Coherence for Weak Units

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Abstract. We define weak units in a semi-monoidal 2-category $\mathscr{C}$ as cancellable pseudo-idempotents: they are pairs (I, α) where I is an object such that tensoring with I from either side constitutes a biequivalence of $\mathscr{C}$, and $\alpha : I \otimes I \to I$ is an equivalence in $\mathscr{C}$. We show that this notion of weak unit has coherence built in: Theorem A: α has a canonical associator 2-cell, which automatically satisfies the pentagon equation. Theorem B: every morphism of weak units is automatically compatible with those associators. Theorem C: the 2-category of weak units is contractible if non-empty. Finally we show (Theorem E) that the notion of weak unit is equivalent to the notion obtained from the definition of tricategory: α alone induces the whole family of left and right maps (indexed by the objects), as well as the whole family of Kelly 2-cells (one for each pair of objects), satisfying the relevant coherence axioms.

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Introduction

The notion of tricategory, introduced by Gordon, Power, and Street [2] in 1995, seems still to represent the highest-dimensional explicit weak categorical structure that can be manipulated by hand (i.e. without methods of homotopy theory), and is therefore an important test bed for higher-categorical ideas. In this work we investigate the nature of weak units at this level. While coherence for weak associativity is rather well understood, thanks to the geometrical insight provided by the Stasheff associahedra [12], coherence for unit structures is more mysterious, and so far there seems to be no clear geometric pattern for the coherence laws for units in higher dimensions. Specific interest in weak units

stems from Simpson's conjecture [11], according to which strict n-groupoids with weak units should model all homotopy n-types.

In the present paper, working in the setting of a strict 2-category $\mathscr{C}$ with a strict tensor product, we define a notion of weak unit by simple axioms that involve only the notion of equivalence, and hence in principle make sense in all dimensions. Briefly, a weak unit is a cancellable pseudo-idempotent. We work out the basic theory of such units, and compare with the notion extracted from the definition of tricategory. In the companion paper *Weak units and homotopy* 3*-types* [4] we employ this notion of unit to prove a version of Simpson's conjecture for 1-connected homotopy 3-types, which is the first nontrivial case. The strictness assumptions of the present paper should be justified by that result.

By cancellable pseudo-idempotent we mean a pair (I, α) where I is an object in $\mathscr{C}$ such that tensoring with I from either side is an equivalence of 2-categories, and $\alpha : I \otimes I \xrightarrow{\sim} I$ is an equi-arrow (i.e. an arrow admitting a pseudo-inverse). The remarkable fact about this definition is that α, viewed as a multiplication map, comes with canonical higher order data built in: it possesses a canonical associator A which automatically satisfies the pentagon equation. This is our Theorem A. The point is that the arrow α alone, thanks to the cancellability of I, induces all the usual structure of left and right constraints with all the 2-cell data that goes into them and the axioms they must satisfy.

As a warm-up to the various constructions and ideas, we start out in Section 1 by briefly running through the corresponding theory for cancellable-idempotent units in monoidal 1-categories. This theory has been treated in detail in [8].

The rest of the paper is dedicated to the case of monoidal 2-categories. In Section 2 we give the definitions and state the main results: Theorem A says that there is a canonical associator 2-cell for α, and that this 2-cell automatically satisfies the pentagon equation. Theorem B states that unit morphisms automatically are compatible with the associators of Theorem A. Theorem C states that the 2-category of units is contractible if non-empty. Hence, 'being unital' is, up to homotopy, a property rather than a structure.

Next follow three sections dedicated to proofs of each of these three theorems. In Section 3 we show how the map $\alpha : II \xrightarrow{\sim} I$ alone induces left and right constraints, which in turn are used to construct the associator and establish the pentagon equation. The left and right constraints are not canonical, but surprisingly the associator does not depend on the choice of them. In Section 4 we prove Theorem B by interpreting it as a statement about units in the 2-category of arrows, where it is possible to derive it from Theorem A. In Section 5 we prove Theorem C. The key ingredient is to use the left and right constraints to link up all the units, and to show that the unit morphisms are precisely those compatible with the left and right constraints; this makes them 'essentially unique' in the required sense.

In Section 6 we go through the basic theory of classical units (i.e. as extracted from the definition of tricategory [2]). Finally, in Section 7 we show that the two notions of unit are equivalent. This is our Theorem E. A curiosity implied by the arguments in this section is that the left and right axioms for the 2-cell data in the Gordon-Power-Street definition (denoted TA2 and TA3 in [2]) imply each other.
(We have no Theorem D.)

This notion of weak units as cancellable idempotents is precisely what can be extracted from the more abstract, Tamsamani-style, theory of fair n-categories [7] by making an arbitrary choice of a fixed weak unit. In the theory of fair categories, the key object is a contractible space of all weak units, rather than any particular point in that space, and handling this space as a whole bypasses coherence issues. However, for the sake of understanding what the theory entails, and for the sake of concrete computations, it is interesting to make a choice and study the ensuing coherence issues, as we do in this paper. The resulting approach is very much in the spirit of the classical theory of monoidal categories, bicategories, and tricategories, and provides some new insight to these theories. To stress this fact we have chosen to formulate everything from scratch in such classical terms, without reference to the theory of fair categories.
In the case of monoidal 1-categories, the cancellable-idempotent viewpoint on units goes back to Saavedra [10]. The importance of this viewpoint in higher categories was first suggested by Simpson [11], in connection with his weak-unit conjecture. He gave an ad hoc definition in this style, as a mere indication of what needed to be done, and raised the question of whether higher homotopical data would have to be specified. The surprising answer is, at least here in dimension 3, that specifying α is enough, then the higher homotopical data is automatically built in.

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1 Units in monoidal categories

It is helpful first briefly to recall the relevant results for monoidal categories, referring the reader to [8] for further details of this case.

1.1. Semi-monoidal categories. A *semi-monoidal category* is a category $\mathscr{C}$ equipped with a tensor product (which we denote by plain juxtaposition),

i.e. an associative functor

$$\begin{array}{rcl} \mathscr{C} \times \mathscr{C} & \longrightarrow & \mathscr{C} \\ (X, Y) & \longmapsto & XY. \end{array}$$

For simplicity we assume strict associativity, $X(YZ) = (XY)Z$.

1.2. Monoidal categories. (Mac Lane [9].) A semi-monoidal category $\mathscr{C}$ is a *monoidal category* when it is furthermore equipped with a distinguished object I and natural isomorphisms

$$IX \xrightarrow{\lambda_X} X \xleftarrow{\rho_X} XI$$

obeying the following rules (cf. [9]):

$$\lambda_I = \rho_I \tag{1}$$
$$\lambda_{XY} = \lambda_X Y \tag{2}$$
$$\rho_{XY} = X\rho_Y \tag{3}$$
$$X\lambda_Y = \rho_X Y \tag{4}$$

Naturality of λ and ρ implies

$$\lambda_{IX} = I\lambda_X, \qquad \rho_{XI} = \rho_X I, \tag{5}$$

independently of Axioms (1)–(4).

