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On a Unimodality Conjecture in Matroid Theory

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A certain unimodal conjecture in matroid theory states the number of rank-r matroids on a set of size n is unimodal in r and attains its maximum at $r = \lfloor n/2 \rfloor$. We show that this conjecture holds up to $r = 3$ by constructing a map from a class of rank-2 matroids into the class of loopless rank-3 matroids. Similar inequalities are proven for the number of non-isomorphic loopless matroids, loopless matroids and matroids.

Keywords: Matroid Theory, Unimodal Conjecture, Rank-2 matroids, Rank-3 matroids

1 Introduction

Let us first recall some elementary definitions (further details may be found in Oxley (1), Welsh (2) and the excellent appendix of White (3)). Let S_n be a finite set of size n. A matroid M on the *ground set* S_n is a collection of subsets $I(M)$ of S_n satisfying

- $I(M) \neq \emptyset$,
- if $X \in I(M)$ and $Y \subseteq X$ then $Y \in I(M)$,
- if $X, Y \in I(M)$ with $|X| = |Y| + 1$, then there exists $x \in X \backslash Y$ such that $Y \cup \{x\} \in I(M)$.

Sets in $I(M)$ are called *independent sets*. The *rank* of a set $X \subseteq S_n$, denoted $r(X)$, is the size of the largest independent set which it contains. The rank of the matroid $r(M) := r(S_n)$. A set X is *closed* (or termed a *flat*) if $r(X \cup \{x\}) = r(X) + 1$ for all $x \in S_n \backslash X$. We denote by $\mathcal{F}(M)$ the closed sets of M. The *loops* of M are the elements of the rank-0 flat. Also note that $x \in S_n$ is a loop if it is not contained in any of the independent sets of M.

A certain unimodal conjecture in matroid theory states that the sequence of the number of non-isomorphic rank-r matroids on S_n, $\{f_r(n) : 1 \leqslant r \leqslant n\}$, is unimodal in r and attains its maximum at $r = \lfloor n/2 \rfloor$ (see Oxley (1) or Welsh (4) p.300). It is easily seen that $f_1(n) \leqslant f_2(n)$ holds since $f_1(n) = n$ and $f_2(n) = p(1) + \cdots + p(n) - n$, where $p(n)$ is the number of integer partitions of n. The step between rank-2 and rank-3 is not as clear since the exact value of $f_3(n)$ remains unknown. We show, through construction of a map between a class of rank-2 matroids and loopless rank-3 matroids and known values of these numbers from the On–line Encyclopedia of Integer Sequences, that this unimodal conjecture holds for these rank-2 versus rank-3 matroids. Furthermore, we show the corresponding inequalities hold for the number of rank-2,3 non-isomorphic loopless matroids, $g_2(n) \leqslant g_3(n)$, loopless matroids, $c_2(n) \leqslant c_3(n)$, and matroids, $m_2(n) < m_3(n)$.

Let $b_i(n)$ be the number of partitions of the set S_n. into i parts and $b(n)$ be the n^{th} Bell number. Let $p_i(n)$ the number of partitions of the integer n into i parts. The number of rank-2 matroids can be enumerated through considering the points and lines of the associated geometry. We have $c_2(n) = b(n) - 1$, $g_2(n) = p(n) - 1$ and $m_2(n) = b(n+1) - 2^n$ (for proofs see Dukes (5)). The main results of this paper are given in Theorems 2.5, 2.6, 2.11 and 2.12.

