Arithmetic Quotients of the Complex Ball and a Conjecture of Lang

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Abstract. We prove that various arithmetic quotients of the unit ball in $\mathbb{C}^n$ are Mordellic, in the sense that they have only finitely many rational points over any finitely generated field extension of $\mathbb{Q}$. In the previously known case of compact hyperbolic complex surfaces, we give a new proof using their Albanese in conjunction with some key results of Faltings, but without appealing to the Shafarevich conjecture. In higher dimension, our methods allow us to solve an alternative of Ullmo and Yafaev. Our strongest result uses in addition Rogawski's theory and establishes the Mordellicity of the Baily-Borel compactifications of Picard modular surfaces of some precise levels related to the discriminant of the imaginary quadratic fields.

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Introduction

Let F be a totally real number field of degree d and ring of integers $\mathfrak{o}$, and let M be a totally imaginary quadratic extension of F with ring of integers $\mathfrak{O}$. Let G be a unitary group over F defined by a hermitian form on M^{n+1} of signature $(n,1)$ at one infinite place ι and $(n+1,0)$ or $(0,n+1)$ at the others. A subgroup $\Gamma \subset G(F)$ is arithmetic if it is commensurable with $G(\mathfrak{o})$, the stabilizer in $G(F)$ of $\mathfrak{O}^{n+1}$, and we will denote by Y_Γ the quotient of the n-dimensional complex hyperbolic space by the natural action of $\iota(\Gamma) \subset G(F_\iota) = \mathrm{U}(n,1)$. If $F \neq \mathbb{Q}$, then the hermitian form is anisotropic and Y_Γ is a projective variety defined over a number field (see Proposition 1.2).
A projective variety X over $\mathbb{C}$ is said to be Mordellic if it has only a finite number of rational points in every finitely generated field extension of $\mathbb{Q}$ over which X is defined. Lang conjectured in [L, Conjecture VIII.1.2] that X is

Mordellic if and only if the corresponding analytic space $X(\mathbb{C})$ is hyperbolic, meaning that any holomorphic map $\mathbb{C} \to X(\mathbb{C})$ is constant, which by Brody [B] is equivalent to requiring the Kobayashi semi-distance on $X(\mathbb{C})$ to be a metric. It is a consequence of a conjecture of Ullmo (see [U, Conjecture 2.1]) that a projective variety X defined over a number field k is Mordellic if it is arithmetically Mordellic, meaning that it has only a finite number of rational points in every finite extension of k.

Our first result establishes that many arithmetic compact surfaces previously only known to be arithmetically Mordellic by [U, Théorème 3.2] are in fact Mordellic. To state it precisely we need to fix a Hecke character λ of M as in Definition 3.1. The existence of such characters is known (see Lemma 3.5). Denote by $\mathfrak{C}$ the conductor of λ and, if the extension M/F is everywhere unramified, multiply $\mathfrak{C}$ by any prime $\mathfrak{q}$ of F which does not split in M. Moreover, fix an auxiliary prime $\mathfrak{p}$ of F which splits in M and is relatively prime to $\mathfrak{C}$. Finally, for every ideal $\mathfrak{N} \subset \mathfrak{O}$ we consider the standard congruence subgroups $\Gamma_0(\mathfrak{N})$, $\Gamma_1(\mathfrak{N})$ and $\Gamma(\mathfrak{N})$ of $G(F)$ (see Definition 1.3).

THEOREM 0.1. *Let $n = 2$ and G over $F \neq \mathbb{Q}$ as above. Then for every choice of $(\mathfrak{C}, \mathfrak{p})$, and for any torsion free subgroup $\Gamma \subset \Gamma_1(\mathfrak{C}) \cap \Gamma_0(\mathfrak{p})$ of finite index, Y_Γ is Mordellic.*

A consequence of this is that for any arithmetic subgroup $\Gamma \subset G(F)$ there exists a finite explicit cover of Y_Γ which is Mordellic. Note also that even though the theorem only concerns arithmetic subgroups, because F and M can vary, it can be applied to infinitely many pairwise non-commensurable cocompact discrete subgroups in $\mathrm{U}(2,1)$. In order to apply our method to the analogous case of a unitary group G'' defined by a division algebra of dimension 9 over M with an involution of a second kind, one would need to find a (cocompact) arithmetic subgroup $\Gamma \subset G''(F)$ such that the Albanese of Y_Γ is non-zero. This is an open question for any G'' since, in contrast to our case, it is known by Rogawski [R1] that the Albanese of Y_Γ is zero for any congruence subgroup $\Gamma \subset G''(F)$. While Ullmo's approach uses the Shafarevich conjecture, ours is based instead on the Mordell-Lang conjecture proved by Faltings [F2] and on the key Proposition 3.6, which we hope is of independent interest.

Consider now the case when the hermitian form is isotropic, which necessarily implies that $F = \mathbb{Q}$ and M is imaginary quadratic. Then Y_Γ is not compact and, for Γ arithmetic, we denote by Y_Γ^* the Baily-Borel compactification which is a normal, projective variety of dimension n. A smooth toroidal compactification X_Γ of Y_Γ can be defined over a number field (see [F3]), and it is not hyperbolic even if Γ is torsion-free; for example, if $n = 2$, then X_Γ is a union of Y_Γ with a finite number of elliptic curves – one above each cusp of Y_Γ^*. However, by a result of Tai and Mumford [Mu, §4], X_Γ is of general type for Γ sufficiently small. The *Bombieri-Lang conjecture* asserts then that the points of X_Γ over any finitely generated field extension of $\mathbb{Q}$ over which X_Γ is defined are not Zariski dense. We prove this in Proposition 4.3 which allows us to solve an alternative of Ullmo and Yafaev [UY] regarding the Lang locus of Y_Γ^*.

Theorem 0.2. *For all $\Gamma \subset G(\mathbb{Q})$ arithmetic and sufficiently small, Y_Γ^* is arithmetically Mordellic.*

Keeping the assumption that M is imaginary quadratic, say of fundamental discriminant $-D$, let us now suppose in addition that $n = 2$. The corresponding locally symmetric spaces Y_Γ are called Picard modular surfaces. We state here our main theorem.

Theorem 0.3. *Let* $\mathfrak{D} = \begin{cases} 3\mathfrak{O} & \text{, if } D = 3, \\ \sqrt{-D}\mathfrak{O} & \text{, if } D \neq 3 \text{ is odd,} \\ 2\sqrt{-D}\mathfrak{O} & \text{, if } 8 \text{ divides } D, \\ \sqrt{-D}\mathfrak{P}_2 & \text{, otherwise, where } \mathfrak{P}_2^2 = 2\mathfrak{O}. \end{cases}$

(i) *Let* $\Gamma = \begin{cases} \Gamma(\mathfrak{D}^2) & \text{, if } D \in \{3,4,7,11,19,43,67,163\}, \\ \Gamma(\mathfrak{D}) & \text{, if } D \in \{8,15,20,23,24,31,39,47,71\}, \\ \Gamma_1(\mathfrak{D}) & \text{, otherwise.} \end{cases}$
Then Y_Γ^* is Mordellic, while X_Γ is a minimal surface of general type.

(ii) *Let $N > 2$ be a prime inert in M and not equal to 3 when $D = 4$. Then $Y^*_{\Gamma(N)\cap\Gamma_1(\mathfrak{D})}$ is Mordellic, while $X_{\Gamma(N)\cap\Gamma_1(\mathfrak{D})}$ is a minimal surface of general type.*

The fact that one can take $\Gamma(\mathfrak{D})$ instead of $\Gamma(\mathfrak{D}^2)$ when $D \in \{15, 20, 23, 24, 31, 39, 47, 71\}$ in Theorem 0.3(i) depends on a preprint of Džambić [D] considered for publication elsewhere (see the proof for details).
At the heart of our proof stand some arithmetical computations using certain key theorems of Rogawski [R1, R2]. They yield, for each imaginary quadratic field M, an explicit congruence subgroup Γ such that the smooth compactification X_Γ does not admit a dominant map to its Albanese variety. A geometric ingredient of the proof is a result of Holzapfel *et al* that X_Γ is of general type, though not hyperbolic, implying by a theorem of Nadel [N] that any curve of genus ≤ 1 on it is contained in the compactifying divisor.
If there is anything new in our approach, it lies in the systematic use of the modern theory of automorphic representations in Diophantine geometry.

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1. Basics: lattices, general type and neatness

For any integer $n > 1$, let $\mathcal{H}^n_{\mathbb{C}}$ be the n-dimensional complex hyperbolic space, represented by the unit ball in $\mathbb{C}^n$ equipped with the Bergman metric of constant holomorphic sectional curvature $-4/(n+1)$, on which the real Lie group $\mathrm{U}(n,1)$ acts in a natural way.
Given a lattice $\Gamma \subset \mathrm{U}(n,1)$ we denote by $\bar{\Gamma} = \Gamma/\Gamma \cap \mathrm{U}(1)$ its image in the adjoint group $\mathrm{PU}(n,1) = \mathrm{U}(n,1)/\mathrm{U}(1)$, where $\mathrm{U}(1)$ is centrally embedded in $\mathrm{U}(n,1)$. Conversely any lattice $\bar{\Gamma} \subset \mathrm{PU}(n,1) = \mathrm{PSU}(n,1)$ is the image of a lattice in $\mathrm{U}(n,1)$, namely the lattice $\mathrm{U}(1)\bar{\Gamma} \cap \mathrm{SU}(n,1)$. We consider the quotient $Y_\Gamma = Y_{\bar{\Gamma}} = \bar{\Gamma}\backslash\mathcal{H}^n_{\mathbb{C}}$.