1.3 Remark. Tensoring with I from either side is an equivalence of categories.

1.4 Lemma. (Kelly [5].) *Axiom (4) implies axioms (1), (2), and (3).*

Proof. (4) implies (2): Since tensoring with I on the left is an equivalence, it is enough to prove $I\lambda_{XY} = I\lambda_X Y$. But this follows from Axiom (4) applied twice (swap λ out for a ρ and swap back again only on the nearest factor):

$$I\lambda_{XY} = \rho_I XY = I\lambda_X Y.$$

Similarly for ρ, establishing (3).
(4) and (2) implies (1): Since tensoring with I on the right is an equivalence, it is enough to prove $\lambda_I I = \rho_I I$. But this follows from (2), (5), and (4):

$$\lambda_I I = \lambda_{II} = I\lambda_I = \rho_I I.$$

□

The following alternative notion of unit object goes back to Saavedra [10]. A thorough treatment of the notion was given in [8].

1.5. UNITS AS CANCELLABLE PSEUDO-IDEMPOTENTS. An object I in a semi-monoidal category $\mathscr{C}$ is called *cancellable* if the two functors $\mathscr{C} \to \mathscr{C}$

$$\begin{aligned} X &\longmapsto IX \\ X &\longmapsto XI \end{aligned}$$

are fully faithful. By definition, a *pseudo-idempotent* is an object I equipped with an isomorphism $\alpha : II \xrightarrow{\sim} I$. Finally we define a *unit object* in $\mathscr{C}$ to be a cancellable pseudo-idempotent.

1.6 LEMMA. [8] *Given a unit object (I, α) in a semi-monoidal category $\mathscr{C}$, for each object X there are unique arrows*

$$IX \xrightarrow{\lambda_X} X \xleftarrow{\rho_X} XI$$

such that

$$\begin{aligned} &(\mathsf{L}) \qquad I\lambda_X = \alpha X \\ &(\mathsf{R}) \qquad \rho_X I = X\alpha. \end{aligned}$$

The λ_X and ρ_X are isomorphisms and natural in X.

Proof. Let $\mathbb{L} : \mathscr{C} \to \mathscr{C}$ denote the functor defined by tensoring with I on the left. Since $\mathbb{L}$ is fully faithful, we have a bijection

$$\mathrm{Hom}(IX, X) \to \mathrm{Hom}(IIX, IX).$$

Now take λ_X to be the inverse image of αX; it is an isomorphism since αX is. Naturality follows by considering more generally the bijection

$$\mathrm{Nat}(\mathbb{L}, \mathrm{id}_{\mathscr{C}}) \to \mathrm{Nat}(\mathbb{L} \circ \mathbb{L}, \mathbb{L});$$

let λ be the inverse image of the natural transformation whose components are αX. Similarly on the right. □

1.7 LEMMA. [8] *For λ and ρ as above, the Kelly axiom* (4) *holds:*

$$X\lambda_Y = \rho_X Y.$$

Therefore, by Lemma 1.6 a semi-monoidal category with a unit object is a monoidal category in the classical sense.

Proof. In the commutative square

$$\begin{array}{ccc} XIIY & \xrightarrow{XI\lambda_Y} & XIY \\ {\scriptstyle \rho_X IY}\downarrow & & \downarrow{\scriptstyle \rho_X Y} \\ XIY & \xrightarrow[X\lambda_Y]{} & XY \end{array}$$

the top arrow is equal to $X\alpha Y$, by X tensor (L), and the left-hand arrow is also equal to $X\alpha Y$, by (R) tensor Y. Since $X\alpha Y$ is an isomorphism, it follows that $X\lambda_Y = \rho_X Y$. □

1.8 Lemma. *For a unit object (I, α) we have: (i) The map $\alpha : II \to I$ is associative. (ii) The two functors $X \mapsto IX$ and $X \mapsto XI$ are equivalences.*

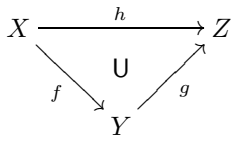
Proof. Since α is invertible, associativity amounts to the equation $I\alpha = \alpha I$, which follows from the previous proof by setting $X = Y = I$ and applying L and R once again. To see that $\mathbb{L}$ is an equivalence, just note that it is isomorphic to the identity via λ. □

1.9. Uniqueness of units. Just as in a semi-monoid a unit element is unique if it exists, one can show [8, 2.20] that in a semi-monoidal category, between any two units there is a unique isomorphism of units. This statement does not involve λ and ρ, but the proof does: the canonical isomorphism $I \xrightarrow{\sim} J$ is the composite $I \xrightarrow{\rho_I^{-1}} IJ \xrightarrow{\lambda_J} J$.

2 Units in monoidal 2-categories: definition and main results

In this section we set up the necessary terminology and notation, give the main definition, and state the main results.

2.1. 2-categories. We work in a strict 2-category $\mathscr{C}$. We use the symbol # to denote composition of arrows and horizontal composition of 2-cells in $\mathscr{C}$, always written from the left to the right, and occasionally decorating the symbol # by the name of the object where the two arrows or 2-cells are composed. By an *equi-arrow* in $\mathscr{C}$ we understand an arrow f admitting an (unspecified) pseudo-inverse, i.e. an arrow g in the opposite direction such that $f\#g$ and $g\#f$ are isomorphic to the respective identity arrows, and such that the comparison 2-cells satisfy the usual triangle equations for adjunctions. (The usual word for 'equi-arrow' is of course 'equivalence'; we reserve the latter word for equivalence of categories and 2-categories. We find it useful to have a different word for the equivalences inside a 2-category.) It is worth pointing out that it is not necessary to insist on the triangle equations. If the 2-cells exist but do not satisfy the triangle equation, they can always be replaced by 2-cells that do.
We shall make extensive use of arguments with pasting diagrams [6]. Our drawings of 2-cells should be read from bottom to top, so that for example



denotes $\mathsf{U} : f \underset{Y}{\#} g \Rightarrow h$. The symbol © will denote identity 2-cells.

The few 2-functors we need all happen to be strict. By *natural transformation* we always mean pseudo-natural transformation. Hence a natural transformation $u : F \Rightarrow G$ between two 2-functors from $\mathscr{D}$ to $\mathscr{C}$ is given by an arrow $u_X : FX \to GX$ for each object $X \in \mathscr{D}$, and an invertible 2-cell

$$\begin{array}{ccc} FX & \xrightarrow{u_X} & GX \\ {\scriptstyle F(x)}\downarrow & u_x & \downarrow{\scriptstyle G(x)} \\ FX' & \xrightarrow[u_{X'}]{} & GX' \end{array}$$

for each arrow $x : X \to X'$ in $\mathscr{D}$, subject to the usual compatibility conditions [6]. The modifications we shall need will happen to be invertible.