2 Mapping rank-2 to rank-3 matroids

Let $\mathcal{M}_r(n)$ be the collection of rank-r matroids on S_n. Let $\mathcal{A}_r(n)$ be the collection of rank-r matroids on S_n with at least one loop and $\mathcal{B}_r(n) := \mathcal{M}_r(n) \backslash \mathcal{A}_r(n)$. We define the map $\sigma : \mathcal{A}_2(n) \to \mathcal{B}_3(n)$ as follows: given $M \in \mathcal{A}_2(n)$ with loops F_0 and rank-1 flats $\mathcal{F}_1(M) = \{F_0 \cup F_1, \dots, F_0 \cup F_m\}$ define $M' = \sigma(M)$ as:

$$\begin{array}{rcl} \mathcal{F}_0(M') & := & \{\emptyset\} \\ \mathcal{F}_1(M') & := & \{F_0, F_1, \dots, F_m\} \\ \mathcal{F}_2(M') & := & \{F_0 \cup F_i | 1 \leqslant i \leqslant m\} \cup \{F_1 \cup \dots \cup F_m\} \\ \mathcal{F}_3(M') & := & \{S_n\}. \end{array}$$

It is easily checked that these collections of flats satisfy the axioms for a loopless rank-3 matroid. For $M \in \mathcal{M}_r(n)$, let us write $d(M)$ for the number of rank-1 flats of M (which we will refer to as the *degree* of M). Let us mention that for any loopless matroid M, the rank-1 flats of M partition the ground set. Similarly, for any matroid, the rank-1 flats partition the ground set less the set of loops. Also note that in the collection $\mathcal{F}_2(M')$, there are precisely $d(M)$ sets containing F_0, 2 sets containing F_i (for any $1 \leqslant i \leqslant d(M)$) and one set containing $F_i \cup F_j$ (for all $0 \leqslant i \neq j \leqslant d(M)$).

The following lemma shows that to each rank-2 matroid with at least one loop, there corresponds a rank-3 loopless matroid (although not necessarily unique). The following lemma classifies those matroids which map to a unique loopless matroid in $\mathcal{B}_3(n)$ and those which do not.

Lemma 2.1 *Let $M_1, M_2 \in \mathcal{A}_2(n)$ be such that $\mathcal{F}_0(M_1) = \{F_0^{(1)}\}$, $\mathcal{F}_0(M_2) = \{F_0^{(2)}\}$, $\mathcal{F}_1(M_1) = \{F_0^{(1)} \cup F_1^{(1)}, \dots, F_0^{(1)} \cup F_{d(M_1)}^{(1)}\}$ and $\mathcal{F}_1(M_2) = \{F_0^{(2)} \cup F_1^{(2)}, \dots, F_0^{(2)} \cup F_{d(M_2)}^{(2)}\}$. Then $\sigma(M_1) = \sigma(M_2)$ if and only if $d(M_1) = d(M_2) = 2$ and*

$$\left\{F_0^{(1)}, F_1^{(1)}, F_2^{(1)}\right\} = \left\{F_0^{(2)}, F_1^{(2)}, F_2^{(2)}\right\}.$$

Proof: ONLY IF: Let $M_1, M_2 \in \mathcal{A}_2(n)$ be such that $M_1 \neq M_2$ and $\sigma(M_1) = \sigma(M_2)$. Let $M_1' := \sigma(M_1)$ and $M_2' := \sigma(M_2)$. Then we must have $\mathcal{F}_1(M_1') = \mathcal{F}_1(M_2')$ and $\mathcal{F}_2(M_1') = \mathcal{F}_2(M_2')$. Now $\mathcal{F}_1(M_1') = \mathcal{F}_1(M_2') \Rightarrow d(M_1) = d(M_2)$ and $\{F_i^{(1)}\}_{i=0}^{d(M_1)} = \{F_i^{(2)}\}_{i=0}^{d(M_2)}$. If $d(M_1) > 2$ then we must have $F_0^{(1)} = F_0^{(2)}$ which would imply $M_1 = M_2$. Hence $d(M_1) = 2 = d(M_2)$. This gives $\mathcal{F}_2(M_1') = \{F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_1^{(1)}\} = \mathcal{F}_2(M_2')$ only if $\{F_0^{(1)}, F_1^{(1)}, F_2^{(1)}\} = \{F_0^{(2)}, F_1^{(2)}, F_2^{(2)}\}$.