LEMMA 1.1. *Let Γ be a lattice in $\mathrm{U}(n,1)$.*

(i) *The analytic variety Y_Γ is an orbifold and one has the following implications:*

$$\Gamma \text{ neat} \Rightarrow \Gamma \text{ torsion-free} \Rightarrow \bar{\Gamma} \text{ torsion-free} \Rightarrow Y_\Gamma \text{ hyperbolic manifold.}$$

(ii) *Assume that $\bar{\Gamma}$ is torsion-free. Then the natural projection $\mathcal{H}^n_{\mathbb{C}} \to Y_\Gamma$ is an etale covering with deck transformation group $\bar{\Gamma}$.*

Proof. The stabilizer in $\mathrm{U}(n,1)$ of any point of $\mathcal{H}^n_{\mathbb{C}}$ is a compact group, hence its intersection with the discrete subgroup Γ is finite, showing that Y_Γ is an orbifold.
Recall that Γ is neat if the subgroup of $\mathbb{C}^\times$ generated by the eigenvalues of any $\gamma \in \Gamma$ is torsion-free. In particular Γ is torsion-free. Since $\Gamma \cap \mathrm{U}(1)$ is finite, this implies that $\bar{\Gamma}$ is torsion-free too. Under the latter assumption, $\Gamma \cap \mathrm{U}(1)$ acts trivially on $\mathcal{H}^n_{\mathbb{C}}$, and $\bar{\Gamma}$ acts freely and properly discontinuously on it, hence Y_Γ is a manifold. Since $\mathcal{H}^n_{\mathbb{C}}$ is simply connected, it is a universal covering space of Y_Γ with group $\bar{\Gamma}$. Hence any holomorphic map from $\mathbb{C}$ to Y_Γ lifts to a holomorphic map from $\mathbb{C}$ to $\mathcal{H}^n_{\mathbb{C}}$ which must be constant because $\mathcal{H}^n_{\mathbb{C}}$ has negative curvature. Thus Y_Γ is hyperbolic. □

Deligne's classification [De] of Shimura varieties implies, when Γ is a congruence subgroup, that Y_Γ admits an embedding in a Shimura variety. Hence, by Shimura's theory of canonical models, Y_Γ can be defined over a finite abelian extension of the reflex field M.
We claim that this is also true for Γ arithmetic, when sufficiently small. Indeed any such Y_Γ is a finite unramified cover of a congruence quotient $Y_{\Gamma'}$ which we have seen is defined over a number field. By Grothendieck, the finite index subgroup $\bar{\Gamma}$ of the topological fundamental group $\bar{\Gamma}'$ of $Y_{\Gamma'}(\mathbb{C})$ gives rise to a finite index subgroup of the algebraic fundamental group of $Y_{\Gamma'}$, yielding a finite algebraic (etale) map from a model of Y_Γ to $Y_{\Gamma'}$.
In the cocompact case, this remains true even when Γ is not arithmetic.

PROPOSITION 1.2. *Assume that $\bar{\Gamma}$ is cocompact and torsion-free. Then the projective variety $Y_{\bar{\Gamma}}$ is of general type and can be defined over a number field.*

Proof. The existence of the positive Bergman metric on $\mathcal{H}^n_{\mathbb{C}}$ implies by the Kodaira embedding theorem that any quotient by a free action such as $Y_{\bar\Gamma}$ has ample canonical bundle, which results in $Y_{\bar\Gamma}$ being of general type; it even implies that any subvariety of $Y_{\bar\Gamma}$ is of general type. For surfaces one may alternately use the hyperbolicity of $Y_{\bar\Gamma}$ to rule out all the cases in the Enriques-Kodaira classification where the Kodaira dimension is less than 2, thus showing that $Y_{\bar\Gamma}$ is of general type.
Calabi and Vesentini [CV] have proved that $Y_{\bar\Gamma}$ is locally rigid, hence by Shimura [S1] it can be defined over a number field.
In order to highlight the importance of rigidity of compact ball quotients, we provide a short second proof when $n = 2$ and Γ is arithmetic, based on Yau's algebro-geometric characterization of compact Kähler surfaces covered by $\mathcal{H}^2_{\mathbb{C}}$. Since $Y_{\bar\Gamma}$ has an ample canonical bundle it can be embedded in some projective space, hence is algebraic over $\mathbb{C}$ by Chow. Since $Y_{\bar\Gamma}$ is uniformized by $\mathcal{H}^2_{\mathbb{C}}$, the Chern numbers c_1, c_2 of its complex tangent bundle satisfy the relation $c_1^2 = 3c_2$. Since everything can be defined algebraically, for any automorphism σ of $\mathbb{C}$, the variety $Y^\sigma_{\bar\Gamma}$ also has ample canonical bundle and $c_1^{\sigma\,2} = 3c_2^\sigma$. By a famous result of Yau [Y, Theorem 4], this is equivalent to the fact that $Y^\sigma_{\bar\Gamma}$ may be realized as $\bar\Gamma^\sigma\backslash\mathcal{H}^2_{\mathbb{C}}$ for some cocompact torsion-free lattice $\bar\Gamma^\sigma$.
Since $\bar\Gamma$ is arithmetic, it has infinite index in its commensurator in $\mathrm{PU}(2,1)$, denoted by $\mathrm{Comm}(\bar\Gamma)$. For every element $g \in \mathrm{Comm}(\bar\Gamma)$ there is a Hecke correspondence

$$Y_{\bar\Gamma} \leftarrow Y_{\bar\Gamma\cap g^{-1}\bar\Gamma g} \xrightarrow[g\cdot]{\sim} Y_{g\bar\Gamma g^{-1}\cap\bar\Gamma} \to Y_{\bar\Gamma} \tag{1}$$

and the correspondences for g and g' differ by an isomorphism $Y_{g\bar\Gamma g^{-1}\cap\bar\Gamma} \xrightarrow{\sim} Y_{g'\bar\Gamma g'^{-1}\cap\bar\Gamma}$ over $Y_{\bar\Gamma}$ if and only if $g' \in \bar\Gamma g$. By Chow (1) is defined algebraically, hence yields a correspondence on $Y^\sigma_{\bar\Gamma} = Y_{\bar\Gamma^\sigma}$:

$$Y_{\bar\Gamma^\sigma} \leftarrow Y_{\bar\Gamma_1} \xrightarrow{\sim} Y_{\bar\Gamma_2} \to Y_{\bar\Gamma^\sigma},$$

for some finite index subgroups $\bar\Gamma_1$ and $\bar\Gamma_2$ of $\bar\Gamma^\sigma$. By the universal property of the covering space $\mathcal{H}^2_{\mathbb{C}}$, the middle isomorphism is given by an element of $g_\sigma \in \mathrm{PU}(2,1) \simeq \mathrm{Aut}(\mathcal{H}^2_{\mathbb{C}})$. Since $\mathrm{Aut}(\mathcal{H}^2_{\mathbb{C}}/Y_{\bar\Gamma_i}) = \bar\Gamma_i$ $(i = 1, 2)$, it easily follows that $\bar\Gamma_2 = g_\sigma\bar\Gamma_1 g_\sigma^{-1}$, and by applying σ^{-1} one sees that $\bar\Gamma_1 = \bar\Gamma^\sigma \cap g_\sigma^{-1}\bar\Gamma^\sigma g_\sigma$. It follows that $g_\sigma \in \mathrm{Comm}_G(\Gamma^\sigma)$ and one can check that $g'_\sigma \in \bar\Gamma^\sigma g_\sigma$ if and only if $g' \in \bar\Gamma g$. Therefore $\mathrm{Comm}(\bar\Gamma^\sigma)/\bar\Gamma^\sigma \simeq \mathrm{Comm}(\bar\Gamma)/\bar\Gamma$ is infinite too, which by a major theorem of Margulis implies that $\bar\Gamma^\sigma$ is arithmetic, providing an alternative proof of a result of Kazhdan.
Thus $\mathrm{Aut}(\mathbb{C})$ acts on the set of isomorphism classes of cocompact arithmetic quotients $Y_{\bar\Gamma}$, or equivalently, on the set of equivalence classes of cocompact arithmetic subgroups $\bar\Gamma$ (up to conjugation by an element of $\mathrm{PU}(2,1)$). The latter set is countable for the following reason. The group $\mathrm{U}(2,1)$ has only countably many $\mathbb{Q}$-forms, classified by central simple algebras of dimension 9 over M, endowed with an involution of a second kind and verifying some conditions at infinity (see [PR, pp. 87-88]). Moreover, there are only countably many

arithmetic subgroups for a given $\mathbb{Q}$-form, since those are all finitely generated and contained in their common commensurator, which is countable.
Finally, by [G, Corollary 2.13], the fact that $Y_{\bar{\Gamma}}$ has a countable orbit under the action of $\mathrm{Aut}(\mathbb{C})$ is equivalent to $Y_{\bar{\Gamma}}$ being defined over a number field. □

It is a well known fact that any orbifold admits a finite cover which is a manifold. In view of Lemma 1.1, the two lemmas below provide such covers explicitly for arithmetic quotients.

DEFINITION 1.3. For every ideal $\mathfrak{N} \subset \mathfrak{O}$ we define the congruence subgroup $\Gamma(\mathfrak{N})$ (resp. $\Gamma_0(\mathfrak{N})$, resp. $\Gamma_1(\mathfrak{N})$) as the kernel (resp. the inverse image of upper triangular, resp. upper unipotent, matrices) of the composite homomorphism:

$$G(\mathfrak{o}) \hookrightarrow \mathrm{GL}(n+1, \mathfrak{O}) \to \mathrm{GL}(n+1, \mathfrak{O}/\mathfrak{N}).$$

The following lemma is well-known (see [H1, Lemma 4.3]).