2.2. SEMI-MONOIDAL 2-CATEGORIES. By *semi-monoidal 2-category* we mean a 2-category $\mathscr{C}$ equipped with a tensor product, i.e. an associative 2-functor

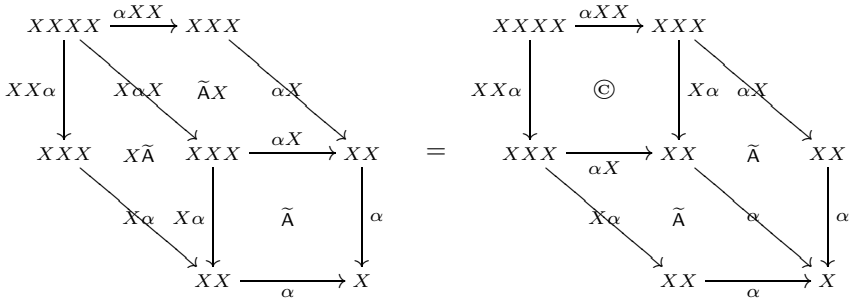
$$\begin{array}{rcl} \otimes : \mathscr{C} \times \mathscr{C} & \longrightarrow & \mathscr{C} \\ (X, Y) & \longmapsto & XY, \end{array}$$

denoted by plain juxtaposition. We already assumed $\mathscr{C}$ to be a strict 2-category, and we also require $\otimes$ to be a strict 2-functor and to be strictly associative: $(XY)Z = X(YZ)$. This is mainly for convenience, to keep the focus on unit issues.
Note that the tensor product of two equi-arrows is again an equi-arrow, since its pseudo-inverse can be taken to be the tensor product of the pseudo-inverses.

2.3. SEMI-MONOIDS. A *semi-monoid* in $\mathscr{C}$ is a triple $(X, \alpha, \widetilde{\mathsf{A}})$ consisting of an object X, a multiplication map $\alpha : XX \to X$, and an invertible 2-cell $\widetilde{\mathsf{A}}$ called the *associator*,

$$\begin{array}{ccc} XXX & \xrightarrow{\alpha X} & XX \\ {\scriptstyle X\alpha}\downarrow & \widetilde{\mathsf{A}} & \downarrow{\scriptstyle \alpha} \\ XX & \xrightarrow[\alpha]{} & X \end{array}$$

required to satisfy the 'pentagon equation':



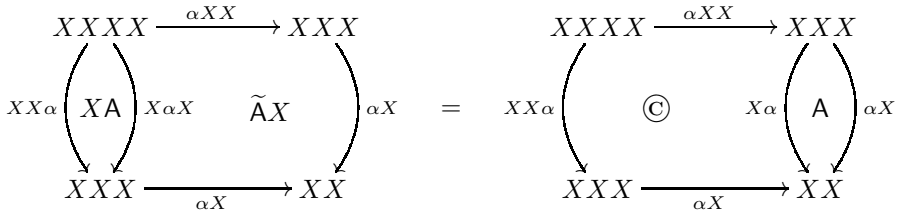
In the applications, α will be an equi-arrow, and hence we will have

$$\widetilde{\mathsf{A}} = \mathsf{A} \underset{XX}{\#} \alpha$$

for a some unique invertible

$$\mathsf{A} : X\alpha \Rightarrow \alpha X,$$

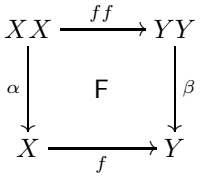
which it will more convenient to work with. In this case, the pentagon equation is equivalent to the more compact equation



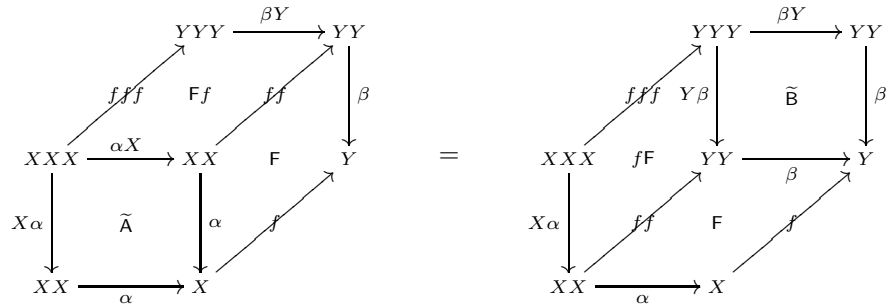
(6)

which we shall also make use of.

2.4. SEMI-MONOID MAPS. A *semi-monoid map* $f : (X, \alpha, \widetilde{\mathsf{A}}) \to (Y, \beta, \widetilde{\mathsf{B}})$ is the data of an arrow $f : X \to Y$ in $\mathscr{C}$ together with an invertible 2-cell

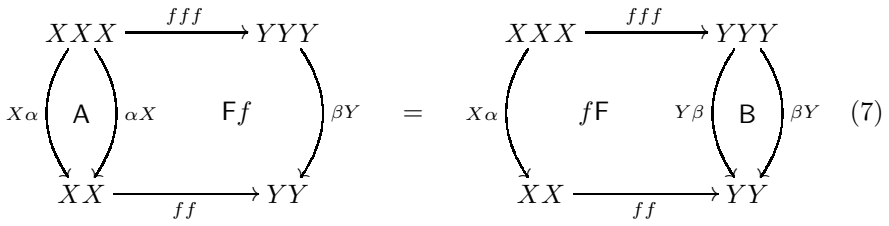


such that this cube commutes:



When β is an equi-arrow, the cube equation is equivalent to the simpler equa-

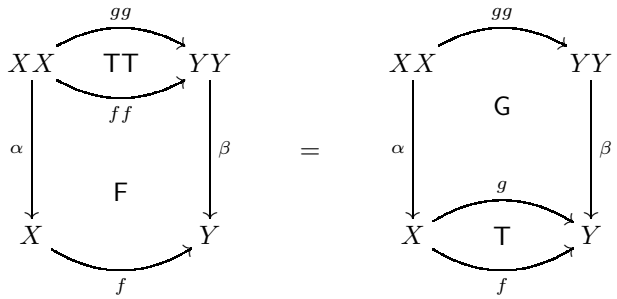
tion:



(7)

which will be useful.

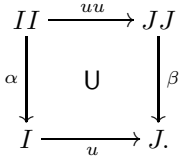
2.5. SEMI-MONOID TRANSFORMATIONS. A *semi-monoid transformation* between two parallel semi-monoid maps (f, F) and (g, G) is a 2-cell $\mathsf{T} : f \Rightarrow g$ in $\mathscr{C}$ such that this cylinder commutes:



2.6 LEMMA. *Let $f : X \to Y$ be a semi-monoid map. If f is an equi-arrow (as an arrow in $\mathscr{C}$) with quasi-inverse $g : Y \to X$, then there is a canonical 2-cell G such that (g, G) is a semi-monoid map.*

Proof. The 2-cell G is defined as the mate [6] of the 2-cell F^{-1}. It is routine to check the cube equation in 2.4. □

2.7. PSEUDO-IDEMPOTENTS. A *pseudo-idempotent* is a pair (I, α) where $\alpha : II \to I$ is an equi-arrow. A *morphism of pseudo-idempotents* from (I, α) to (J, β) is a pair (u, U) consisting of an arrow $u : I \to J$ in $\mathscr{C}$ and an invertible 2-cell



If (u, U) and (v, V) are morphisms of pseudo-idempotents from (I, α) to (J, β), a 2-*morphism of pseudo-idempotents* from (u, U) to (v, V) is a 2-cell $\mathsf{T} : u \Rightarrow v$ satisfying the cylinder equation of 2.5.