IF: This is trivial as $\{F_0^{(1)},F_1^{(1)},F_2^{(1)}\} = \{F_0^{(2)},F_1^{(2)},F_2^{(2)}\}$ gives $\mathcal{F}_1(M_1') = \mathcal{F}_1(M_2')$ and

$$\begin{aligned}\mathcal{F}_2(M_1') &:= \left\{F_0^{(1)}\cup F_1^{(1)},F_0^{(1)}\cup F_1^{(1)},F_0^{(1)}\cup F_1^{(1)}\}\right\} \\ &= \left\{F_0^{(2)}\cup F_1^{(2)},F_0^{(2)}\cup F_1^{(2)},F_0^{(2)}\cup F_1^{(2)}\}\right\} =: \mathcal{F}_2(M_2').\end{aligned}$$

□

Thus it is seen for each matroid $M \in \sigma(\mathcal{A}_2(n))$ such that $d(M) = 2$, there are precisely three different matroids $M_1,M_2,M_3 \in \mathcal{A}_2(n)$ such that $\sigma(M_1) = \sigma(M_2) = \sigma(M_3) = M$.

Lemma 2.2 *For all $n \geqslant 3$, $c_3(n) \geqslant b(n+1) - b(n) - 3^{n-1}$.*

Proof: We show that the number of unique matroids in the image of $\mathcal{A}_2(n)$ under σ is given by $b(n+1) - b(n) - 3^{n-1}$, thereby lower-bounding $c_3(n)$. In the enumeration below, we divide the matroids to be counted in the image into two classes, those matroids M with $d(M) = 2$ and those with $d(M) > 2$. The former class projects different matroids to the same matroid in $\mathcal{B}_3(n)$ and through the use of the previous lemma we take care of this over-counting, hence

$$\begin{aligned}&\#\{\sigma(M)|M \in \mathcal{A}_2(n)\} \\ &= \#\{\sigma(M)|M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} + \sum_{i=3}^{n}\#\{\sigma(M)|M \in \mathcal{A}_2(n) \text{ and } d(M) = i\} \\ &= \frac{1}{3}\#\{M|M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} + \sum_{i=3}^{n}\#\{\sigma(M)|M \in \mathcal{A}_2(n) \text{ and } d(M) = i\} \\ &= \sum_{i=2}^{n}\#\{\sigma(M)|M \in \mathcal{A}_2(n) \text{ and } d(M) = i\} - \frac{2}{3}\#\{M|M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} \\ &= \#\mathcal{A}_2(n) - \frac{2}{3}\#\{M|M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} \\ &= b(n+1) - 2^n - (b(n) - 1) - \frac{2}{3}\#\{M|M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\}.\end{aligned}$$

Note that $\#\{M|M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} = \sum_{l=2}^{n-1}\binom{n}{l}b_2(l)$ and $b_2(l) = \frac{1}{2}\sum_{j=1}^{l-1}\binom{l}{j} = 2^{l-1} - 1$, giving:

$$\begin{aligned}\#\{M|M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} &= \sum_{l=2}^{n-1}\binom{n}{l}(2^{l-1} - 1) \\ &= \frac{1}{2}(3^n - 2^n - 2n - 1) - (2^n - n - 2) \\ &= \frac{3}{2}(3^{n-1} - 2^n + 1).\end{aligned}$$

Thus

$$\begin{aligned}\#\{\sigma(M)|M \in \mathcal{A}_2(n)\} &= b(n+1) - 2^n - b(n) + 1 - \frac{2}{3}\frac{3}{2}(3^{n-1} - 2^n + 1) \\ &= b(n+1) - b(n) - 3^{n-1}.\end{aligned}$$

□

The corresponding inequality for the number of non-isomorphic loopless matroids is proved in Lemma 2.3. We do this in a similar manner as before, by showing that each rank-2 matroid (which is not a loopless matroid) of degree greater than 3 corresponds uniquely to a rank-3 loopless matroid.