LEMMA 1.4. *For any integer $N > 2$ the group $\Gamma(N)$ is neat.*

LEMMA 1.5. *Suppose that $n = 2$ and that M is an imaginary quadratic field of fundamental discriminant $-D \notin \{-3, -4, -7, -8, -24\}$. Then $\Gamma_1(\sqrt{-D}\mathfrak{O})$ is neat.*

Proof. Suppose that the subgroup of $\mathbb{C}^\times$ generated by the eigenvalues of some $\gamma \in \Gamma_1(\sqrt{-D}\mathfrak{O})$ contains a non-trivial root of unity. Note first that $\det(\gamma) \in \mathfrak{O}^\times \cap (1 + \sqrt{-D}\mathfrak{O}) = \{1\}$.
If γ is elliptic then it is necessarily of finite order. Otherwise γ fixes a boundary point of $\mathcal{H}^2_{\mathbb{C}} \subset \mathbb{P}^2(\mathbb{C})$ and is therefore conjugated in $\mathrm{GL}(3, \mathbb{C})$ to a matrix of the form $\begin{pmatrix} \bar{\alpha} & * & * \\ 0 & \beta & * \\ 0 & 0 & \alpha^{-1} \end{pmatrix}$, where β is necessarily a root of unity. If $\beta = 1$, then $\det(\gamma) = 1$ implies that $\alpha \in \mathbb{R}$, leading to $\alpha = -1$. Hence, in all cases, one may assume γ has a non-trivial root of unity ζ as an eigenvalue.
By the Cayley-Hamilton theorem we have $[M(\zeta) : M] \leq 3$ and since $D \neq 7$ we may assume (after possibly raising γ to some power) that ζ has order 2 or 3. By the congruence condition, each prime p dividing D has to divide also the norm of $\zeta - 1$, hence D can be divisible only by the primes 2 or 3. Thus $D \in \{3, 4, 8, 24\}$, leading to a contradiction. □

2. Irregularity of arithmetic varieties

Let $z \mapsto \bar{z}$ be the non-trivial automorphism of M/F and let ω be the quadratic character of M/F, viewed as a Hecke character of F. Put $M^1 = \{z \in M^\times \mid z\bar{z} = 1\}$, which we will view as an algebraic torus over F and denote by $\mathbb{A}^1_M$ its $\mathbb{A}_F$-points.
We denote by $q(X)$ the *irregularity* of X, given by the dimension of $\mathrm{H}^0(X, \Omega^1_X)$.

2.1. AUTOMORPHIC FORMS CONTRIBUTING TO THE IRREGULARITY. Fix a maximal compact subgroup $K_\infty \simeq (\mathrm{U}(n) \times \mathrm{U}(1)) \times \mathrm{U}(n+1)^{d-1}$ of the real linear Lie group $G_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R}) \simeq \mathrm{U}(n, 1) \times \mathrm{U}(n+1)^{d-1}$. Let $\Gamma \subset G(F)$ be a lattice such that $\bar{\Gamma}$ is torsion free.

Since Y_Γ is the Eilenberg-MacLane space of $\overline{\Gamma}$, there is a decomposition:

$$\mathrm{H}^1(Y_\Gamma,\mathbb{C}) \simeq \mathrm{H}^1(\bar{\Gamma},\mathbb{C}) \simeq \bigoplus_{\pi_\infty} \mathrm{H}^1(\mathrm{Lie}(G_\infty),K_\infty;\pi_\infty)^{\oplus m(\pi_\infty,\Gamma)}, \tag{2}$$

where π_∞ runs over irreducible unitary representations of G_∞ occurring in the discrete spectrum of $\mathrm{L}^2(\Gamma\backslash G_\infty)$ with multiplicity $m(\pi_\infty,\Gamma)$, and $\mathrm{H}^*(\mathrm{Lie}(G_\infty),K_\infty;\pi_\infty)$ is the relative Lie algebra cohomology. When Γ is cocompact, the entire L^2-spectrum is discrete and this decomposition follows from [BW, XIII]. When Γ is non-cocompact, one gets by [BC, §4.4-4.5] such a decomposition, but only for the L^2-cohomology of Y_Γ. However, one knows (see [MR, §1]) that $\mathrm{H}^1(Y_\Gamma,\mathbb{C})$ is isomorphic to the middle intersection cohomology (in degree 1) of Y_Γ^*, which is in turn isomorphic to the L^2-cohomology (in degree 1) of Y_Γ.

By [BW, VI.4.11] there are exactly two irreducible non-tempered unitary representations of $\mathrm{SU}(n,1)$ with trivial central character, denoted $J_{1,0}$ and $J_{0,1}$, each of whose relative Lie algebra cohomology in degree 1 does not vanish and is in fact one dimensional. Since $\mathrm{U}(n,1)$ is the product of its center with $\mathrm{SU}(n,1)$, $J_{1,0}$ and $J_{0,1}$ can be uniquely extended to representations π^+ and π^-, say, of $\mathrm{U}(n,1)$ with trivial central characters (when $n=2$ those are the representations J^+ and J^- from [R1, p.178]). It follows that at the distinguished Archimedean place ι, where $G(F_\iota)=\mathrm{U}(n,1)$, we have

$$\mathrm{H}^1(\mathrm{Lie}(\mathrm{U}(n,1)),\mathrm{U}(n)\times\mathrm{U}(1);\pi_\iota) = \begin{cases} \mathbb{C}\ , \text{ if } \pi_\iota=\pi^+ \text{ or } \pi^-, \\ 0\ , \text{ otherwise.} \end{cases}$$

Moreover the only irreducible unitary representation with non-zero relative Lie algebra cohomology in degree 0 is the trivial representation $\mathbb{1}$, which does not contribute in degree 1; in particular $\pi^+\neq\mathbb{1}$ and $\pi^-\neq\mathbb{1}$. This allows us to deduce from (2) the following formula

$$\dim_{\mathbb{C}}\mathrm{H}^1(Y_\Gamma,\mathbb{C}) = m(\pi^+\otimes\mathbb{1}^{\otimes d-1},\Gamma)+m(\pi^-\otimes\mathbb{1}^{\otimes d-1},\Gamma), \tag{3}$$

where π^+ and π^- are viewed as representations of $G(F_\iota)=\mathrm{U}(n,1)$ and $\mathbb{1}^{\otimes d-1}$ denotes the trivial representation of $\mathrm{U}(n+1)^{d-1}$.

By [MR, §1], $\mathrm{H}^1(Y_\Gamma,\mathbb{C})$ is isomorphic to $\mathrm{H}^1(X_\Gamma,\mathbb{C})$, hence admits a pure Hodge structure of weight 1 and its dimension is given by $2q(X_\Gamma)$. In particular, the natural map $\mathrm{H}^0(X_\Gamma,\Omega^1_{X_\Gamma})\to\mathrm{H}^0(Y_\Gamma,\Omega^1_{Y_\Gamma})$ is an isomorphism, *i.e.*,

$$q(Y_\Gamma)=q(X_\Gamma). \tag{4}$$

It is known that $\pi^+\otimes\mathbb{1}^{\otimes d-1}$ (resp. $\pi^-\otimes\mathbb{1}^{\otimes d-1}$) contributes to $\mathrm{H}^0(Y_\Gamma,\Omega^1_{Y_\Gamma})$ (resp. $\mathrm{H}^1(Y_\Gamma,\Omega^0_{Y_\Gamma})$). Since the latter two groups have the same dimension, it follows from (3) that

$$q(Y_\Gamma)=m(\pi^+\otimes\mathbb{1}^{\otimes d-1},\Gamma)=m(\pi^-\otimes\mathbb{1}^{\otimes d-1},\Gamma). \tag{5}$$

We will now focus on the case when Γ is a congruence subgroup and switch to the adelic setting which is better suited for computing the irregularity. For any

open compact subgroup K of $G(\mathbb{A}_{F,f})$, where $\mathbb{A}_{F,f}$ denotes the ring of finite adeles of F, we consider the adelic quotient

$$Y_K = G(F)\backslash G(\mathbb{A}_F)/KK_\infty. \tag{6}$$

Let $G^1 = \ker(\det : G \to M^1)$ be the derived group of G. Since G^1 is simply connected and G^1_∞ is non-compact, $G^1(F)$ is dense in $G^1(\mathbb{A}_{F,f})$ by strong approximation (see [PR, Theorem 7.12]). It follows that the group of connected components of Y_K is isomorphic to the idele class group:

$$\pi_0(Y_K) \simeq \mathbb{A}^1_M/M^1 \det(K)M^1_\infty. \tag{7}$$

To describe each connected component of Y_K, choose $t_i \in G(\mathbb{A}_{F,f})$, $1 \leq i \leq h$, such that $(\det(t_i))_{1\leq i\leq h}$ forms a complete set of representatives of $\mathbb{A}^1_M/M^1 \det(K)M^1_\infty$, and let $\Gamma_i = G(F) \cap t_i K t_i^{-1} G_\infty$. Then

$$G(F)\backslash G(\mathbb{A}_F)/K = \coprod_{i=1}^{h} \Gamma_i\backslash G_\infty \text{ and } Y_K = \coprod_{i=1}^{h} Y_{\Gamma_i}. \tag{8}$$

Therefore (2) and (5) can be rewritten as:

$$\mathrm{H}^1(Y_K, \mathbb{C}) \simeq \bigoplus_{\pi=\pi_\infty\otimes\pi_f} \left(\mathrm{H}^1(\mathrm{Lie}(G_\infty), K_\infty; \pi_\infty) \otimes \pi_f^K\right)^{\oplus m(\pi)} \text{ and} \tag{9}$$

$$q(Y_K) = \sum_{\substack{\pi=\pi_\infty\otimes\pi_f \\ \pi_\infty=\pi_\iota\otimes\mathbb{1}^{\otimes d-1}, \pi_\iota\simeq\pi^+}} m(\pi)\dim(\pi_f^K) = \sum_{\substack{\pi=\pi_\infty\otimes\pi_f \\ \pi_\infty=\pi_\iota\otimes\mathbb{1}^{\otimes d-1}, \pi_\iota\simeq\pi^-}} m(\pi)\dim(\pi_f^K), \tag{10}$$

where π runs over all automorphic representation of $G(\mathbb{A}_F)$ occurring discretely, with multiplicity $m(\pi)$, in $\mathrm{L}^2(G(F)\backslash G(\mathbb{A}_F))$.