2.8. CANCELLABLE OBJECTS. An object I in $\mathscr{C}$ is called *cancellable* if the two 2-functors $\mathscr{C} \to \mathscr{C}$

$$\begin{aligned} X &\longmapsto IX \\ X &\longmapsto XI \end{aligned}$$

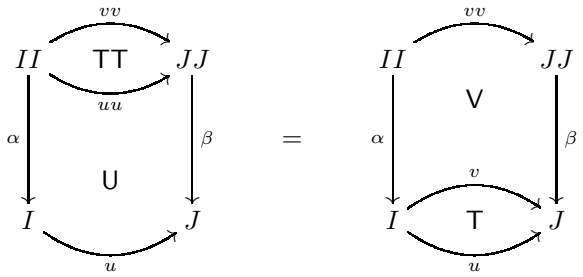
are fully faithful. (Fully faithful means that the induced functors on hom categories are equivalences.) A *cancellable morphism* between cancellable objects I and J is an equi-arrow $u : I \to J$. (Equivalently it is an arrow such that the functors on hom cats defined by tensoring with u on either side are equivalences, cf. 5.1.) A *cancellable* 2*-morphism* between cancellable arrows is any invertible 2-cell.

We are now ready for the main definition and the main results.

2.9. UNITS. A *unit object* is by definition a cancellable pseudo-idempotent. Hence it is a pair (I, α) consisting of an object I and an equi-arrow $\alpha : II \to I$, with the property that tensoring with I from either side define fully faithful 2-functors $\mathscr{C} \to \mathscr{C}$.
A *morphism* of units is a cancellable morphism of pseudo-idempotents. In other words, a unit morphism from (I, α) to (J, β) is a pair (u, U) where $u : I \to J$ is an equi-arrow and U is an invertible 2-cell

$$\begin{array}{ccc} II & \xrightarrow{uu} & JJ \\ {\scriptstyle\alpha}\downarrow & \mathsf{U} & \downarrow{\scriptstyle\beta} \\ I & \xrightarrow[u]{} & J. \end{array}$$

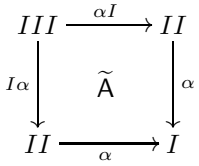
A 2*-morphism* of units is a cancellable 2-morphism of pseudo-idempotents. Hence a 2-morphism from (u, U) to (v, V) is an invertible 2-cell $\mathsf{T} : u \Rightarrow v$ such that



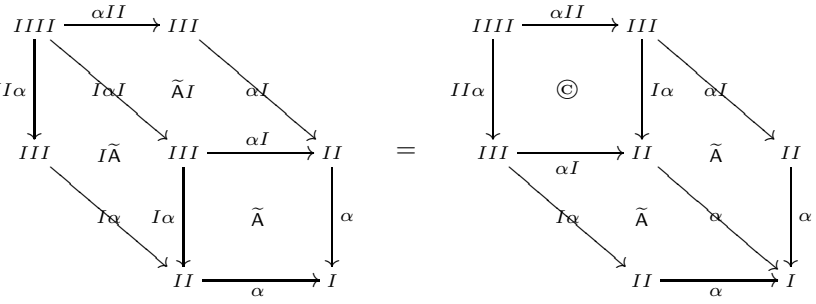
This defines the 2*-category of units*.

In the next section we'll see how the notion of unit object induces left and right constraints familiar from standard notions of monoidal 2-category. It will then turn out (Lemmas 5.1 and 5.2) that unit morphisms and 2-morphisms can be characterised as those morphisms and 2-morphisms compatible with the left and right constraints.

THEOREM A (ASSOCIATIVITY). *Given a unit object (I, α), there is a canonical invertible 2-cell*



which satisfies the pentagon equation



(8)

In other words, a unit object is automatically a semi-monoid. The 2-cell A is characterised uniquely in 3.7.

THEOREM B. *A unit morphism $(u, \mathsf{U}) : (I, \alpha) \to (J, \beta)$ is automatically a semi-monoid map, when I and J are considered semi-monoids in virtue of Theorem A.*

THEOREM C (CONTRACTIBILITY). *The 2-category of units in $\mathscr{C}$ is contractible, if non-empty.*

In other words, between any two units there exists a unit morphism, and between any two parallel unit morphisms there is a unique unit 2-morphism. Theorem C shows that units objects are unique up to homotopy, so in this sense 'being unital' is a property not a structure.

The proofs of these three theorems rely on the auxiliary structure of left and right constraints which we develop in the next section, and which also displays the relation with the classical notion of monoidal 2-category: in Section 7 we show that the cancellable-idempotent notion of unit is equivalent to the notion extracted from the definition of tricategory of Gordon, Power, and Street [2]. This is our Theorem E.

3 LEFT AND RIGHT ACTIONS, AND ASSOCIATIVITY OF THE UNIT (THEOREM A)

Throughout this section we fix a unit object (I, α).

3.1 LEMMA. *For each object X there exists pairs $(\lambda_X, \mathsf{L}_X)$ and (ρ_X, R_X),*

$$\lambda_X : IX \to X, \qquad \mathsf{L}_X : I\lambda_X \Rightarrow \alpha X$$
$$\rho_X : XI \to X, \qquad \mathsf{R}_X : X\alpha \Rightarrow \rho_X I$$

where λ_X and ρ_X are equi-arrows, and L_X are R_X are invertible 2-cells. For every such family, there is a unique way to assemble the λ_X into a natural transformation (this involves defining 2-cells λ_f for every arrow f in $\mathscr{C}$) in such a way that L is a natural modification. Similarly for the ρ_X and R_X.

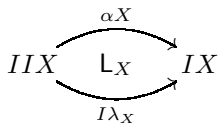
The λ_X is an action of I on each X, and the 2-cell L_X expresses an associativity constraint on this action. Using these structures we will construct the associator for α, and show it satisfies the pentagon equation. Once that is in place we will see that the actions λ and ρ are coherent too (satisfying the appropriate pentagon equations).

We shall treat the left action. The right action is of course equivalent to establish.