Lemma 2.3 *For all $n \geqslant 4$, $g_3(n) \geqslant \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2+6n+3) - 1$.*

Proof: We show the number of non-isomorphic matroids in the image of $\mathcal{A}_2(n)$ under σ is given by $\sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2+6n+3) - 1$ which lower bounds $g_3(n)$.

Let us identify $\mathcal{A}_2^\star(n) \subseteq \mathcal{A}_2(n)$ by placing an ordering on the elements of $S_n = \{x_1,\dots,x_n\}$. Given $M \in \mathcal{A}_2(n)$ with $d(M) = m$, loops F_0 and rank-1 flats $\{F_0 \cup F_1, \dots, F_0 \cup F_m\}$, let $M \in \mathcal{A}_2^\star(n)$ if and only if $F_0 = \{x_1,\dots,x_{|F_0|}\}$, F_1 contains the next $|F_1|$ elements of S_n, i.e. $\{x_{|F_0|+1},\dots,x_{|F_0|+|F_1|}\}$ and so forth. Define

$$\mathcal{T}(n) \quad := \quad \left\{M \in \mathcal{A}_2^\star(n) \middle| d(M) = 2 \text{ and } |F_0| \leqslant |F_1| \leqslant |F_2|\right\}$$

and for $3 \leqslant i \leqslant n-1$, $3 \leqslant j \leqslant i$, define

$$\Omega_{i,j}(n) \quad := \quad \left\{M \in \mathcal{A}_2^\star(n) \middle| d(M) = j, |F_0| = n-i \text{ and } |F_1| \leqslant \cdots \leqslant |F_j|\right\}.$$

Let us now write

$$\mathcal{A}_2^{\star\star}(n) \quad := \quad \mathcal{T}(n) \cup \bigcup_{i=3}^{n-1} \bigcup_{j=3}^{i} \Omega_{i,j}(n) \subseteq \mathcal{A}_2^\star(n).$$

It is obvious that no two matroids in $\mathcal{T}(n)$ are isomorphic to one-another. Similarly with $\Omega_{k,l}(n)$. We have simply reduced our class of matroids from $\mathcal{A}_2(n)$ to $\mathcal{A}_2^{\star\star}(n)$ in the same manner as one moves from the set of partitions of a finite set of size n to the set of integer partitions of n.

The unions in the definition of $\mathcal{A}_2^{\star\star}(n)$ are strictly disjoint and no isomorphisms may occur between matroids in different classes or matroids in the same class. The same is true of the image of $\mathcal{A}_2^{\star\star}(n)$ under the map σ. We may directly enumerate the number of non-isomorphic matroids in $\mathcal{B}_3(n)$ in the image of $\mathcal{A}_2^{\star\star}(n)$ under σ as

$$p_3(n) + \sum_{i=3}^{n-1} \sum_{j=3}^{i} p_j(i).$$

The rightmost term is bounded below;

$$\begin{aligned}
\sum_{i=3}^{n-1}\sum_{j=3}^{i} p_j(i) &= \sum_{i=2}^{n-1}\{p(i)-p_1(i)-p_2(i)\} \\
&= \sum_{i=3}^{n-1}\{p(i)-1-\lfloor i/2\rfloor\} \\
&= -(n-3)+\sum_{i=3}^{n-1}p(i)-\sum_{i=3}^{n-1}\lfloor i/2\rfloor \\
&= \begin{cases} \sum_{i=1}^{n-1}p(i)-\frac{n(n+2)}{4}+1, & n \text{ even}, \\ \sum_{i=1}^{n-1}p(i)-\frac{(n+1)^2}{4}+1, & n \text{ odd}, \end{cases} \\
&\geqslant \sum_{i=1}^{n-1}p(i)-\frac{(n+1)^2}{4},
\end{aligned}$$