2.2. IRREGULARITY GROWTH. Non-vanishing of $q(Y_\Gamma)$ for sufficiently small congruence subgroups is known by a theorem of Shimura [S2, Theorem 8.1], extending earlier works of Kazhdan and Borel-Wallach [BW, VIII]. Our Diophantine results require however the stronger assumption that $q(Y_\Gamma) > n$, which we establish as a corollary of the next proposition.

PROPOSITION 2.1. *For every open compact subgroup $K \subset G(\mathbb{A}_{F,f})$ such that $q(Y_K) \neq 0$ there exist infinitely many primes $\mathfrak{p}$ of F for which one can find an explicit finite index subgroup $K' \subset K$ differing form K only at $\mathfrak{p}$, such that $\pi_0(Y_{K'}) = \pi_0(Y_K)$ and $q(Y_{K'}) > q(Y_K)$.*

Proof. Since $q(Y_K) \neq 0$ by assumption, formula (10) implies that there exists an automorphic representation π with $\pi_\infty = \pi_\iota \otimes \mathbb{1}^{\otimes d-1}$, $\pi_\iota \simeq \pi^+$, such that $m(\pi) \neq 0$ and $\pi_f^K \neq 0$.

Let $\mathfrak{p}$ be a prime of F which splits in M, so that $G(F_\mathfrak{p}) = \mathrm{GL}(n+1, F_\mathfrak{p})$. Assume that $K = K^0_\mathfrak{p} \times K^{(\mathfrak{p})}$ where $K^0_\mathfrak{p} = \mathrm{GL}(n+1, \mathfrak{o}_\mathfrak{p})$ is the standard maximal compact subgroup of $G(F_\mathfrak{p})$ and $K^{(\mathfrak{p})}$ is the part of K away from $\mathfrak{p}$. In particular $\pi_\mathfrak{p}$ is unramified. Moreover $\pi_\mathfrak{p}$ is a unitary representation, since it is a local component of an automorphic representation. By the main

result of [T], $\pi_{\mathfrak{p}}$ is then the full induced representation of $\mathrm{GL}(n+1, F_{\mathfrak{p}})$ from an unramified character μ of a parabolic subgroup $P(F_{\mathfrak{p}})$.
We claim that, in our case, P is a proper parabolic subgroup. Otherwise $\pi_{\mathfrak{p}}$ will be one dimensional, hence $G^1(F_{\mathfrak{p}})$ will act trivially. Since by strong approximation $G^1(F)G^1(F_{\mathfrak{p}})$ is dense in $G^1(\mathbb{A}_F)$, the latter will act trivially on any smooth vector in π, contradicting the fact that $\pi_\iota \not\simeq \mathbb{1}$.
Let $\mathbb{F}_q = \mathfrak{o}/\mathfrak{p}$ be the residue field of $F_{\mathfrak{p}}$ and denote by $P(\mathbb{F}_q)$ the corresponding parabolic subgroup of $G(\mathbb{F}_q)$. Let $K_{0,P}(\mathfrak{p})$ is the inverse image of $P(\mathbb{F}_q)$ under the reduction modulo $\mathfrak{p}$ homomorphism $\mathrm{GL}(n+1, \mathfrak{o}_{\mathfrak{p}}) \to \mathrm{GL}(n+1, \mathbb{F}_q)$.
Consider $K' = K_{0,P}(\mathfrak{p})K^{(\mathfrak{p})}$. Since $\det(K') = \det(K)$, (7) implies that $\pi_0(Y_{K'}) = \pi_0(Y_K)$. Moreover, formula (10) implies that

$$q(Y_{K'}) \geq q(Y_K) + \dim(\pi_{\mathfrak{p}}^{K_{0,P}(\mathfrak{p})}) - \dim(\pi_{\mathfrak{p}}^{K_{\mathfrak{p}}^0}) = q(Y_K) + \dim(\pi_{\mathfrak{p}}^{K_{0,P}(\mathfrak{p})}) - 1,$$

hence it suffices to show that the $K_{0,P}(\mathfrak{p})$-invariants in $\pi_{\mathfrak{p}}$ form at least a 2-dimensional space. We claim that we even have

$$\dim(\pi_{\mathfrak{p}}^{K_{0,P}(\mathfrak{p})}) \geq n+1. \tag{11}$$

Indeed, since μ is unramified, its restriction to $P \cap K_{\mathfrak{p}}^0$ is trivial. Therefore by the Iwasawa decomposition $G(F_{\mathfrak{p}}) = P(F_{\mathfrak{p}}) \cdot K_{\mathfrak{p}}^0$ the restriction of $\pi_{\mathfrak{p}}$ to $K_{\mathfrak{p}}^0$ is isomorphic to $\mathrm{Ind}_{P(F_{\mathfrak{p}})\cap K_{\mathfrak{p}}^0}^{K_{\mathfrak{p}}^0}(\mathbb{1})$.

The subspace of $K_{0,P}(\mathfrak{p})$-invariant vectors in $\mathrm{Ind}_{P(F_{\mathfrak{p}})\cap K_{\mathfrak{p}}^0}^{K_{\mathfrak{p}}^0}(\mathbb{1})$ identifies naturally with the space of $\mathbb{C}$-valued functions on the set:

$$\left(P(F_{\mathfrak{p}}) \cap K_{\mathfrak{p}}^0\right) \backslash K_{\mathfrak{p}}^0 / K_{0,P}(\mathfrak{p}) \simeq P(\mathbb{F}_q) \backslash G(\mathbb{F}_q) / P(\mathbb{F}_q).$$

and the number of such double cosets is the number of double cosets of the Weyl group of G relative to the subgroup attached to P. We may assume, for getting a lower bound, that P is maximal (and proper). The smallest number appears for P of type $(n, 1)$ and it is $n+1$. The claim follows. □

Corollary 2.2. *For every arithmetic subgroup $\Gamma \subset G(F)$ there exists an explicit subgroup Γ' of finite index in Γ such that $q(Y_{\Gamma'}) > q(Y_\Gamma)$.*

Proof. By [S2, Theorem 8.1] there exists an open compact subgroup K of $G(\mathbb{A}_{F,f})$ such that $q(Y_K) \neq 0$ (one might take the principal congruence subgroup of level $2N$, which is included in the index 4 subgroup of the principal congruence subgroup of level N considered by Shimura). Denote by h the cardinality of $\pi_0(Y_K)$. Applying recursively Proposition 2.1 yields a finite index subgroup $K' \subset K$, such that $\pi_0(Y_{K'}) = \pi_0(Y_K)$ and

$$q(Y_{K'}) > h \cdot q(Y_\Gamma).$$

Write $Y_{K'} = \coprod_{i=1}^h Y_{\Gamma_i'}$ as in (8), and let $\Gamma' = \cap_{i=1}^h \Gamma_i'$. Since the irregularity cannot decrease by going to a finite cover, one has:

$$q(Y_{\Gamma' \cap \Gamma}) \geq q(Y_{\Gamma'}) \geq \max_{1 \leq i \leq h} q(Y_{\Gamma_i'}) \geq \frac{1}{h} \sum_{i=1}^h q(Y_{\Gamma_i'}) = \frac{1}{h} q(Y_{K'}) > q(Y_\Gamma).$$

□

One can simplify the final step of the proof above and use any Γ'_i instead of $\cap_{i=1}^h \Gamma'_i$, since Shimura's theory of canonical models implies that the connected components of $Y_{K'}$ are all Galois conjugates, hence share the same irregularity.

3. Irregularity of arithmetic surfaces

The positivity of $q(Y_\Gamma)$ is an essential ingredient in the proof of our Diophantine results.

The starting point for the arithmetic application of this paper was our knowledge that Rogawski's classification [R1, R2] of cohomological automorphic forms on G, combined with some local representation theory, would allow us to compute $q(Y_\Gamma)$ precisely and show that it does not vanish for some explicit congruence subgroups Γ. Marshall [Ma] gives sharp asymptotic bounds for $q(Y_\Gamma)$ when Γ shrinks, also by using Rogawski's theory.

In this section we assume that $n = 2$.

3.1. Rogawski's theory. Rogawski [R1, R2] gives an explicit description, in terms of global Arthur packets, of the automorphic representations π of $G(\mathbb{A}_F)$ occurring in (10), which we will now present.

Let T denote the maximal torus of the standard upper-triangular Borel subgroup B of G.

Let G' denote the quasi-split unitary group associated to M/F, so that G is an inner form of G'. Note that $G_v \simeq G'_v$ for any finite place v and that $G \simeq G'$ only for $d = 1$.

Let λ be a unitary Hecke character of M whose restriction to F is ω, and let ν be a unitary character of $\mathbb{A}_M^1/M^1$.

At a place v of F which does not split in M, which includes any Archimedean v, the local Arthur packet $\Pi'(\lambda_v, \nu_v)$ consists of a square-integrable representation $\pi_s(\lambda_v, \nu_v)$ and a non-tempered representation $\pi_n(\lambda_v, \nu_v)$ of $G'(F_v)$. These constituents of the packet can be described (see [R1, §12.2]) as the unique subrepresentation and the corresponding (Langlands) quotient representation of the induction of the character of $B(F_v)$ which is trivial on the unipotent subgroup and given on $T(F_v)$ by:

$$(\bar\alpha, \beta, \alpha^{-1}) \mapsto \lambda_v(\bar\alpha)|\alpha|_{M_v}^{3/2}\nu_v(\beta), \text{ where } \alpha \in M_v^\times, \beta \in M_v^1. \tag{12}$$

If one considers unitary induction, then one has to divide the above character by the square root of the modular character of $B(F_v)$, that is to say by $(\bar\alpha, \beta, \alpha^{-1}) \mapsto |\alpha|_{M_v}$.