3.2. CONSTRUCTION OF THE LEFT ACTION. Since tensoring with I is a fully faithful 2-functor, the functor

$$\mathrm{Hom}(IX, X) \to \mathrm{Hom}(IIX, IX)$$

is an equivalence of categories. In the second category there is the canonical object αX. Hence there is a pseudo pre-image which we denote $\lambda_X : IX \to X$, together with an invertible 2-cell $\mathsf{L}_X : I\lambda_X \Rightarrow \alpha X$:



Since α is an equi-arrow, also αX is equi, and since L_X is invertible, we conclude that also $I\lambda_X$ is an equi-arrow. Finally since the 2-functor 'tensoring with I' is fully faithful, it reflects equi-arrows, so already λ_X is an equi-arrow.

3.3. NATURALITY. A slight variation in the formulation of the construction gives directly a natural transformation λ and a modification L: Let $\mathbb{L} : \mathscr{C} \to \mathscr{C}$ denote the 2-functor 'tensoring with I on the left'. Since $\mathbb{L}$ is fully faithful, there is an equivalence of categories

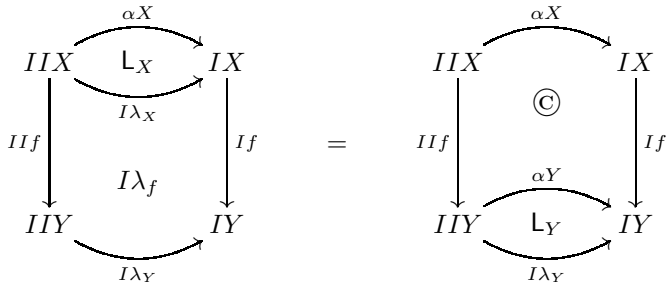
$$\mathrm{Nat}(\mathbb{L}, \mathrm{Id}_{\mathscr{C}}) \to \mathrm{Nat}(\mathbb{L} \circ \mathbb{L}, \mathbb{L}).$$

Now in the second category we have the canonical natural transformation whose X-component is αX (and with trivial components on arrows). Hence there is a pseudo pre-image natural transformation $\lambda : \mathbb{L} \to \mathrm{id}_{\mathscr{C}}$, together with a modification L whose X-component is $\mathsf{L}_X : I\lambda_X \Rightarrow \alpha X$.

However, we wish to stress the fact that the construction is completely objectwise. This fact is of course due to the presence of the isomorphism L_X: something isomorphic to a natural transformation is again natural. More precisely, to provide the 2-cell data λ_f needed to make λ into a natural transformation, just pull back the 2-cell data from the natural transformation αX. In detail, we need invertible 2-cells

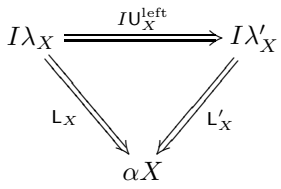
$$\begin{array}{ccc} IX & \xrightarrow{\lambda_X} & X \\ {\scriptstyle If}\downarrow & \lambda_f & \downarrow{\scriptstyle f} \\ IY & \xrightarrow[\lambda_Y]{} & Y. \end{array}$$

To say that the L_X constitute a modification (from λ to the identity) is to have this compatibility for every arrow $f : X \to Y$:



(Here the commutative cell is actually the 2-cell part of the natural transformation αX.) Now the point is that each 2-cell λ_f is uniquely defined by this compatibility: indeed, since the other three 2-cells in the diagram are invertible, there is a unique 2-cell that can fill the place of $I\lambda_f$, and since I is cancellable this 2-cell comes from a unique 2-cell λ_f. The required compatibilities of λ_f with composition, identities, and 2-cells now follows from its construction: λ_f is just the translation via L of the identity 2-cell αX.

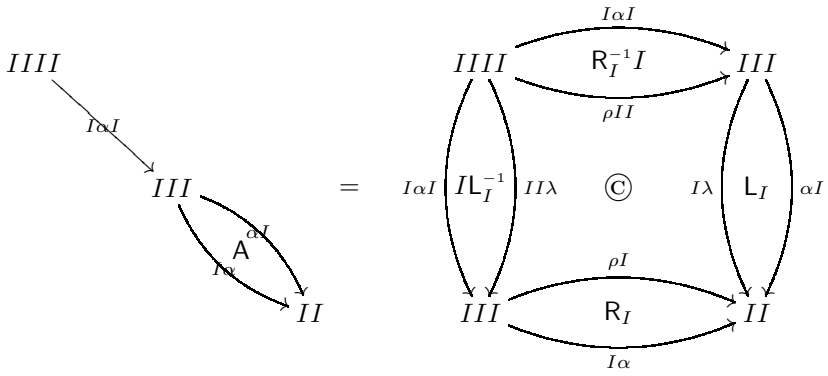
3.4. UNIQUENESS OF THE LEFT CONSTRAINTS. There may be many choices for λ_X, and even for a fixed λ_X, there may be many choices for L_X. However, between any two pairs $(\lambda_X, \mathsf{L}_X)$ and $(\lambda'_X, \mathsf{L}'_X)$ there is a unique invertible 2-cell $\mathsf{U}^{\text{left}}_X : \lambda_X \Rightarrow \lambda'_X$ such that this compatibility holds:



Indeed, this diagram defines uniquely an invertible 2-cell $I\lambda_X \Rightarrow I\lambda'_X$, and since I is cancellable, this 2-cell comes from a unique 2-cell $\lambda_X \Rightarrow \lambda'_X$ which we then call $\mathsf{U}^{\text{left}}_X$.

There is of course a completely analogous statement for right constraints.

3.5. CONSTRUCTION OF THE ASSOCIATOR. We define $\mathsf{A} : I\alpha \Rightarrow \alpha I$ as the unique 2-cell satisfying the equation



(9)

This definition is meaningful: since $I\alpha I$ is an equi-arrow, pre-composing with $I\alpha I$ is a 2-equivalence, hence gives a bijection on the level of 2-cells, so A is determined by the right-hand side of the equation. Note that A is invertible since all the 2-cells in the construction are.

The associator $\widetilde{\mathsf{A}}$ is defined as A-followed-by-α:

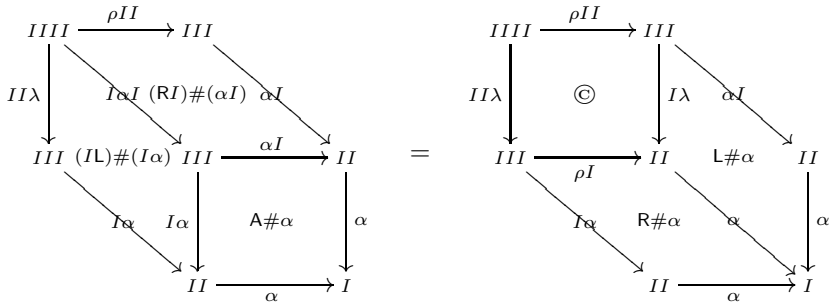
$$\widetilde{\mathsf{A}} \;:=\; \mathsf{A} \underset{II}{\#} \alpha,$$

but it will be more convenient to work with A.

3.6 PROPOSITION. *The definition of* A *does not depend on the choices of left constraint* (λ, L) *and right constraint* (ρ, R).

