) \;<\; \mathsf{qn}(G) \;\leq\; d-1 \ .$$

Proof: The lower bound follows from Lemma 1 and Equation (4) with $k = 1$.

For the upper bound, we have $\mathsf{qn}(P_{n_1} \,\square\, P_{n_2}) = 1$ by Lemma 9. Obviously $\mathsf{sqn}(P_{n_i}) = 1$ for all $i \geq 3$. Thus $\mathsf{qn}(G) \leq d-1$ by Corollary 1.

We now give an alternative proof of the upper bound using a different construction. The graph G can be thought of as having vertex set $\{((x_1, x_2, \ldots, x_d) : 1 \leq x_i \leq n_i, 1 \leq i \leq d\}$, where two vertices $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ are adjacent if and only if $|x_i - y_i| = 1$ for some i, and $x_j = y_j$ for all $j \neq i$. We say this edge is in the i-th dimension. For all $s \geq 0$, let V_s be the set of vertices

$$V_s = \{(x_1, x_2, \ldots, x_d) : \sum_{i=1}^{d} x_i = s\} \ .$$

Order the vertices $(V_0, V_1, \ldots)$, where each V_s is ordered lexicographically. If vw is an edge then v and w differ in exactly one coordinate, and $v \in V_s$ and $w \in V_{s+1}$ for some s. Thus if two edges vw and pq are nested then $v, p \in V_s$ and $w, q \in V_{s+1}$ for some s. Let Q_i be the set of edges in the i-th dimension. Consider two edges e and f in Q_i. Say

$$e = (x_1, x_2, \ldots, x_d)(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_d) \ ,$$

and

$$f = (y_1, y_2, \ldots, y_d)(y_1, \ldots, y_{i-1}, y_i + 1, y_{i+1}, \ldots, y_d) \ .$$

Without loss of generality $(x_1, x_2, \ldots, x_d) \prec (y_1, y_2, \ldots, y_d)$, which implies that

$$(x_1, \ldots, x_{i-1}, x_i + j, x_{i+1}, \ldots, x_d) \prec (y_1, \ldots, y_{i-1}, y_i + j, y_{i+1}, \ldots, y_d) \ .$$

Thus e and f are not nested, and Q_i is a queue. Hence we have a d-queue layout. (At this point we have in fact proved that the lexicographical order admits a d-queue layout.)

We now prove that $Q_{d-1} \cup Q_d$ is a queue, and thus we obtain the claimed $(d-1)$-queue layout. Suppose two edges $e \in Q_{d-1}$ and $f \in Q_d$ are nested. Say

$$e = (x_1, x_2, \ldots, x_d)(x_1, x_2, \ldots, x_{d-1} + 1, x_d) \ ,$$

and

$$f = (y_1, y_2, \ldots, y_d)(y_1, y_2, \ldots, y_{d-1}, y_d + 1) \ .$$

Then for some s, both $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ are in V_s, and both $(x_1, x_2, \ldots, x_{d-1} + 1, x_d)$ and $(y_1, y_2, \ldots, y_{d-1}, y_d + 1)$ are in V_{s+1}.

Case 1. $(x_1, x_2, \ldots, x_d) \prec (y_1, y_2, \ldots, y_d)$: Let j be the first dimension for which $x_j < y_j$. If $j \leq d-2$ then

$$(x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \ldots, y_{d-1}, y_d + 1) \ ,$$

which implies that e and f are not nested. Observe that $j \neq d$ as $(x_1, x_2, \dots, x_d)$ and $(y_1, y_2, \dots, y_d)$ differ in at least two coordinates, since $\sum_i x_i = \sum_i y_i$. Thus $j = d - 1$. That is,

$$x_{d-1} \leq y_{d-1} - 1 \ . \tag{5}$$

Since e and f are nested, we have $(y_1, y_2, \dots, y_{d-1}, y_d + 1) \prec (x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d)$, which implies that $y_{d-1} \leq x_{d-1} + 1$. By Equation (5), $x_{d-1} = y_{d-1} - 1$. Since $x_{d-1} + x_d = y_{d-1} + y_d$, we have $x_d = y_d + 1$, which implies that

$$(y_1, y_2, \dots, y_{d-1}, y_d + 1) = (x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d) \ .$$

That is, the right-hand endpoints of e and f are the same vertex. Hence e and f are not nested.

Case 2. $(y_1, y_2, \dots, y_d) \prec (x_1, x_2, \dots, x_d)$: By the same argument employed above, the first coordinate for which $(y_1, y_2, \dots, y_d)$ and $(x_1, x_2, \dots, x_d)$ differ is $d - 1$. That is,

$$y_{d-1} < x_{d-1} \ . \tag{6}$$

Since e and f are nested, we have $(x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \dots, y_{d-1}, y_d + 1)$. Thus $x_{d-1} + 1 < y_{d-1}$, which contradicts Equation (6). Hence e and f are not nested.