At any finite place v of F which splits in M, $G_v \simeq G'_v$ also splits and is isomorphic to $\mathrm{GL}(3, F_v)$. The local Arthur packet $\Pi'(\lambda_v, \nu_v)$ has a unique element $\pi_n(\lambda_v, \nu_v)$ which is induced from the character:

$$\left(\begin{array}{cc|c} & h_2 & \begin{matrix}**\end{matrix} \\ \hline 0 & 0 & h_1 \end{array}\right) \mapsto \lambda_v(\det(h_2))|\det(h_2)|_v^{3/2}\nu_v(h_1)$$

of the maximal parabolic of type $(2,1)$ in $\mathrm{GL}(3,F_v)$ (see [R2, §1]).
For almost all v, $\pi_n(\lambda_v,\nu_v)$ is necessarily unramified. We set

$$\Pi'(\lambda,\nu)=\{\otimes_v\pi_v|\pi_v\in\Pi'(\lambda_v,\nu_v)\text{ for all }v\,,\\ \text{and }\pi_v\simeq\pi_n(\lambda_v,\nu_v)\text{ for almost all }v\}\,.$$

Recall that a CM type Φ on M is the choice, for each Archimedean place v of F, of an isomorphism $M\otimes_{F,v}\mathbb{R}\simeq\mathbb{C}$.

Definition 3.1. Let Ξ denote the set of pairs (λ,ν) where λ is a unitary Hecke character of M whose restriction to F is ω, and ν is a unitary character of $\mathbb{A}_M^1/M^1$, such that

$$\lambda_\infty(z)=\prod_{v\in\Phi}\frac{\bar z_v}{|z_v|}\text{ , for all }z\in M_\infty\text{ and}\tag{13}$$

$$\nu_\infty(z)=\prod_{v\in\Phi}z_v\text{ , for all }z\in M_\infty^1,$$

for some CM type Φ on M.

Theorem 3.2 (Rogawski [R1, R2]). (i) *For every $(\lambda,\nu)\in\Xi$, $\Pi'(\lambda,\nu)$ is a global Arthur packet for G' such that for all infinite v, $\pi_n(\lambda_v,\nu_v)=\pi^+$ or π^-.*

(ii) *$\Pi'(\lambda,\nu)$ can be transferred to an Arthur packet $\Pi(\lambda,\nu)$ on G such that $\Pi(\lambda_v,\nu_v)=\{\mathbb{1}\}$ at all the Archimedean places $v\neq\iota$, and $\Pi(\lambda_v,\nu_v)=\Pi'(\lambda_v,\nu_v)$ at the remaining places.*

(iii) *Denote by $W(\lambda\nu_M)\in\{\pm1\}$ the root number of the Hecke character $\lambda\nu_M$, where $\nu_M(z)=\nu(\bar z/z)$ for $z\in\mathbb{A}_M^\times$, and by $s(\pi)$ the number of finite places v such that $\pi_v\simeq\pi_s(\lambda_v,\nu_v)$. Then*

$\pi\in\Pi(\lambda,\nu)$ is automorphic if and only if $W(\lambda\nu_M)=(-1)^{d-1+s(\pi)}$.

Moreover, in this case the global multiplicity $m(\pi)$ is 1.

(iv) *Any automorphic representation π of $G(\mathbb{A}_F)$ such that $\pi_\iota=\pi^+$ or π^-, and $\pi_v=\mathbb{1}$ at all the Archimedean places $v\neq\iota$, belongs to $\Pi(\lambda,\nu)$ for some $(\lambda,\nu)\in\Xi$.*

Proof. Let $H=\mathrm{U}(2)\times\mathrm{U}(1)$ be the unique elliptic endoscopic group, shared by G' and all its inner forms over F. The embedding of L-groups ${}^LH\hookrightarrow{}^LG={}^LG'$ depends on the choice of a Hecke character μ of M, whose restriction to F is ω, and allows one to transfer discrete L-packets on H to automorphic L-packets on G (see [R2, §13.3]). The character μ being fixed, any pair of characters $(\lambda,\nu)\in\Xi$ uniquely determines a (one-dimensional) character of H, whose endoscopic transfer is $\Pi'(\lambda,\nu)$ (see [R2, §1]).
Denote by W_F (resp. W_M) the global Weil group of F (resp. M). By *loc.cit.*, the restriction to W_M of the global Arthur parameter

$$W_F\times\mathrm{SL}(2,\mathbb{C})\to{}^LG=\mathrm{GL}(3,\mathbb{C})\rtimes\mathrm{Gal}(M/F)$$

of $\Pi'(\lambda,\nu)$ is given by the 3-dimensional representation $(\lambda\otimes\mathrm{St})\oplus(\nu_M\otimes\mathbb{1})$, where St (resp. $\mathbb{1}$) is the standard 2-dimensional (resp. trivial) representation

of $\mathrm{SL}(2,\mathbb{C})$. By [La, p.62] the restrictions to $\mathbb{C}^\times$ of the Langlands parameters of π^+ and π^- are given by $z \mapsto \begin{pmatrix} \bar{z} & 0 & 0 \\ 0 & z/\bar{z} & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}$ and its complex conjugate, hence for every Archimedean place v one has $\pi_n(\lambda_v, \nu_v) = \pi^+$ or π^-, depending on the choice of isomorphism $M \otimes_{F,v} \mathbb{R} \simeq \mathbb{C}$ in the CM type Φ. It follows that for every Archimedean place v, $\Pi'(\lambda_v, \nu_v)$ is a packet containing a discrete series representation of G'_v, and thus by [R1, §14.4], there will be a corresponding Arthur packet $\Pi(\lambda, \nu)$ of representations of $G(\mathbb{A}_F)$ such that at any Archimedean place $v \neq \iota$, $\Pi(\lambda_v, \nu_v)$ is a singleton consisting of a finite-dimensional representation of the compact real group $G(F_v) = \mathrm{U}(3)$. In the notation of [R2, p.397] the representations π^+ and π^- have parameters $(r,s) = (1,-1)$ and $(r,s) = (0,1)$, respectively, and hence, by the recipe on the same page, the highest weight of the associated finite-dimensional representation equals $(1,0,-1)$. Therefore at every Archimedean $v \neq \iota$ we have $\Pi(\lambda_v, \nu_v) = \{\mathbb{1}\}$.
So far we have established (i) and (ii), while (iii) is the content of [R2, Theorem 1.1].
Conversely, any π as in (iv) is discrete, hence belongs to an Arthur packet Π on G, which can be transferred to an Arthur packet Π' on G' (see [R1, §14.4 and Proposition 14.6.2]). By definition, $\Pi_v = \Pi'_v$ at $v = \iota$ and at all the finite places v (where, as noted earlier, $G_v = G'_v$). In particular π^+ or π^- belongs to $\Pi_\iota = \Pi'_\iota$, hence Π' arises by endoscopy from H, that is to say equals $\Pi'(\lambda, \nu)$ for some unitary Hecke character λ of M whose restriction to F is ω, and some unitary character of $\mathbb{A}^1_M / M^1$ (see [R1, Theorem 13.3.6]). Since $\Pi_v = \{\mathbb{1}\}$ for all the Archimedean places $v \neq \iota$, by the above mentioned recipe $\Pi(\lambda_v, \nu_v)$ contains either π^+ or π^-, implying that $(\lambda, \nu) \in \Xi$ (see Definition 3.1). □

3.2. IRREGULARITY OF THE CONNECTED COMPONENTS. Let K be an open compact subgroup of $G(\mathbb{A}_{F,f})$. Using Theorem 3.2(iii) one can transform (10) into the formula:

$$4q(Y_K) = \sum_{(\lambda,\nu)\in\Xi} \sum_{\pi\in\Pi(\lambda,\nu)} \dim(\pi_f^K)(W(\lambda\nu_M) + (-1)^{d-1+s(\pi)}). \tag{14}$$

We will now deduce a similar formula for the irregularity of the connected component of identity Y_Γ of Y_K, where $\Gamma = G(F) \cap KG_\infty$.
Recall (see (7)) that $\pi_0(Y_K) \simeq \mathbb{A}^1_M / M^1 \det(K) M^1_\infty$, and denote by $\widehat{\pi_0(Y_K)}$ its (finite, abelian) group of characters. Consider the free action of $\widehat{\pi_0(Y_K)}$ on the set Ξ given by $(\chi, (\lambda, \nu)) \mapsto (\lambda\chi_M^{-1}, \nu\chi)$, and denote by $\Xi/\widehat{\pi_0(Y_K)}$ the quotient set. Since for any $\pi \in \Pi(\lambda,\nu)$ and any $\chi \in \widehat{\pi_0(Y_K)}$ one has $\pi \otimes \chi \in \Pi(\lambda\chi_M^{-1}, \nu\chi)$, the group $\widehat{\pi_0(Y_K)}$ acts freely on the set of automorphic representations contributing to $q(Y_K)$. Moreover this action preserves $\lambda\nu_M$, $s(\pi)$ and the dimension of π_f^K. Hence, in the notations of (8), for any $1 \leq i \leq h$, the image of the composite map

$$\mathrm{H}^1(\mathrm{Lie}(G_\infty), K_\infty; \pi_\infty) \otimes \pi_f^K \to \mathrm{H}^1(Y_K, \mathbb{C}) \to \mathrm{H}^1(Y_{\Gamma_i}, \mathbb{C}),$$

where the first map comes from (9) and the second from the inclusion $Y_{\Gamma_i} \subset Y_K$, does not change when replacing π by $\pi \otimes \chi$ for any $\chi \in \widehat{\pi_0(Y_K)}$.
It follows that $q(Y_{\Gamma_i}) \leq \frac{1}{h} q(Y_K)$ for all $1 \leq i \leq h$. Since $\sum_{i=1}^{h} q(Y_{\Gamma_i}) = q(Y_K)$, we deduce that $q(Y_{\Gamma_i}) = \frac{1}{h} q(Y_K)$ for all $1 \leq i \leq h$. This establishes the formula:

$$4q(Y_\Gamma) = \sum_{(\lambda,\nu) \in \Xi/\widehat{\pi_0(Y_K)}} \sum_{\pi \in \Pi(\lambda,\nu)} \dim(\pi_f^K)(W(\lambda\nu_M) + (-1)^{d-1+s(\pi)}). \tag{15}$$

This formula shows the importance of calculating $\dim(\pi_f^K)$ which, when K is of the form $\prod_v K_v$ with v running over all the finite places of F, can be reduced to a local computation of $\dim(\pi_v^{K_v})$. This will be taken up in the following section at places v where K_v is not the hyperspecial maximal compact subgroup K_v^0.