Proof. Write down the right-hand side of (9) in terms of different left and right constraints. Express these cells in terms of the original L_I and R_I, using the comparison 2-cells $\mathsf{U}_I^{\text{left}}$ and $\mathsf{U}_I^{\text{right}}$ of 3.4. Finally observe that these comparison cells can be moved across the commutative square to cancel each other pairwise. □

3.7. UNIQUENESS OF A. Equation (9) may not appear familiar, but it is equivalent to the following 'pentagon' equation (after post-whiskering with α):



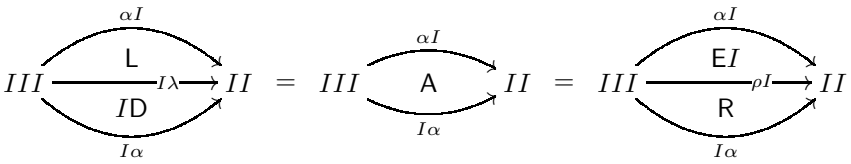
(10)

From this pentagon equation we shall derive the pentagon equation for A, asserted in Theorem A. To this end we need comparison between α, λ_I, and ρ_I, which we now establish, in analogy with Axiom (1) of monoidal category: the left and right constraints coincide on the unit object, up to a canonical 2-cell:

3.8 LEMMA. *There are unique invertible* 2*-cells*

$$\rho_I \overset{\mathsf{E}}{\Rightarrow} \alpha \overset{\mathsf{D}}{\Rightarrow} \lambda_I,$$

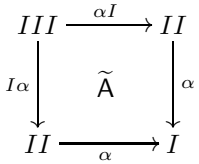
such that



(11)

Proof. The left-hand equation defines uniquely a 2-cell $I\alpha \Rightarrow I\lambda_I$, and since I is cancellable, this cell comes from a unique 2-cell $\alpha \Rightarrow \lambda_I$ which we then call D. Same argument for E. □

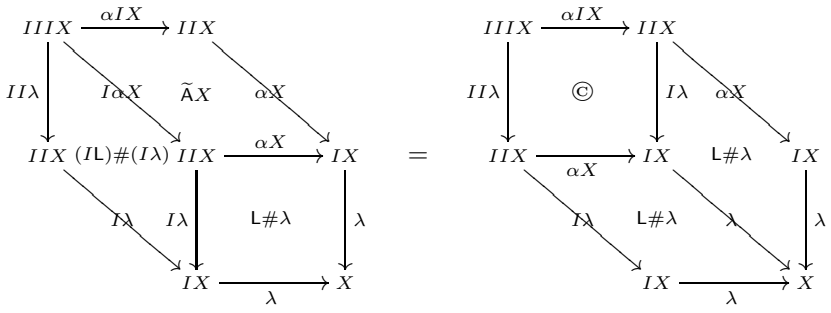
THEOREM A (ASSOCIATIVITY). *Given a unit object* (I, α)*, there is a canonical invertible* 2*-cell*



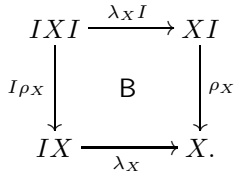
which satisfies the pentagon Equation (8)*.*

Proof. On each side of the cube equation (10), paste the cell $\mathsf{E}II$ on the top, and the cell $II\mathsf{D}$ on the left. On the left-hand side of the equation we can use Equations (11) directly, while on the right-hand side we first need to move those cells across the commutative square before applying (11). The result is precisely the pentagon cube for $\widetilde{\mathsf{A}} = \mathsf{A}\#\alpha$. □

3.9. COHERENCE OF THE ACTIONS. We have now established that $(I, \alpha, \widetilde{\mathsf{A}})$ is a semi-monoid, and may observe that the left and right constraints are coherent actions, i.e. that their 'associators' L and R satisfy the appropriate pentagon equations. For the left action this equation is:



Establishing this (and the analogous equation for the right action) is a routine calculation which we omit since we will not actually need the result. We also note that the two actions are compatible—i.e. constitute a two-sided action. Precisely this means that there is a canonical invertible 2-cell



This 2-cell satisfies two pentagon equations, one for $IIXI$ and one for $IXII$.

4 UNITS IN THE 2-CATEGORY OF ARROWS IN $\mathscr{C}$, AND THEOREM B

In this section we prove Theorem B, which asserts that a morphism of units $(u, \mathsf{U}) : (I, \alpha) \to (J, \beta)$ is automatically a semi-monoid map (with respect to the canonical associators A and B of the two units). We have to establish the cube equation of 2.4, or in fact the reduced version (7). The strategy to establish Equation (7) is to interpret everything in the 2-category of arrows of $\mathscr{C}$. The key point is to prove that a morphism of units is itself a unit in the 2-category of arrows. Then we invoke Theorem A to get an associator for this unit, and a pentagon equation, whose short form (6) will be the sought equation.

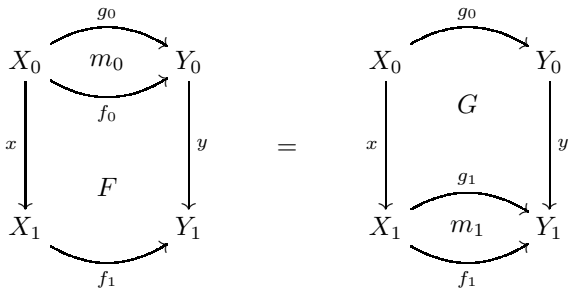
4.1. THE 2-CATEGORY OF ARROWS. The 2-*category of arrows* in $\mathscr{C}$, denoted $\mathscr{C}^{\mathbf{2}}$, is the 2-category described as follows. The objects of $\mathscr{C}^{\mathbf{2}}$ are the arrows of $\mathscr{C}$,

$$X_0 \xrightarrow{x} X_1 \ .$$

The arrows from (X_0, X_1, x) to (Y_0, Y_1, y) are triples (f_0, f_1, F) where $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ are arrows in $\mathscr{C}$ and F is a 2-cell

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle x}\downarrow & F & \downarrow{\scriptstyle y} \\ X_1 & \xrightarrow[f_1]{} & Y_1. \end{array}$$

If (g_0, g_1, G) is another arrow from (X_0, X_1, x) to (Y_0, Y_1, y), a 2-cell from (f_0, f_1, F) to (g_0, g_1, G) is given by a pair (m_0, m_1) where $m_0 : f_0 \Rightarrow g_0$ and $m_1 : f_1 \Rightarrow g_1$ are 2-cells in $\mathscr{C}$ compatible with F and G in the sense that this cylinder commutes:



Composition of arrows in $\mathscr{C}^{\mathbf{2}}$ is just pasting of squares. Vertical composition of 2-cells is just vertical composition of the components (the compatibility is guaranteed by pasting of cylinders along squares), and horizontal composition of 2-cells is horizontal composition of the components (compatibility guaranteed by pasting along the straight sides of the cylinders). Note that $\mathscr{C}^{\mathbf{2}}$ inherits a tensor product from $\mathscr{C}$: this follows from functoriality of the tensor product on $\mathscr{C}$.

