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A note on t-designs with t intersection numbers

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We use the Ray-Chaudhuri and Wilson inequality for a 0-design with t intersection numbers to prove that 'For a fixed block size k, there exist finitely many parametrically feasible t-designs with t intersection numbers and $\lambda > 1$'.

Keywords: t-designs

1 Introduction

Let X be a finite set of v elements, called *points*, and let β be a finite family of distinct k-subsets of X, called *blocks*. Then the pair $D = (X,\beta)$ is called a *t-design with parameters* (v,k,λ) if any t-subset of X is contained in exactly λ members of β. If λ_i denotes the number of blocks containing i points, $i = 0,1,2,\ldots,t-1$, then λ_i is independent of the choice of the i points and $\lambda_i\binom{k-i}{t-i} = \lambda\binom{v-i}{t-i}$. In particular, $b = \lambda_0$ is the number of blocks, and $\lambda_1 = r$ is the number of blocks through any point of D. A 0-design is a pair (X,β) where β is a collection of k-subsets of X. For $0 \le x < k$, x is called *intersection number of* D if there exists $B,B' \in \beta$ such that $|B \cap B'| = x$. A 2-design with two intersection numbers is said to be a *quasi-symmetric design*.

For a 0-design with t intersection numbers, Ray-Chaudhuri and Wilson proved that $b \le \binom{v}{t}$ (see [1]). This result is used by Sane and Shrikhande [7] to prove that, for a fixed value of the block size k, there exist finitely many quasi-symmetric designs with $\lambda > 1$.

In the literature many finiteness results for quasi-symmetric designs and quasi-symmetric 3-designs are proved by using this result by Sane and Shrikhande (see [5], [6], [7], [8]).

Our main aim in this paper is to extend the result by Sane and Shrikhande to t-designs with t intersection numbers. More specifically, we obtain a relation between the parameters of a t-design with t intersection numbers, and we use it to show that, for a fixed value of the block size k, v takes finitely many values. Finally, we use the result by Ray-Chaudhuri and Wilson to complete the proof.

2 Main Results

Theorem 2.1 (Ray-Chaudhuri and Wilson) *Let D be a 0-design with t $(0 < t \leq k \leq v-t)$ intersection numbers $x_1, x_2, \ldots, x_t$ $(0 \leq x_1 < x_2 < \ldots < x_t < k)$. Then $b \leq \binom{v}{t}$.*

Lemma 2.2 *Let D be a t-design with t intersection numbers $x_1, x_2, \ldots, x_t$ $(0 \leq x_1 < x_2 < \ldots < x_t < k)$. Then the following relation holds:*

$$\begin{vmatrix} x_2 - x_1 & \cdots & x_t - x_1 & x_1(b-1)-(r-1) \\ \binom{x_2}{2} - \binom{x_1}{2} & \cdots & \binom{x_t}{2} - \binom{x_1}{2} & \binom{x_1}{2}(b-1) - \binom{k}{2}(\lambda_2 - 1) \\ \vdots & \vdots & \vdots & \vdots \\ \binom{x_2}{t} - \binom{x_1}{t} & \cdots & \binom{x_t}{t} - \binom{x_1}{t} & \binom{x_1}{t}(b-1) - \binom{k}{t}(\lambda_t - 1) \end{vmatrix} = 0. \tag{1}$$

Proof: Let a_i, $2 \leq i \leq t$, be the number of blocks intersecting B_0 in x_i points. Fix a block B_0 and count in two ways the number of $(j+1)$-tuples $(\{p_1, p_2, \ldots, p_j\}, B)$, where B is a block of D other that B_0, and where $p_1, p_2, \ldots, p_j$ are distinct points of D contained in $B \cap B_0$. For $j = 1, 2, \ldots, t$, we have

$$\sum_{i=2}^{t} \binom{x_i}{j} a_i + \binom{x_1}{j} \left(b - 1 - \sum_{i=2}^{t} a_i \right) - \binom{k}{j} (\lambda_j - 1) = 0.$$

We rewrite these equations as follows:

$$\sum_{i=2}^{t} \left[\binom{x_i}{j} - \binom{x_1}{j} \right] a_i + \binom{x_1}{j}(b-1) - \binom{k}{j} (\lambda_j - 1) = 0.$$

These are t equations in $t-1$ unknowns $a_2, a_3, \ldots, a_t$. Since $a_2, a_3, \ldots, a_t, 1$ cannot be simultaneously zero, the coefficient matrix is singular. Hence the determinant given in equation (1) must be zero. □

We would like to point out that equation (1) is an extension of equation (1) of [7]. The latter is found to be useful in the study of quasi-symmetric designs. An immediate application of (1) is given in the following theorem.

Theorem 2.3 *For a fixed block size k, there exist finitely many parametrically feasible t-designs with t intersection numbers $x_1, x_2, \ldots, x_t$ $(0 \leq x_1 < x_2 \ldots < x_t < k)$ and $\lambda > 1$.*

Proof: By Theorem 2.1, the inequality $b \leq \binom{v}{t}$ holds for any t-design with t intersection numbers. Hence it suffices to show that for a fixed k, v takes finitely many values.

We divide the proof in two parts, by considering the cases $x_1 \neq 0$ and $x_1 = 0$ separately.

For $x_1 \neq 0$, we have the equality

$$\sum_{i=2}^{t} (x_i - x_1) a_i = k(r-1) - x_1(b-1).$$

This implies $k(r-1) - x_1(b-1) > 0$. Hence, $x_1 < \frac{k(r-1)}{b-1} < \frac{kr}{b} = \frac{k^2}{v}$. Now it is easy to see that $v < \frac{k^2}{x_1}$.