3.3. Levels of induced representations. Let $\mathfrak{p}$ be a prime of F divisible by a unique prime $\mathfrak{P}$ of M and let $\mathbb{F}_q$ be the residue field $\mathfrak{o}/\mathfrak{p}$. In this section we exhibit open compact subgroups $K_\mathfrak{p}$ of $G(F_\mathfrak{p})$ for which $\pi_n(\lambda_\mathfrak{p}, \nu_\mathfrak{p})$ (resp. $\pi_s(\lambda_\mathfrak{p}, \nu_\mathfrak{p})$) admit a non-zero $K_\mathfrak{p}$-invariant subspace, and compute in some cases the exact dimension of this space.
For every integer $m \geq 1$, we define the open compact subgroup $K(\mathfrak{P}^m)$ (resp. $K_0(\mathfrak{P}^m)$, resp. $K_1(\mathfrak{P}^m)$) of $G(F_\mathfrak{p})$ as the kernel (resp. the inverse image of upper triangular, resp. upper unipotent, matrices) of the composite homomorphism:

$$G(\mathfrak{o}_\mathfrak{p}) \hookrightarrow \mathrm{GL}(3, \mathfrak{O}_\mathfrak{P}) \to \mathrm{GL}(3, \mathfrak{O}/\mathfrak{P}^m). \tag{16}$$

Lemma 3.3. *Let $m \in \mathbb{Z}_{>0}$ be such that the character (12) is trivial on $K_1(\mathfrak{P}^m) \cap T(F_\mathfrak{p})$. Then both $\pi_n(\lambda_\mathfrak{p}, \nu_\mathfrak{p})$ and $\pi_s(\lambda_\mathfrak{p}, \nu_\mathfrak{p})$ have non-zero fixed vectors under $K_1(\mathfrak{P}^m)$.*

Proof. Let J denote the Jacquet functor sending admissible $G(F_\mathfrak{p})$-representations to admissible $T(F_\mathfrak{p})$-representations. The Jacquet functor is exact and its basic properties imply:

$$\begin{aligned} J(\pi_s(\lambda_\mathfrak{p}, \nu_\mathfrak{p})) : (\bar{\alpha}, \beta, \alpha^{-1}) &\mapsto \lambda_\mathfrak{p}(\bar{\alpha})\nu_\mathfrak{p}(\beta)|\alpha|_{M_\mathfrak{p}}^{3/2} = \lambda_\mathfrak{p}(\bar{\alpha})\nu_\mathfrak{p}(\beta)|\alpha|_{M_\mathfrak{p}}^{1/2}|\alpha|_{M_\mathfrak{p}}, \\ J(\pi_n(\lambda_\mathfrak{p}, \nu_\mathfrak{p})) : (\bar{\alpha}, \beta, \alpha^{-1}) &\mapsto \lambda_\mathfrak{p}(\bar{\alpha})\nu_p(\beta)|\alpha|_{M_\mathfrak{p}}^{1/2} = \lambda_\mathfrak{p}(\bar{\alpha})\nu_\mathfrak{p}(\beta)|\alpha|_{M_\mathfrak{p}}^{-1/2}|\alpha|_{M_\mathfrak{p}}. \end{aligned} \tag{17}$$

One knows that $K_1(\mathfrak{P}^m)$ admits an Iwahori decomposition:

$$K_1(\mathfrak{P}^m) = (K_1(\mathfrak{P}^m) \cap N(F_\mathfrak{p})) \cdot (K_1(\mathfrak{P}^m) \cap T(F_\mathfrak{p})) \cdot (K_1(\mathfrak{P}^m) \cap \bar{N}(F_\mathfrak{p})),$$

where $N(F_\mathfrak{p})$ (resp. $\bar{N}(F_\mathfrak{p})$) denotes the unipotent of the standard (resp. opposite) Borel containing $T(F_\mathfrak{p})$. This is proved for the principal congruence subgroup $K(\mathfrak{P}^m)$ in [C, Proposition 1.4.4] and the extension to $K_1(\mathfrak{P}^m)$ is straightforward. Now by the proof of [C, Proposition 3.3.6], given any admissible $G(F_\mathfrak{p})$-representation V, one has a canonical surjection:

$$V^{K_1(\mathfrak{P}^m)} \twoheadrightarrow J(V)^{K_1(\mathfrak{P}^m) \cap T(F_\mathfrak{p})}.$$

Since both characters in (17) are trivial on $K_1(\mathfrak{P}^m) \cap T(F_\mathfrak{p})$, the claim follows. □

LEMMA 3.4. *Suppose that $\mathfrak{p}$ is inert in M and that $(\lambda_\mathfrak{p}, \nu_\mathfrak{p})$ is unramified. Then the dimension of the $K_0(\mathfrak{p})$-fixed subspace of both $\pi_s(\lambda_\mathfrak{p}, \nu_\mathfrak{p})$ and $\pi_n(\lambda_\mathfrak{p}, \nu_\mathfrak{p})$ is 1. Moreover the dimension of the $K(\mathfrak{p})$-fixed subspace of $\pi_s(\lambda_\mathfrak{p}, \nu_\mathfrak{p})$ (resp. $\pi_n(\lambda_\mathfrak{p}, \nu_\mathfrak{p})$) is q^3 (resp. 1).*

Proof. Since $(\lambda_\mathfrak{p}, \nu_\mathfrak{p})$ is unramified, restriction to the standard hyperspecial maximal compact subgroup $K^0_\mathfrak{p}$ of $G(F_\mathfrak{p})$ yields, by Iwasawa decomposition $G(F_\mathfrak{p}) = B(F_\mathfrak{p}) \cdot K^0_\mathfrak{p}$, the following exact sequence:

$$0 \to \pi_s(\lambda_\mathfrak{p}, \nu_\mathfrak{p})_{|K^0_\mathfrak{p}} \to \mathrm{Ind}_{B(F_\mathfrak{p})\cap K^0_\mathfrak{p}}^{K^0_\mathfrak{p}}(\mathbb{1}) \to \pi_n(\lambda_\mathfrak{p}, \nu_\mathfrak{p})_{|K^0_\mathfrak{p}} \to 0.$$

The subspace of $K(\mathfrak{p})$-invariant vectors in $\mathrm{Ind}_{B(F_\mathfrak{p})\cap K^0_\mathfrak{p}}^{K^0_\mathfrak{p}}(\mathbb{1})$ identifies naturally with the space of $\mathbb{C}$-valued functions on the set:

$$\left(B(F_\mathfrak{p}) \cap K^0_\mathfrak{p}\right) \backslash K^0_\mathfrak{p} / K(\mathfrak{p}) \simeq B(\mathbb{F}_q) \backslash G(\mathbb{F}_q),$$

on which $K^0_\mathfrak{p}/K(\mathfrak{p}) = G(\mathbb{F}_q)$ acts by right translation. By the Iwahori decomposition, since $G(\mathbb{F}_q)$ has rank 1, the representation $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\mathbb{1})$ has exactly two irreducible constituents which are the trivial representation and the Steinberg representation, implying that both $\pi_n(\lambda_\mathfrak{p}, \nu_\mathfrak{p})^{K_0(\mathfrak{p})}$ and $\pi_s(\lambda_\mathfrak{p}, \nu_\mathfrak{p})^{K_0(\mathfrak{p})}$ are one-dimensional. Since $\pi_s(\lambda_\mathfrak{p}, \nu_\mathfrak{p})^{K^0_\mathfrak{p}} = 0$, it follows that $\pi_n(\lambda_\mathfrak{p}, \nu_\mathfrak{p})^{K(\mathfrak{p})}$ (resp. $\pi_s(\lambda_\mathfrak{p}, \nu_\mathfrak{p})^{K(\mathfrak{p})}$) is isomorphic to the trivial (resp. Steinberg) representation of $G(\mathbb{F}_q)$, hence its dimension equals 1 (resp. q^3). □

3.4. SURFACES WITH POSITIVE IRREGULARITY. The existence of Hecke characters λ of M satisfying (13) goes back to Chevalley and Weil. We will show that there are still such characters if one further imposes their restriction to F to be ω.

LEMMA 3.5. *For any CM extension M/F and any CM type Φ on M, there exist Hecke characters λ of M whose restriction to F equals ω, such that $\lambda_\infty(z) = \prod_{v\in\Phi} \frac{\bar{z}_v}{|z_v|}$ for all $z \in M^\times_\infty$.*

Proof. Since M is totally imaginary, λ_∞ and ω agree on $F^\times_\infty$, hence there is a character λ_0 of $\mathbb{A}_F^\times M^\times_\infty$ extending both.