4.2 LEMMA. *If I_0 and I_1 are cancellable objects in $\mathscr{C}$ and $i : I_0 \to I_1$ is an equi-arrow, then i is cancellable in $\mathscr{C}^{\mathbf{2}}$.*

Proof. We have to show that for given arrows $x : X_0 \to X_1$ and $y : Y_0 \to Y_1$, the functor

$$\mathrm{Hom}_{\mathscr{C}^{\mathbf{2}}}(x, y) \to \mathrm{Hom}_{\mathscr{C}^{\mathbf{2}}}(ix, iy)$$

defined by tensoring with i on the left is an equivalence of categories (the check for tensoring on the right is analogous).

Let us first show that this functor is essentially surjective. Let

$$\begin{array}{ccc} I_0X_0 & \xrightarrow{s_0} & I_0Y_0 \\ {\scriptstyle ix}\downarrow & S & \downarrow{\scriptstyle iy} \\ I_1X_1 & \xrightarrow[s_1]{} & I_1Y_1 \end{array}$$

be an object in $\mathrm{Hom}_{\mathscr{C}^{\mathbf{2}}}(ix, iy)$. We need to find a square

$$\begin{array}{ccc} X_0 & \xrightarrow{k_0} & Y_0 \\ {\scriptstyle x}\downarrow & K & \downarrow{\scriptstyle y} \\ X_1 & \xrightarrow[k_1]{} & Y_1 \end{array}$$

and an isomorphism (m_0, m_1) from (s_0, s_1, S) to (I_0k_0, I_1k_1, iK), i.e. a cylinder

$$\begin{array}{ccc} I_0X_0 & \overset{I_0k_0}{\underset{s_0}{\Downarrow m_0}} & I_0Y_0 \\ {\scriptstyle ix}\downarrow & S & \downarrow{\scriptstyle iy} \\ I_1X_1 & \underset{s_1}{\Uparrow} & I_1Y_1 \end{array} \quad = \quad \begin{array}{ccc} I_0X_0 & \overset{I_0k_0}{\Uparrow} & I_0Y_0 \\ {\scriptstyle ix}\downarrow & iK & \downarrow{\scriptstyle iy} \\ I_1X_1 & \underset{s_1}{\Downarrow m_1} & I_1Y_1 \end{array}$$

Since I_0 is a cancellable object, the arrow s_0 is isomorphic to I_0k_0 for some $k_0 : X_0 \to Y_0$. Let the connecting invertible 2-cell be denoted $m_0 : s_0 \Rightarrow I_0k_0$. Similarly we find k_1 and $m_1 : s_1 \Rightarrow I_1k_1$. Since m_0 and m_1 are invertible, there is a unique 2-cell

$$\begin{array}{ccc} I_0X_0 & \xrightarrow{I_0k_0} & I_0Y_0 \\ {\scriptstyle ix}\downarrow & T & \downarrow{\scriptstyle iy} \\ I_1X_1 & \xrightarrow[I_1k_1]{} & I_1Y_1 \end{array}$$

that can take the place of iK in the cylinder equation; it remains to see that T is of the form iK for some K. But this follows since the map

$$\begin{array}{rcl} \mathrm{2Cell}_{\mathscr{C}}(k_0\#y, x\#k_1) & \longrightarrow & \mathrm{2Cell}_{\mathscr{C}}(i(k_0\#y), i(x\#k_1)) \\ K & \longmapsto & iK \end{array} \tag{12}$$

is a bijection. Indeed, the map factors as 'tensoring with I_0 on the left' followed by 'post-composing with iY_1'; the first is a bijection since I_0 is cancellable, the second is a bijection since i (and hence iY_1) is an equi-arrow).

Now for the fully faithfulness of $\mathrm{Hom}_{\mathscr{C}^{\mathbf{2}}}(x,y) \to \mathrm{Hom}_{\mathscr{C}^{\mathbf{2}}}(ix,iy)$. Fix two objects in the left-hand category, P and Q:

$$\begin{array}{ccc} X_0 & \xrightarrow{p_0} & Y_0 \\ {\scriptstyle x}\downarrow & P & \downarrow{\scriptstyle y} \\ X_1 & \xrightarrow[p_1]{} & Y_1 \end{array} \qquad \begin{array}{ccc} X_0 & \xrightarrow{q_0} & Y_0 \\ {\scriptstyle x}\downarrow & Q & \downarrow{\scriptstyle y} \\ X_1 & \xrightarrow[q_1]{} & Y_1. \end{array}$$

The arrows from P to Q are pairs (m_0, m_1) consisting of

$$m_0 : p_0 \Rightarrow q_0 \qquad m_1 : p_1 \Rightarrow q_1$$

cylinder-compatible with the 2-cells P and Q. The image of these two objects are

$$\begin{array}{ccc} I_0X_0 & \xrightarrow{I_0p_0} & I_0Y_0 \\ {\scriptstyle ix}\downarrow & iP & \downarrow{\scriptstyle iy} \\ I_1X_1 & \xrightarrow[I_1p_1]{} & I_1Y_1 \end{array} \qquad \begin{array}{ccc} I_0X_0 & \xrightarrow{I_0q_0} & I_0Y_0 \\ {\scriptstyle ix}\downarrow & iQ & \downarrow{\scriptstyle iy} \\ I_1X_1 & \xrightarrow[I_1q_1]{} & I_1Y_1. \end{array}$$

The possible 2-cells from iP to iQ are pairs (n_0, n_1) consisting of

$$n_0 : I_0p_0 \Rightarrow I_0q_0 \qquad n_1 : I_1p_1 \Rightarrow I_1q_1$$

cylinder-compatible with the 2-cells iP and iQ. Now since I_0 is cancellable, every 2-cell n_0 like this is uniquely of the form I_0n_0 for some n_0. Hence there is a bijection between the possible m_0 and the possible n_0. Similarly for m_1 and n_1. So there is a bijection between pairs (m_0, m_1) and pairs (n_0, n_1). Now by functoriality of tensoring with i, all images of compatible (m_0, m_1) are again compatible. It remains to rule out the possibility that some (n_0, n_1) pair could be compatible without (m_0, m_1) being so, but this follows again from the argument that 'tensoring with i on the left' is a bijection on hom sets, just like argued for (12). □

4.3 LEMMA. *An arrow in* $\mathscr{C}^{\mathbf{2}}$,

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle x}\downarrow & F & \downarrow{\scriptstyle y} \\ X_1 & \xrightarrow[f_1]{} & Y_1 \end{array}$$

is an equi-arrow in $\mathscr{C}^{\mathbf{2}}$ *if the components* f_0 *and* f_1 *are equi-arrows in* $\mathscr{C}$ *and* F *is invertible.*

Proof. We can construct an explicit quasi-inverse by choosing quasi-inverses to the components. □