For $x_1 = 0$, by Lemma 2.2 we have

$$\begin{vmatrix} x_2 & \cdots & x_t & k(r-1) \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} & \binom{k}{2}(\lambda_2 - 1) \\ \vdots & \vdots & \vdots & \vdots \\ \binom{x_2}{t-1} & \cdots & \binom{x_t}{t-1} & \binom{k}{t-1}(\lambda_{t-1} - 1) \\ \binom{x_2}{t} & \cdots & \binom{x_t}{t} & \binom{k}{t}(\lambda_t - 1) \end{vmatrix} = 0.$$

Now we put $\lambda_j = \binom{v-j}{t-j}\lambda / \binom{k-j}{t-j}$, $j = 1, 2, \ldots, t-1$, to get

$$\begin{vmatrix} & k \\ & \binom{k}{2} \\ A & \vdots \\ & \binom{k}{t-1} \\ & \binom{k}{t} \end{vmatrix} = \lambda \begin{vmatrix} & k\binom{v-1}{t-1} \Big/ \binom{k-1}{t-1} \\ & \binom{k}{2}\binom{v-2}{t-2} \Big/ \binom{k}{t-1} \\ A & \vdots \\ & \binom{k}{t-1}\binom{v-t+1}{1} \Big/ \binom{k-t+1}{1} \\ & \binom{k}{t} \end{vmatrix},$$

where

$$A = \begin{bmatrix} x_2 & \cdots & x_t \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} \\ \vdots & \vdots & \vdots \\ \binom{x_2}{t-1} & \cdots & \binom{x_t}{t-1} \\ \binom{x_2}{t} & \cdots & \binom{x_t}{t} \end{bmatrix}.$$

We simplify the above equation as follows:

$$\begin{vmatrix} & k \\ & \binom{k}{2} \\ A & \vdots \\ & \binom{k}{t-1} \\ & \binom{k}{t} \end{vmatrix} = \lambda \begin{vmatrix} & k\dfrac{(v-1)(v-2)\cdots(v-t+2)(v-t+1)}{(k-1)(k-2)\cdots(k-t+2)(k-t+1)} \\ & \binom{k}{2}\dfrac{(v-2)(v-3)\cdots(v-t+2)(v-t+1)}{(k-2)(k-3)\cdots(k-t+2)(k-t+1)} \\ A & \vdots \\ & \binom{k}{t-1}\dfrac{v-t+1}{k-t+1} \\ & \binom{k}{t} \end{vmatrix}.$$

Now we multiply both sides by $(k-1)(k-2)\cdots(k-t+2)(k-t+1)$, to get

$$\left|\begin{array}{cc} & k \\ & \binom{k}{2} \\ A & \vdots \\ & \binom{k}{t-1} \\ & \binom{k}{t} \end{array}\right|(k-1)(k-2)\cdots(k-t+2)(k-t+1)$$

$$=\lambda\left|\begin{array}{cc} & k(v-1)(v-2)\cdots(v-t+2)(v-t+1) \\ & \binom{k}{2}(v-2)(v-3)\cdots(v-t+2)(v-t+1)(k-1) \\ A & \vdots \\ & \binom{k}{t-1}(v-t+1)(k-1)(k-2)\cdots(k-t+2) \\ & \binom{k}{t}(k-1)(k-2)\cdots(k-t+2)(k-t+1) \end{array}\right|.$$

Note that λ divides the left hand side. Hence λ takes finitely many values only. It is clear from the above equation, by expanding the right hand side determinant with respect to the last column of the matrix, that $v-t+1$ divides

$$(k-1)(k-2)\cdots(k-t+1)\left(\left|\begin{array}{cc} & k \\ & \binom{k}{2} \\ A & \vdots \\ & \binom{k}{t-1} \\ & \binom{k}{t} \end{array}\right| - \lambda\binom{k}{t}\left|\begin{array}{ccc} x_2 & \cdots & x_t \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} \\ \vdots & \vdots & \vdots \\ \binom{x_2}{t-1} & \cdots & \binom{x_t}{t-1} \end{array}\right|\right).$$

For a fixed k, this is a finite number. If it is non-zero, then v takes only finitely many values, otherwise $v-t+2$ divides

$$\binom{k}{t-1}(k-1)(k-2)\cdots(k-t+2)\left|\begin{array}{ccc} x_2 & \cdots & x_t \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} \\ \vdots & \vdots & \vdots \\ \binom{x_2}{t-2} & \cdots & \binom{x_t}{t-2} \\ \binom{x_2}{t} & \cdots & \binom{x_t}{t} \end{array}\right|.$$

This can be simplified to

$$\binom{k}{t-1}(k-1)(k-2)\cdots(k-t+2)\frac{x_2x_3\cdots x_t}{2!3!\cdots(t-2)!t!}\begin{vmatrix} 1 & \cdots & 1 \\ x_2 & \cdots & x_t \\ \vdots & \vdots & \vdots \\ x_2^{t-3} & \ldots & x_t^{t-3} \\ x_2^{t-1} & \ldots & x_t^{t-1} \end{vmatrix}.$$

It is easy to check that the above determinant is non-zero. This completes the proof. □

In [7], Sane and Shrikhande proved analogues of Theorem 2.3 for quasi-symmetric designs. As in our paper, they also divide their proof in two parts, depending on whether or not the smaller intersection number, x, is non-zero or not. The case $x_1 \neq 0$ of Theorem 2.3 is similar to that of $x \neq 0$ of [7]. But to prove their result for the case $x = 0$, they use a rather long argument, whereas our proof is elementary and short.

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