Therefore $Q_1, Q_2, \dots, Q_{d-2}, Q_{d-1} \cup Q_d$ is the desired $(d-1)$-queue layout. □

More generally we have the following.

Theorem 4 *The queue-number of a generalised d-dimensional grid $G = P^k_{n_1} \square P^k_{n_2} \square \cdots \square P^k_{n_d}$ (where $n_i \geq k + 1$) satisfies:*

$$\frac{dk}{4} \ \leq \ \frac{dk}{2} - \frac{k(k+1)}{4} \sum_{i=1}^{d} \frac{1}{n_i} \ < \ \mathsf{qn}(G) \ \leq \ \left\lceil (d - \tfrac{1}{2})k \right\rceil \ .$$

Proof: By Lemma 6, $\mathsf{qn}(P^k_n) = \lceil \frac{k}{2} \rceil$ and $\mathsf{sqn}(P^k_n) \leq k$. Thus, the upper bound follows from Corollary 1. Thus the lower bound follows from Lemma 1 and Equation (4). □

By Theorem 4 with $k = n - 1$ we have the following.

Corollary 2 *The queue-number of the d-dimensional Hamming graph $G = K_n \square K_n \square \cdots \square K_n$ satisfies:*

$$\frac{d(n-1)}{4} \ < \ \mathsf{qn}(G) \ \leq \ \left\lceil (d - \tfrac{1}{2})(n-1) \right\rceil \ .$$

A *generalised d-dimensional toroidal grid* is a graph $C^k_{n_1} \square C^k_{n_2} \square \cdots \square C^k_{n_d}$ for all $k \geq 1$ and $n_i \geq 2k + 1$.

Theorem 5 *The queue-number of a generalised toroidal grid $G = C^k_{n_1} \square C^k_{n_2} \square \cdots \square C^k_{n_d}$ (where $n_i \geq 2k + 1$) satisfies:*

$$\frac{kd}{2} < \mathsf{qn}(G) \leq (2d - 1)k \ .$$

Proof: Since $\eta(G) = kd$, we have that $\mathsf{qn}(G) > \frac{kd}{2}$ by Lemma 1. Thus $\mathsf{qn}(G) \geq \lfloor \frac{d}{2} \rfloor + 1$. By Lemma 7, $\mathsf{qn}(C^k_{n_1}) \leq k$ and $\mathsf{sqn}(C^k_{n_1}) \leq 2k$. By Corollary 1, $\mathsf{qn}(G) \leq 2k(d-1) + k = (2d-1)k$ □

6 Direct and Strong Products

We have the following bounds on the queue-number of direct and strong products.

Theorem 6 *For all graphs G and H,*

$$\mathsf{qn}(G \times H) \leq 2\,\mathsf{sqn}(G) \cdot \mathsf{qn}(H) \ .$$

Furthermore, if $\mathsf{sqn}(G) \leq c \cdot \eta(G)$ and $\mathsf{qn}(H) \leq c \cdot \eta(H)$, then

$$\mathsf{qn}(G \times H) \ > \ \tfrac{1}{c^2}\,\mathsf{sqn}(G) \cdot \mathsf{qn}(H) \ .$$

Proof: First we prove the upper bound. Let $k := \mathsf{sqn}(G)$, and let $(\sigma, \{Q_1, Q_2, \dots, Q_k\})$ be a strict k-queue layout of G. Let $\ell := \mathsf{qn}(H)$, and let $(\pi, \{P_1, P_2, \dots, P_\ell\})$ be an ℓ-queue layout of H. For $1 \leq i \leq k$ and $1 \leq j \leq \ell$, let

$$\begin{aligned}
E'_{i,j} &:= \{(v,a)(w,b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(a) < \pi(b)\}\\
E''_{i,j} &:= \{(v,a)(w,b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(b) < \pi(a)\}
\end{aligned}$$

Then $\{E'_{i,j}, E''_{i,j} : 1 \leq i \leq k, 1 \leq j \leq \ell\}$ is a partition of $E(G \times H)$ into $2k\ell$ sets. Let ϕ be the vertex ordering of $G \times H$ in which $\phi(v,a) < \phi(w,b)$ if and only if $\sigma(v) < \sigma(w)$, or $v = w$ and $\pi(a) < \pi(b)$. We claim that each set $E'_{i,j}$ and $E''_{i,j}$ is a queue in ϕ.