for all $n \geqslant 2$. As for $p_3(n)$, from Hall (6) [p.32], we have

$$\begin{aligned}
p_3(n) &= \begin{cases} \lfloor n^2/12\rfloor, & \text{for } n \not\equiv 3 (\text{mod } 6), \\ \lceil n^2/12\rceil, & \text{for } n \equiv 3 (\text{mod } 6), \end{cases} \\
&\geqslant \frac{n^2}{12}-1,
\end{aligned}$$

and so

$$\begin{aligned}
g_3(n) &\geqslant \sum_{i=1}^{n-1}p(i)+\frac{n^2}{12}-\frac{(n+1)^2}{4}-1 \\
&= \sum_{i=1}^{n-1}p(i)-\frac{1}{12}(2n^2+6n+3)-1.
\end{aligned}$$

□

2.1 Matroids

The following lemma is needed in order to support the theorem which follows it.

Lemma 2.4 *For all $n \geqslant 1$, $b(n+1)-2^n \geqslant 2^n-(1+n)$.*

Proof: We have that $b(i) \geqslant 2$ for all $i \geqslant 2$. Since $n \geqslant 2$, it follows that

$$\begin{aligned}
b(n+1)-2^n &= \sum_{i=0}^{n}\binom{n}{i}(b(i)-1) \\
&\geqslant \sum_{i=2}^{n}\binom{n}{i}1 \\
&= 2^n-(1+n).
\end{aligned}$$

□

Theorem 2.5 *For all $n > 4$, $c_3(n) \geqslant c_2(n)$.*

Proof: From Lemma 2.2 we have that $c_3(n) \geqslant b(n+1) - b(n) - 3^{n-1}$. We know $c_2(n) = b(n) - 1$. It suffices to show that

$$b(n+1) - b(n) - 3^{n-1} \quad \geqslant \quad b(n) - 1$$

for all $n > 4$. Let us look at the value $b(n+1) - 2b(n) - 3^{n-1} + 2^{n-1}$:

$$\begin{aligned} b(n+1) - 2b(n) - 3^{n-1} + 2^{n-1} &= \sum_{i=0}^{n-1} \binom{n}{i} b(i) - \sum_{i=0}^{n-1} \binom{n-1}{i} b(i) - \sum_{i=0}^{n-1} \binom{n-1}{i} 2^i + \sum_{i=0}^{n-1} \binom{n-1}{i} \\ &= \sum_{i=0}^{n-2} \binom{n-1}{i} \left(b(i+1) - 2^i\right) \end{aligned}$$

and using Lemma 2.4,

$$\begin{aligned} &\geqslant \sum_{i=1}^{n-2} \binom{n-1}{i} \left(2^i - (1+i)\right) \\ &= 3^{n-1} - 2^n + 1 - \sum_{i=1}^{n-2} \binom{n-1}{i} i \\ &= 3^{n-1} - (n+3)2^{n-2} + n. \end{aligned}$$

The problem has been reduced to showing $3^{n-1} - (n+3)2^{n-2} + n \geqslant 2^{n-1} - 1$ for all $n \geqslant 5$, which is easily shown by induction. □

We now show the number of rank-3 matroids dominates the number of rank-2 matroids by using two things: the first is the result proved previously, that the number of rank-3 loopless matroids is at least as large as the number of rank-2 loopless matroids; the second is the first few known values of the numbers $c_2(n)$ and $c_3(n)$. The latter knowledge makes the inequality strict.