We show now that λ_0 can be extended to a Hecke character λ of M, which will obviously satisfy the assumptions of the lemma. Since M/F is totally imaginary, $\mathfrak{o}^{\times 2}$ has finite index in $\mathfrak{O}^\times$. By [Ch, Théorème 1] there exists an open compact subgroup U of $\mathbb{A}^\times_{M,f}$ such that $U \cap \mathfrak{O}^\times \subset \mathfrak{o}^{\times 2}$. We may, and we will, assume that U is contained in the congruence subgroup whose level is the relative different of M/F and by replacing U by $U \cap \bar{U}$ we can further assume that $U = \bar{U}$. Since the Artin conductor of ω is the relative discriminant of M/F, it follows that ω is trivial on $U \cap \mathbb{A}^\times_{F,f}$. Hence one can extend λ_0 to a character of $\mathbb{A}_F^\times U M^\times_\infty$ by letting it be trivial on U.

Suppose we knew that

$$M^\times \cap \mathbb{A}_F^\times U M^\times_\infty = F^\times. \tag{18}$$

Then there is a unique character of $M^\times \mathbb{A}_F^\times U M_\infty^\times$ extending both λ_0 and the trivial character of $M^\times U$. Since $\mathbb{A}_M^\times / M^\times \mathbb{A}_F^\times U M_\infty^\times$ is a finite abelian (idele class) group, the character above can be further extended to a character λ of $\mathbb{A}_M^\times / M^\times$, and any such extension has the desired properties.
It remains to prove (18). Let $x \in M^\times \cap \mathbb{A}_F^\times U M_\infty^\times$. Then

$$\bar{x}/x \in M^1 \cap U M_\infty^1 = \mathfrak{O}^\times \cap U M_\infty^1 \subset \mathfrak{o}^{\times 2}.$$

Since $F^{\times 2} \cap M^1 = \{1\}$, we have $x = \bar{x} \in F^\times$. □

PROPOSITION 3.6. *Fix any Hecke character λ of M satisfying* (13) *whose restriction to F is ω. Let $\mathfrak{p}$ be a prime of F which splits in M and is relatively prime to the conductor $\mathfrak{C}$ of λ. If $W(\lambda^3) = (-1)^d$ we choose a prime $\mathfrak{q}$ of F which does not split in M; if not, we take $\mathfrak{q} = \mathfrak{o}$. Then $q(Y_{\Gamma_1(\mathfrak{C}) \cap \Gamma_0(\mathfrak{p}\mathfrak{q})}) > 2$.*

Proof. Let $K = K_1(\mathfrak{C}) \cap K_0(\mathfrak{p}\mathfrak{q})$, so that $\Gamma = G(F) \cap K K_\infty$. Let $\Pi(\lambda, \lambda_{|M^1}^{-1})$ be the global Arthur packet on G associated to λ. Let $\pi = \otimes_v \pi_v \in \Pi(\lambda, \lambda_{|M^1}^{-1})$ be such that $\pi_\iota = \pi^+$, $\pi_v = \mathbb{1}$ for every infinite place $v \neq \iota$, $\pi_v = \pi_{n,v}$ for every finite $v \neq \mathfrak{q}$, and finally if $W(\lambda^3) = (-1)^d$ then $\pi_\mathfrak{q} = \pi_{s,\mathfrak{q}}$. By Theorem 3.2(iii), π is automorphic, and by Lemma 3.3, we have $\pi_{n,v}^{K_v} \neq 0$ for all finite places $v \neq \mathfrak{p}, \mathfrak{q}$. Moreover, if $\mathfrak{q} \neq \mathfrak{o}$, then $\pi_{s,\mathfrak{q}}^{K_\mathfrak{q}} \neq 0$ by Lemma 3.3 (resp. Lemma 3.4) if $\mathfrak{q}$ divides (resp. does not divide) $\mathfrak{C}$. Finally (11) implies that $\dim(\pi_\mathfrak{p}^{K_0(\mathfrak{p})}) \geq 3$, hence $q(Y_\Gamma) \geq 3$ by (15) as claimed. □

REMARK 3.7. Since restriction of λ to $\mathbb{A}_F^\times$ equals ω, its conductor $\mathfrak{C}$ is divisible by the different of M/F. Hence, unless M/F is unramified everywhere, one might take as $\mathfrak{q}$ a place where M/F is ramified and Proposition 3.6 applies then to $K = K_1(\mathfrak{C}) \cap K_0(\mathfrak{p})$. Given a totally real number field F, there exists a totally imaginary quadratic extension M/F unramified everywhere if and only if all the units in F have norm 1.

For the rest of this section we assume that $F = \mathbb{Q}$, so that G is quasi-split.

PROPOSITION 3.8. *Any Γ as in Theorem 0.3 is neat and $q(Y_\Gamma) > 2$.*

Proof. There exists an open compact subgroup K of $G(\mathbb{A}_{\mathbb{Q},f})$ such that $\Gamma = G(\mathbb{Q}) \cap K G(\mathbb{R})$.
We claim that there exists a Hecke character λ of M of conductor $\mathfrak{D}$ satisfying (13), whose restriction to F equals ω. If $D \neq 3$ is odd or if 8 divides D such characters, called canonical, are proved to exists by Rohrlich [Ro]. If $D > 4$ is even but not divisible by 8, then by Yang (see [Ya, p.88]) there are such characters, called the *simplest*. Finally for $D = 3$ (resp. D=4) the claim follows from the existence of a CM elliptic curve over $\mathbb{Q}$ of conductor 27 (resp. 32).
By definition, $(\lambda, \lambda_{|M^1}^{-1}) \in \Xi$ and is trivial on $K_1(\mathfrak{D}) \cap T(\mathbb{A}_{\mathbb{Q},f})$. Then Lemma 3.3 implies that:

$$\pi_f^{K_1(\mathfrak{D})} \neq 0 \text{ , for all } \pi \in \Pi(\lambda, \lambda_{|M^1}^{-1}). \tag{19}$$

In case (ii) of Theorem 0.3 where $\Gamma = \Gamma(N) \cap \Gamma_1(\mathfrak{D})$ we fix a prime p dividing D and $\pi = \otimes_v \pi_v \in \Pi(\lambda, \lambda_{|M^1}^{-1})$ such that $\pi_v = \pi_n(\lambda_v, \lambda_{|M_v^1}^{-1})$ for all $v \neq p, N$, $\pi_N = \pi_s(\lambda_N, \lambda_{|M_N^1}^{-1})$ and

$$\pi_p = \begin{cases} \pi_n(\lambda_p, \lambda_{|M_p^1}^{-1}) \text{ , if } W(\lambda^3) = -1, \\ \pi_s(\lambda_p, \lambda_{|M_p^1}^{-1}) \text{ , if } W(\lambda^3) = 1. \end{cases}$$

Since $\Gamma(N)$ is neat by Lemma 1.4, we can apply (15) which, when combined with Lemma 3.4, yields

$$q(Y_{\Gamma(N)\cap\Gamma_1(\mathfrak{D})}) \geq \dim(\pi_N^{K(N)}) \geq N^3 \geq 3.$$

We now turn to case (i) and suppose first that M has class number $h \geq 3$. For any class character ξ one has $(\lambda\xi, \lambda_{|M^1}^{-1}) \in \Xi$ giving h pairwise distinct elements in $\Xi/\pi_0(\widehat{Y_{K_1(\mathfrak{D})}})$. Fix a prime p dividing D and consider $\pi = \otimes_v \pi_v \in \Pi(\lambda\xi, \lambda_{|M^1}^{-1})$ such that $\pi_v = \pi_n(\lambda_v\xi_v, \lambda_{|M_v^1}^{-1})$ for all $v \neq p$ and

$$\pi_p = \begin{cases} \pi_n(\lambda_p\xi_p, \lambda_{|M_p^1}^{-1}) \text{ , if } W(\lambda^3) = 1, \\ \pi_s(\lambda_p\xi_p, \lambda_{|M_p^1}^{-1}) \text{ , if } W(\lambda^3) = -1. \end{cases}$$

Since $\Gamma_1(\mathfrak{D})$ is neat by Lemma 1.5, one can apply (15) which combined with (19) yields $q(Y_{\Gamma_1(\mathfrak{D})}) \geq h \geq 3$.

If M is one of the 18 imaginary quadratic fields of class number 2, then its fundamental discriminant D has (exactly) two distinct prime divisors $p < q$. For each character λ on M as above, consider $\pi \in \Pi(\lambda, \lambda_{|M^1}^{-1})$ such that $\pi_v = \pi_n(\lambda_v, \lambda_{|M_v^1}^{-1})$ for all $v \neq p, q$, and moreover if $W(\lambda^3) = 1$, then

$$(\pi_p, \pi_q) = (\pi_n(\lambda_p, \lambda_{|M_p^1}^{-1}), \pi_n(\lambda_q, \lambda_{|M_q^1}^{-1})) \text{ or } (\pi_s(\lambda_p, \lambda_{|M_p^1}^{-1}), \pi_s(\lambda_q, \lambda_{|M_q^1}^{-1})),$$

whereas if $W(\lambda^3) = -1$, then

$$(\pi_p, \pi_q) = (\pi_n(\lambda_p, \lambda_{|M_p^1}^{-1}), \pi_s(\lambda_q, \lambda_{|M_q^1}^{-1})) \text{ or } (\pi_s(\lambda_p, \lambda_{|M_p^1}^{-1}), \pi_n(\lambda_q, \lambda_{|M_q^1}^{-1})).$$

If $D \neq 24$ then $\Gamma_1(\mathfrak{D})$ is neat by Lemma 1.5 and (15) implies that $q(Y_{\Gamma_1(\mathfrak{D})}) \geq 2 \cdot 2 = 4$. If $D = 24$ then $\Gamma(\mathfrak{D})$ is neat by Lemma 1.4, since 4 divides $\mathfrak{D}$, and again $q(Y_{\Gamma(\mathfrak{D})}) \geq 4$.

Finally, we consider the nine imaginary quadratic fields of class number 1.