Suppose that two edges $(v,a)(w,b), (x,c)(y,d) \in E'_{i,j}$ are nested. Without loss of generality, $\phi(v,a) < \phi(x,c) < \phi(y,d) < \phi(w,b)$. If $v \neq x$ and $y \neq w$, then $\sigma(v) < \sigma(x) < \sigma(y) < \sigma(w)$, and the edges $vw, xy \in Q_i$ are nested in σ. If $v \neq x$ and $y = w$, then $\sigma(v) < \sigma(x) < \sigma(y) = \sigma(w)$, and the edges $vw, xy \in Q_i$ overlap in σ. If $v = x$ and $y \neq w$, then $\sigma(v) = \sigma(x) < \sigma(y) < \sigma(w)$, and the edges $vw, xy \in Q_i$ overlap in σ. Each of these outcomes contradict the assumption that Q_i is a strict queue in σ. Otherwise $v = x$ and $y = w$, in which case $\pi(a) < \pi(c) < \pi(d) < \pi(b)$, and ab and cd are nested in π. This contradicts the assumption that P_j is a queue in π. Thus each $E'_{i,j}$ is queue in ϕ. By symmetry, each $E''_{i,j}$ is also a queue in ϕ.

Now we prove the lower bound. Lemmata 1 and 8(b) imply that

$$\mathsf{qn}(G \times H) > \eta(G \times H)/2 = \eta(G) \cdot \eta(H) \geq \tfrac{1}{c}\,\mathsf{sqn}(G) \cdot \tfrac{1}{c}\,\mathsf{qn}(H) \ .$$

□

Theorem 7 *For all graphs G and H,*

$$\mathsf{qn}(G \boxtimes H) \leq 2\,\mathsf{sqn}(G) \cdot \mathsf{qn}(H) + \mathsf{sqn}(G) + \mathsf{qn}(H) \ .$$

Furthermore, if $\mathsf{sqn}(G) \leq c \cdot \eta(G)$ and $\mathsf{qn}(H) \leq c \cdot \eta(H)$, then

$$\mathsf{qn}(G \boxtimes H) \ > \ \tfrac{1}{c^2}\,\mathsf{sqn}(G) \cdot \mathsf{qn}(H) + \tfrac{1}{2c}\big(\mathsf{sqn}(G) + \mathsf{qn}(H)\big) \ .$$

Proof: To prove the upper bound, observe that the vertex ordering ϕ defined in Theorems 2 and 6 is the same. By Theorem 2, ϕ admits a $\mathsf{sqn}(G) + \mathsf{qn}(H)$-queue layout of $G \,\square\, H$. By Theorem 6, ϕ admits a

$2\,\mathsf{sqn}(G) \cdot \mathsf{qn}(H)$-queue layout of $G \times H$. Since $G \boxtimes H = (G \,\square\, H) \cup (G \times H)$, ϕ admits the claimed queue layout of $G \boxtimes H$.

For the lower bound, Lemmata 1 and 8(c) imply that

$$\mathsf{qn}(G \boxtimes H) > \tfrac{1}{2}\eta(G \boxtimes H) = \eta(G) \cdot \eta(H) + \tfrac{1}{2}(\eta(G) + \eta(H)) \geq \frac{1}{c}\,\mathsf{sqn}(G) \cdot \frac{1}{c}\,\mathsf{qn}(H) \enspace .$$

□

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References

[1] N. ALON. The linear arboricity of graphs. *Israel J. Math.*, 62(3):311–325, 1988.

[2] NOGA ALON, V. J. TEAGUE, AND N. C. WORMALD. Linear arboricity and linear k-arboricity of regular graphs. *Graphs Combin.*, 17(1):11–16, 2001.

[3] SANDEEP N. BHATT, FAN R. K. CHUNG, F. THOMSON LEIGHTON, AND ARNOLD L. ROSENBERG. Scheduling tree-dags using FIFO queues: A control-memory trade-off. *J. Parallel Distrib. Comput.*, 33:55–68, 1996.

[4] EMILIO DI GIACOMO, GIUSEPPE LIOTTA, AND HENK MEIJER. Computing straight-line 3D grid drawings of graphs in linear volume. *Comput. Geom.*, 32(1):26–58, 2005.

[5] EMILIO DI GIACOMO, GIUSEPPE LIOTTA, HENK MEIJER, AND STEPHEN K. WISMATH. Volume requirements of 3D upward drawings. In PATRICK HEALY, ed., *Proc. 13th International Symp. on Graph Drawing* (GD '05), Lecture Notes in Comput. Sci. Springer, to appear.

[6] VIDA DUJMOVIĆ, PAT MORIN, AND DAVID R. WOOD. Layout of graphs with bounded tree-width. *SIAM J. Comput.*, 34(3):553–579, 2005.

[7] VIDA DUJMOVIĆ, ATTILA PÓR, AND DAVID R. WOOD. Track layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):497–522, 2004.