Theorem 2.6 *For all $n \geqslant 5$, $m_3(n) \geqslant m_2(n)$.*

Proof: The number of rank-r matroids on S_n is related to the number of loopless matroids on S_n by

$$m_r(n) \quad = \quad \sum_{i=r}^{n} \binom{n}{i} c_r(i).$$

In Theorem 2.5 we showed that $c_3(n) \geqslant c_2(n)$ for all $n \geqslant 5$. Replacing $r = 3$ in the above expression and using the first few values of $c_3(n)$ (taken from row 3, table A058710, of Sloane (7)),

$$\begin{aligned} m_3(n) &= \sum_{i=3}^{n} \binom{n}{i} c_3(i) \\ &= 1\binom{n}{3} + 11\binom{n}{4} + 106\binom{n}{5} + 1232\binom{n}{6} + \sum_{i=7}^{n} \binom{n}{i} c_3(i) \\ &\geqslant 1\binom{n}{3} + 11\binom{n}{4} + 106\binom{n}{5} + 1232\binom{n}{6} + \sum_{i=7}^{n} \binom{n}{i} c_2(i) \\ &= 830\binom{n}{6} + 75\binom{n}{5} - 3\binom{n}{4} - 3\binom{n}{3} - \binom{n}{2} + \sum_{i=2}^{n} \binom{n}{i} c_2(i) \end{aligned}$$

A simple check shows that $830\binom{n}{6}+75\binom{n}{5}-3\binom{n}{4}-3\binom{n}{3}-\binom{n}{2}$ is greater than zero and increasing for all $n \geqslant 7$. From Table 1 (see Appendix), the result is also seen to hold for $n=5,6$. Equality holds only for $n=5$, for all other values of n the inequality is strict. □

2.2 Non-isomorphic matroids

Proving the corresponding inequalities for the non-isomorphic numbers is more difficult. We first prove several lemmas related to the numbers $p(n)$ which we will need in the proofs of the two remaining theorems.

Lemma 2.7 *For all $n \geqslant 1$, $p(n+1) \geqslant p(n)+\left\lfloor \frac{n+1}{2} \right\rfloor$.*

Proof: The number of partitions of the integer $n+1$ whose first part contains the integer 1 is precisely $p(n)$. The number beginning with i, for any $2 \leqslant i \leqslant \left\lfloor \frac{n+1}{2} \right\rfloor$ is at least 1 since we can have the partition $n+1 = i+(n+1-i)$. Also, the number $n+1$ is a partition by itself, hence,

$$\begin{aligned} p(n+1) &\geqslant p(n)+\left(\left\lfloor \frac{n+1}{2} \right\rfloor - 1\right)+1 \\ &= p(n)+\left\lfloor \frac{n+1}{2} \right\rfloor . \end{aligned}$$

□

Lemma 2.8 *For all $n \geqslant 1$, $p(n) \geqslant 1+\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \geqslant \frac{n^2+3}{4}$.*

Proof: From Lemma 2.7 we have

$$p(n+1) \geqslant p(n)+\left\lfloor \frac{n+1}{2} \right\rfloor$$

for all $n \geqslant 1$. Applying this lemma recursively gives

$$\begin{aligned} p(n) &\geqslant p(n-1)+\left\lfloor \frac{n}{2} \right\rfloor \\ &\geqslant p(n-2)+\left\lfloor \frac{n-1}{2} \right\rfloor+\left\lfloor \frac{n}{2} \right\rfloor \\ &\vdots \\ &\geqslant p(1)+\left\lfloor \frac{1+1}{2} \right\rfloor+\cdots+\left\lfloor \frac{n-1}{2} \right\rfloor+\left\lfloor \frac{n}{2} \right\rfloor \\ &\geqslant 1+\left\lfloor \frac{1+1}{2} \right\rfloor+\cdots+\left\lfloor \frac{n-1}{2} \right\rfloor+\left\lfloor \frac{n}{2} \right\rfloor . \end{aligned} \tag{1}$$

Now we wish to evaluate the sum $\sum_{i=2}^{n} \left\lfloor \frac{i}{2} \right\rfloor$. Let $n=2m+1$ for some $m \geqslant 1$, then

$$\begin{aligned} \sum_{i=2}^{n} \left\lfloor \frac{i}{2} \right\rfloor &= \sum_{i=2}^{2m+1} \left\lfloor \frac{i}{2} \right\rfloor \\ &= \sum_{i=1}^{m} \left\lfloor \frac{2i}{2} \right\rfloor+\left\lfloor \frac{2i+1}{2} \right\rfloor \\ &= \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil . \end{aligned}$$