For $D \in \{7, 11, 19, 43, 67, 163\}$ there is a unique character λ as in the beginning of the proof. Any character of $(1 + \sqrt{-D}\mathfrak{O}/1 + D\mathfrak{O}) \simeq \mathbb{Z}/D\mathbb{Z}$ lifts to a finite order Hecke character ξ of M with trivial restriction to $\mathbb{Q}$, hence $(\lambda\xi, \lambda_{|M^1}^{-1}) \in \Xi$. Let $\pi = \otimes_v \pi_v \in \Pi(\lambda\xi, \lambda_{|M^1}^{-1})$ be such that $\pi_v = \pi_n(\lambda_v\xi_v, \lambda_{|M_v^1}^{-1})$ for all $v \neq D$ and

$$\pi_D = \begin{cases} \pi_n(\lambda_D\xi_D, \lambda_{|M_D^1}^{-1}) \text{ , if } W(\lambda^3) = 1, \\ \pi_s(\lambda_D\xi_D, \lambda_{|M_D^1}^{-1}) \text{ , if } W(\lambda^3) = -1. \end{cases}$$

Since $\Gamma(D)$ is neat by Lemma 1.4, by (15) we get $q(Y_{\Gamma(D)}) \geq D \cdot \dim(\pi_D^{K(D)}) \geq D$.

For $D = 3$ the same argument with D^2 instead of D, implies that $q(Y_{\Gamma(9\mathfrak{O})}) \geq 3$. For $D = 4$ (resp. $D = 8$) the group $\Gamma(8\mathfrak{O})$ (resp. $\Gamma(2\sqrt{-8}\mathfrak{O})$) is neat by Lemma 1.4 and it is an exercise on idele class groups to show that there are at least three Hecke characters of M satisfying (13) whose restriction to $\mathbb{Q}$ is ω, and whose conductor divides 8 (resp. $2\sqrt{-8}$). It follows then from (15) and (19) that for $D = 4$ (resp. $D = 8$) one has $q(Y_{\Gamma(8\mathfrak{O})}) \geq 3$ (resp. $q(Y_{\Gamma(2\sqrt{-8}\mathfrak{O})}) \geq 3$). □

Remark 3.9. The computation of the smallest level K for which there exists an automorphic representation $\pi \in \Pi(\lambda, \nu)$ such that $\pi_f^K \neq 0$ is analyzed in detail in [DR]. In particular, if λ is a canonical character, we check that the level subgroup at any p dividing D is precisely the one conjectured by B. Gross, namely the index 2 subgroup of the maximal parahoric subgroup with reductive quotient PGL(2).

4. The Albanese map and Mordellicity

A major ingredient in the proof of our theorems is the Mordell-Lang conjecture for abelian varieties in characteristic zero, established by Faltings [F2] using some earlier work of himself [F1] and Vojta [V] (see Mazur's detailed account [M]).

Theorem 4.1 (Mordell-Lang conjecture : theorem of Faltings). *Suppose A is an abelian variety over $\mathbb{C}$ and $Z \subset A$ a closed subvariety. Then for any finitely generated field extension k of $\mathbb{Q}$ over which $Z \subset A$ is defined, the set $Z(k)$ is contained in a union of finitely many translates of abelian subvarieties of A, each of which is defined over k and contained in Z.*

The following corollary was proved in Moriwaki [Mo, Theorem 1.1]. He stated it for number fields, but the proof is the same for finitely generated fields over $\mathbb{Q}$.

Corollary 4.2. *Let X be a connected smooth projective variety over $\mathbb{C}$ which does not admit a dominant map to its Albanese variety. Then for any finitely generated field extension k of $\mathbb{Q}$ over which X is defined, the set $X(k)$ is not Zariski dense in X.*

Proof. The conclusion is obvious if $X(k)$ is empty so we may choose a point of $X(k)$ to define the Albanese map over k:

$$j : X \to \mathrm{Alb}(X).$$

Applying Theorem 4.1 to the closed subvariety $Z = j(X)$ of $\mathrm{Alb}(X)$ we get a finite number, say $m \geq 1$, of translates Z_i of abelian subvarieties of $\mathrm{Alb}(X)$ defined over k and such that

$$Z(k) \subset \bigcup_{i=1}^{m} Z_i(k) \quad \text{and} \quad Z_i \subset Z.$$

Since j is defined over k, each k-rational point of X is contained in $j^{-1}(Z_i)$ for some i. If $j^{-1}(Z_i)$ were not a proper closed subvariety of X, the universal property of the Albanese map would imply that $Z = Z_i = \mathrm{Alb}(X)$, contradicting the assumption that j is not dominant. □

PROPOSITION 4.3. *For every arithmetic subgroup $\Gamma \subset G(F)$ there exists a finite cover of Y_Γ whose points over any finitely generated field extension of $\mathbb{Q}$ are not Zariski dense, i.e., the Bombieri-Lang conjecture holds for that cover.*

Proof. Applying Corollary 2.2 recursively yields a finite index subgroup $\Gamma' \subset \Gamma$, which one can assume to be torsion free, such that $q(Y_{\Gamma'}) > n = \dim(Y_{\Gamma'})$. It suffices then to apply Corollary 4.2 to $Y_{\Gamma'}$ in the compact case and, in view of (4), to $X_{\Gamma'}$ in the non-compact case. □

Proof of Theorem 0.1. By Proposition 3.6, we have $q(Y_\Gamma) > 2$, hence Y_Γ does not admit a dominant map to its Albanese variety. Moreover Y_Γ is a geometrically irreducible smooth projective surface, hence by Corollary 4.2 $Y_\Gamma(k)$ is not Zariski dense in Y_Γ for any finitely generated field extension k of $\mathbb{Q}$ over which Y_Γ is defined. If $Y_\Gamma(k)$ were infinite, then Y_Γ would contain an irreducible curve C defined over k and such that $C(k)$ infinite. Since $C(k)$ is Zariski dense in C, the curve C is geometrically irreducible and its geometric genus is at most 1 by Theorem 4.1 applied to the Albanese map of C. Taking a complex uniformization of C would provide a non-constant holomorphic map from $\mathbb{C}$ to Y_Γ, which is impossible by Lemma 1.1(i) which we can apply as Γ is torsion free. Therefore Y_Γ is Mordellic. □

Proof of Theorem 0.2. The Lang locus of a quasi-projective irreducible variety Z over a number field is defined as the Zariski closure of the union, over all number fields k, of irreducible components of positive dimension of the Zariski closure of $Z(k)$. It is clear that Z is arithmetically Mordellic if and only if its Lang locus is empty. The main theorem in [UY] asserts that, for Γ neat and sufficiently small, the Lang locus of Y_Γ^* is either empty or everything.
By Corollary 2.2 one can assume by further shrinking Γ that $q(Y_\Gamma) > n$, and by (4) we also have $q(X_\Gamma) > n$ for X_Γ a smooth toroidal compactification of Y_Γ. By Corollary 4.2 the Lang locus of X_Γ is not everything, which forces the Lang locus of Y_Γ^* to be empty. □

Proof of Theorem 0.3. Let us first show that X_Γ is of general type, hence its canonical divisor $\mathcal{K}_X$ is big in the sense of [N, Definition 1.1]. Note that just like irregularity, the Kodaira dimension cannot decrease when going to a finite covering. By Holzapfel [H2, Theorem 5.4.15] and Feustel [Fe] the surface $X_{\Gamma_1(\mathfrak{D})}$ is of general type for all

$$D \notin \{3, 4, 7, 8, 11, 15, 19, 20, 23, 24, 31, 39, 47, 71\}.$$

Also by [H1, Proposition 4.13], $X_{\Gamma(N)}$ is of general type for all integers $N > 2$, with the possible exceptions of $N = 3$ and $N = 4$ when $D = 4$, implying in particular that $X_{\Gamma(\mathfrak{D}^2)}$ is of general type for $D \in \{3, 4, 7, 8, 11, 19\}$. Finally, the argument from *loc. cit.* transports in a straight forward way to the case

when the level is an ideal of $\mathfrak{O}$, yielding that the remaining varieties $X_{\Gamma(\mathfrak{D})}$, $D \in \{15, 20, 23, 24, 31, 39, 47, 71\}$, are of general type as well. See [D] where this has been carried out (it should be noted that his argument works also when $D = 24$).

If $g = \sum_{i,j=1}^{2} g_{i\bar{j}} dz_i d\bar{z}_j$ denotes the Bergman metric of $\mathcal{H}^2_{\mathbb{C}}$ viewed as the unit ball $\{z = (z_1, z_2) \in \mathbb{C}^2\,,\, |z| < 1\}$, normalized by requiring that

$$\mathrm{Ric}(g) = \sum_{i,j=1}^{2} -\frac{\partial^2 \log(g_{1\bar{1}} g_{2\bar{2}} - g_{2\bar{1}} g_{1\bar{2}})}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j = -g,$$

then the holomorphic sectional curvature is constant and equals $-4/3$ (see [GKK, §3.3]), where $g_{i\bar{j}} = \frac{3\left((1-|z|^2)\delta_{ij} + \bar{z}_i z_j\right)}{(1-|z|^2)^2}$.

By (4) and Proposition 3.8 we have that Γ is neat and $q(X_\Gamma) = q(Y_\Gamma) > 2$. Corollary 4.2 then implies that $X_\Gamma(k)$ is not Zariski dense in X_Γ for any finitely generated field extension k of $\mathbb{Q}$ over which X_Γ is defined. If $X_\Gamma(k)$ is infinite, arguing as in the proof of Theorem 0.1 shows that X_Γ contains a geometrically irreducible curve C whose geometric genus is at most one. Now applying a result of Nadel [N, Theorem 2.1] with $\gamma = 1$ (so that $-\gamma \geq -4/3$), we see that the bigness of $\mathcal{K}_X$ implies that C is contained in the compactifying divisor, which is a finite union of elliptic curves indexed by the cusps. It follows that $Y^*_\Gamma(k)$ is finite and that X_Γ does not contain any rational curves at all, let alone just those of self intersection -1, hence it is a minimal surface of general type. □

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