[8] VIDA DUJMOVIĆ AND DAVID R. WOOD. Tree-partitions of k-trees with applications in graph layout. Tech. Rep. TR-2002-03, School of Computer Science, Carleton University, Ottawa, Canada, 2002.

[9] VIDA DUJMOVIĆ AND DAVID R. WOOD. On linear layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):339–358, 2004.

[10] VIDA DUJMOVIĆ AND DAVID R. WOOD. Stacks, queues and tracks: Layouts of graph subdivisions. *Discrete Math. Theor. Comput. Sci.*, 7:155–202, 2005.

[11] Vida Dujmović and David R. Wood. Three-dimensional upward grid drawings of graphs. 2005.

[12] Shimon Even and A. Itai. Queues, stacks, and graphs. In Zvi Kohavi and Azaria Paz, eds., *Proc. International Symp. on Theory of Machines and Computations*, pp. 71–86. Academic Press, 1971.

[13] Joseph L. Ganley. Stack and queue layouts of Halin graphs, 1995. Manuscript.

[14] Toru Hasunuma. Laying out iterated line digraphs using queues. In Guiseppe Liotta, ed., *Proc. 11th International Symp. on Graph Drawing* (GD '03), vol. 2912 of *Lecture Notes in Comput. Sci.*, pp. 202–213. Springer, 2004.

[15] Lenwood S. Heath, F. Thomson Leighton, and Arnold L. Rosenberg. Comparing queues and stacks as mechanisms for laying out graphs. *SIAM J. Discrete Math.*, 5(3):398–412, 1992.

[16] Lenwood S. Heath and Sriram V. Pemmaraju. Stack and queue layouts of posets. *SIAM J. Discrete Math.*, 10(4):599–625, 1997.

[17] Lenwood S. Heath and Sriram V. Pemmaraju. Stack and queue layouts of directed acyclic graphs. part II. *SIAM J. Comput.*, 28(5):1588–1626, 1999.

[18] Lenwood S. Heath, Sriram V. Pemmaraju, and Ann N. Trenk. Stack and queue layouts of directed acyclic graphs. part I. *SIAM J. Comput.*, 28(4):1510–1539, 1999.

[19] Lenwood S. Heath and Arnold L. Rosenberg. Laying out graphs using queues. *SIAM J. Comput.*, 21(5):927–958, 1992.

[20] Atsumi Imamiya and Akihiro Nozaki. Generating and sorting permutations using restricted-deques. *Information Processing in Japan*, 17:80–86, 1977.

[21] Sandi Klavžar. Coloring graph products—a survey. *Discrete Math.*, 155(1-3):135–145, 1996.

[22] Edward T. Ordman and William Schmitt. Permutations using stacks and queues. In *Proc. 24th Southeastern International Conf. on Combinatorics, Graph Theory, and Computing*, vol. 96 of *Congr. Numer.*, pp. 57–64. 1993.

[23] Sriram V. Pemmaraju. *Exploring the Powers of Stacks and Queues via Graph Layouts*. Ph.D. thesis, Virginia Polytechnic Institute and State University, U.S.A., 1992.

[24] Vaughan R. Pratt. Computing permutations with double-ended queues. Parallel stacks and parallel queues. In *Proc. 5th Annual ACM Symp. on Theory of Computing* (STOC '73), pp. 268–277. ACM, 1973.

[25] S. Rengarajan and C. E. Veni Madhavan. Stack and queue number of 2-trees. In Ding-Zhu Du and Ming Li, eds., *Proc. 1st Annual International Conf. on Computing and Combinatorics* (COCOON '95), vol. 959 of *Lecture Notes in Comput. Sci.*, pp. 203–212. Springer, 1995.

[26] FARHAD SHAHROKHI AND WEIPING SHI. On crossing sets, disjoint sets, and pagenumber. *J. Algorithms*, 34(1):40–53, 2000.

[27] ROBERT E. TARJAN. Sorting using networks of queues and stacks. *J. Assoc. Comput. Mach.*, 19:341–346, 1972.

[28] VADIM G. VIZING. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz No.*, 3:25–30, 1964.

[29] DAVID R. WOOD. Queue layouts, tree-width, and three-dimensional graph drawing. In MANINDRA AGRAWAL AND ANIL SETH, eds., *Proc. 22nd Foundations of Software Technology and Theoretical Computer Science* (FST TCS '02), vol. 2556 of *Lecture Notes in Comput. Sci.*, pp. 348–359. Springer, 2002.

[30] JIAN-LIANG WU. On the linear arboricity of planar graphs. *J. Graph Theory*, 31(2):129–134, 1999.

[31] JIANLIANG WU. The linear arboricity of series-parallel graphs. *Graphs Combin.*, 16(3):367–372, 2000.