For the $n = 2m$ case with $m \geqslant 1$, we simply remove the last term in the previous expression, thus

$$\begin{aligned}\sum_{i=2}^{n}\left\lfloor\frac{i}{2}\right\rfloor &= \sum_{i=2}^{2m+1}\left\lfloor\frac{i}{2}\right\rfloor - \left\lfloor\frac{2m+1}{2}\right\rfloor \\ &= \left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil.\end{aligned}$$

Continuing to the inequality in Equation 1 above,

$$\begin{aligned}p(n) &\geqslant 1+\left\lfloor\frac{1+1}{2}\right\rfloor+\cdots+\left\lfloor\frac{n-1}{2}\right\rfloor+\left\lfloor\frac{n}{2}\right\rfloor \\ &= 1+\left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil,\end{aligned}$$

for all $n \geqslant 1$. If n is even, then $\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil = \frac{n^2}{4}$. If n is odd, then $\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil = \frac{n-1}{2}\frac{n+1}{2}$. In either case, $1+\lfloor\frac{n}{2}\rfloor\lceil\frac{n}{2}\rceil \geqslant 1+\frac{n^2-1}{4}$. □

Lemma 2.9 *For all* $n \geqslant 5$, $p(n+1) < 2p(n) - \frac{n+2}{3}$.

Proof: Let $x_1 + x_2 + ... + x_k = n+1$ be a partition of $n+1$ with $1 \leqslant x_1 < \ldots < x_k$. There are precisely $p(n)$ partitions with $x_1 = 1$, since $x_2 + \ldots + x_k = n+1-1$.

For all those partitions with $x_1 \geqslant 2$, we see that reducing x_1 by 1 will yield a partition of n. Thus an upper bound on the number beginning with $x_2 \geqslant 2$ is $p(n)$. For all partitions starting with $x_1 = 2$, we see that $x_2 \neq 1$, thus we may remove all those sequences with $x_2 = 1 \leqslant x_3 \leqslant \cdots \leqslant x_k$ such that $2+1+x_3+\ldots+x_k = n+1$. Reformulated, this means all those partitions with $x_3+\ldots+x_k = n-2$ and $1 \leqslant x_3 \leqslant \cdots \leqslant x_k$ of which there are $p(n-2)$.

Thus we see that $p(n+1) < p(n)+p(n)-p(n-2) = 2p(n)-p(n-2)$. From lemma 2.8 we know that for $n \geqslant 3$,

$$p(n-2) \geqslant \frac{(n-2)^2+3}{4} = \frac{n^2-4n+7}{4}.$$

Now, we see that the simple inequality $(3n-13)(n-1) \geqslant 0$ holds for all $n \geqslant \frac{13}{3}$, i.e. $\frac{(n^2-4n+7)}{4} \geqslant \frac{(n+2)}{3}$. From above, this gives

$$\begin{aligned}p(n+1) &< 2p(n)-p(n-2) \\ &\leqslant 2p(n)-\frac{(n^2-4n+7)}{4} \\ &\leqslant 2p(n)-\frac{(n+2)}{3},\end{aligned}$$

for all $n \geqslant 5$ and we are done. A check of the first few vales of $p(n)$ shows the stated inequality to hold for all $n \geqslant 2$. □

Lemma 2.10 *For all* $n \geqslant 7$, $\sum_{i=1}^{n-1} p(i) > p(n) + \frac{1}{12}(2n^2+6n+3)$.

Proof: By simple induction. The result is true for $n = 7$ since $p(1) + p(2) + \cdots + p(6) = 30$ and $p(7) + \frac{1}{12}(2(7)^2 + 6(7) + 3) < 27$. Suppose it to be true for some $n = m \geqslant 7$, then:

$$\begin{aligned}\sum_{i=1}^{m} p(i) &= p(m) + \sum_{i=1}^{m-1} p(i)\\ &\geqslant p(m) + p(m) + \frac{1}{12}(2m^2 + 6m + 3)\\ &= 2p(m) + \frac{1}{12}(2m^2 + 6m + 3)\end{aligned}$$

and using Lemma 2.9,

$$\begin{aligned}&> p(m+1) + \frac{m+2}{3} + \frac{1}{12}(2m^2 + 6m + 3)\\ &= p(m+1) + \frac{1}{12}(2(m+1)^2 + 6(m+1) + 3).\end{aligned}$$

□

Theorem 2.11 *For all $n \geqslant 5$, $g_3(n) \geqslant g_2(n)$.*

Proof: We have that $g_2(n) = p(n) - 1$. Also, we know from Theorem 2.6 that $g_3(n) \geqslant \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1$. From Lemma 2.10, we have $\sum_{i=1}^{n-1} p(i) > p(n) + \frac{1}{12}(2n^2 + 6n + 3)$ for all $n \geqslant 7$. Combining these facts gives

$$\begin{aligned}g_3(n) &\geqslant \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1\\ &\geqslant p(n) - 1\end{aligned}$$

which is $g_2(n)$. From Table 1, the result is also seen to hold for $n = 5, 6$. □

Theorem 2.12 *For all $n \geqslant 5$, $f_3(n) \geqslant f_2(n)$.*

Proof: The number of non-isomorphic rank-3 matroids on S_n in terms of loopless non-isomorphic rank-3 matroids is given through the relation $f_3(n) = \sum_{i=3}^{n} g_3(i)$ for all $n \geqslant 3$. The value $f_2(n) = p(1) + p(2) + \ldots + p(n) - n$ for all $n \geqslant 2$. From Theorem 2.11 we have $g_3(n) \geqslant g_2(n)$ for all $n \geqslant 7$. Applying the above expression for $f_3(n)$, using the known value for $g_3(n)$ (from Sloane (7), row 3 of A058716) and assuming $n \geqslant 7$,

$$\begin{aligned}f_3(n) &= 38 + \sum_{i=7}^{n} g_3(i)\\ &> 23 + \sum_{i=7}^{n} g_2(i)\\ &= \sum_{i=2}^{n} g_2(i)\end{aligned}$$

which is precisely $f_2(n)$. From Table 1, the result is seen to hold for $n = 5, 6$. Note that the above inequality is strict for $n \geqslant 6$ and equality holds only for $n = 5$. □

Note that, by duality, an immediate Corollary of Theorems 2.6 and 2.12 is the following.

Corollary 2.13 *For all $n \geqslant 6$,*

$$\begin{array}{ccccccc} f_n(n) & \leqslant & f_{n-1}(n) & \leqslant & f_{n-2}(n) & \leqslant & f_{n-3}(n) \\ m_n(n) & \leqslant & m_{n-1}(n) & \leqslant & m_{n-2}(n) & \leqslant & m_{n-3}(n). \end{array}$$

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Appendix

n	2	3	4	5	6	7	8	OLEIS Number	Row
$g_2(n)$	1	2	4	6	10	14	21	A058716	2
$g_3(n)$		1	3	9	25	70	217	A058716	3
$f_2(n)$	1	3	7	13	23	37	58	A053534	2
$f_3(n)$		1	4	13	38	108	325	A053534	3
$c_2(n)$	1	4	14	31	202	876	4139	A058710	2
$c_3(n)$		1	11	106	1232	22172	803583	A058710	3
$m_2(n)$	1	7	36	171	813	4012	20891	A058669	2
$m_3(n)$		1	15	171	2053	33442	1022217	A058669	3

Table 1: Known values for the number of rank-2 and rank-3 matroids taken from Sloane (7).