4.4 COROLLARY. *If* (I_0, α_0) *and* (I_1, α_1) *are units in* $\mathscr{C}$*, and* $(u, \mathsf{U}) : I_0 \to I_1$ *is a unit map between them, then*

$$u : I_0 \to I_1$$

is a unit object in $\mathscr{C}^{\mathbf{2}}$ *with structure map*

$$\begin{array}{ccc} I_0I_0 & \xrightarrow{\alpha_0} & I_0 \\ {\scriptstyle uu}\downarrow & \mathsf{U}^{-1} & \downarrow{\scriptstyle u} \\ I_1I_1 & \xrightarrow[\alpha_1]{} & I_1. \end{array}$$

Proof. The object u is cancellable by Lemma 4.2, and the morphism $(\alpha_0, \alpha_1, \mathsf{U}^{-1})$ from uu to u is an equi-arrow by Lemma 4.3. □

THEOREM B. *Let* (I_0, α_0) *and* (I_1, α_1) *be units, with canonical associators* A_0 *and* A_1*, respectively. If* (u, U) *is a unit map from* I_0 *to* I_1 *then it is automatically a semi-monoid map. That is,*

$$\begin{array}{ccccc} I_0I_0I_0 & \xrightarrow{uuu} & I_1I_1I_1 & & \\ {\scriptstyle I_0\alpha_0}\downarrow\ \mathsf{A}_0\ \downarrow{\scriptstyle \alpha_0I_0} & & \mathsf{U}u & \downarrow{\scriptstyle \alpha_1I_1} & \\ I_0I_0 & \xrightarrow[uu]{} & I_1I_1 & & \end{array} \quad = \quad \begin{array}{ccccc} I_0I_0I_0 & \xrightarrow{uuu} & I_1I_1I_1 & & \\ {\scriptstyle I_0\alpha_0}\downarrow & u\mathsf{U} & {\scriptstyle I_1\alpha_1}\downarrow\ \mathsf{A}_1\ \downarrow{\scriptstyle \alpha_1I_1} & & \\ I_0I_0 & \xrightarrow[uu]{} & I_1I_1 & & \end{array}$$

Proof. By the previous Corollary, (u, U^{-1}) is a unit object in $\mathscr{C}^{\mathbf{2}}$. Hence there is a canonical associator

$$\mathsf{B} : u\mathsf{U}^{-1} \Leftrightarrow \mathsf{U}^{-1}u.$$

By definition of 2-cells in $\mathscr{C}^{\mathbf{2}}$, this is a pair of 2-cells in $\mathscr{C}$

$$\mathsf{B}_0 : I_0\alpha_0 \Rightarrow \alpha_0I_0 \qquad \mathsf{B}_1 : I_1\alpha_1 \Rightarrow \alpha_1I_1,$$

fitting the cylinder equation

$$\begin{array}{ccccc} & & {\scriptstyle \alpha_0I_0}\frown & & \\ I_0I_0I_0 & \mathsf{B}_0 & & I_0I_0 & \\ & & {\scriptstyle I_0\alpha_0}\smile & & \\ {\scriptstyle uuu}\downarrow & & & \downarrow{\scriptstyle uu} & \\ & u\mathsf{U}^{-1} & & & \\ I_1I_1I_1 & & & I_1I_1 & \\ & & {\scriptstyle I_1\alpha_1}\smile & & \end{array} \quad = \quad \begin{array}{ccccc} & & {\scriptstyle \alpha_0I_0}\frown & & \\ I_0I_0I_0 & & & I_0I_0 & \\ & \mathsf{U}^{-1}u & & & \\ {\scriptstyle uuu}\downarrow & & & \downarrow{\scriptstyle uu} & \\ & & {\scriptstyle \alpha_1I_1}\frown & & \\ I_1I_1I_1 & \mathsf{B}_1 & & I_1I_1 & \\ & & {\scriptstyle I_1\alpha_1}\smile & & \end{array}$$

This is precisely the cylinder diagram we are looking for—provided we can show that $\mathsf{B}_0 = \mathsf{A}_0$ and $\mathsf{B}_1 = \mathsf{A}_1$. But this is a consequence of the characterising property of the associator of a unit: first note that as a unit object in $\mathscr{C}^{\mathbf{2}}$, u induces left and right constraints: for each object $x : X_0 \to X_1$ in $\mathscr{C}^{\mathbf{2}}$ there is a left action of the unit u, and this left action will induce a left action of (I_0, α_0) on X_0 and a left action of (I_1, α_1) on X_1 (the ends of the cylinders). Similarly there is a right action of u which induces right actions at the ends of the cylinder. Now the unique B that exists as associator for the unit object u compatible with the left and right constraints induces B_0 and B_1 at the ends of the cylinder, and these will of course be compatible with the induced left and right constraints. Hence, by uniqueness of associators compatible with left and right constraints, these induced associators B_0 and B_1 must coincide with A_0 and A_1. Note that this does not dependent on choice of left and right constraints, cf. Proposition 3.6. □

5 Contractibility of the space of weak units (Theorem C)

The goal of this section is to prove Theorem C, which asserts that the 2-category of units in $\mathscr{C}$ is contractible if non-empty. First we describe the unit morphisms and unit 2-morphisms in terms of compatibility with left and right constraints. This will show that there are not too many 2-cells. Second we use the left and right constraints to connect any two units.

The following lemma shows that just as the single arrow α induces all the λ_X and ρ_X, the single 2-cell U of a unit map induces families $\mathsf{U}^{\text{left}}_X$ and $\mathsf{U}^{\text{right}}_X$ expressing compatibility with λ_X and ρ_X.

5.1 Lemma. *Let (I, α) and (J, β) be units, and let (u, U) be a morphism of pseudo-idempotents from (I, α) to (J, β). The following are equivalent.*

(i) u is an equi-arrow (i.e. u is a morphism of units).

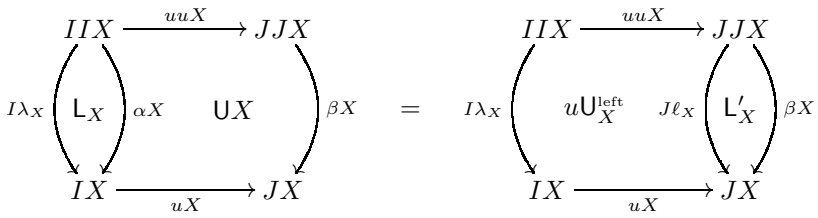
(ii) u is left cancellable, i.e. tensoring with u on the left is an equivalence of categories $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(IX, JY)$.

(ii') u is right cancellable, i.e. tensoring with u on the right is an equivalence of categories $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(XI, YJ)$.

(iii) For fixed left actions $(\lambda_X, \mathsf{L}_X)$ for the unit (I, α) and (ℓ_X, L'_X) for the unit (J, β), there is a unique invertible 2-cell $\mathsf{U}^{\text{left}}_X$, natural in X:

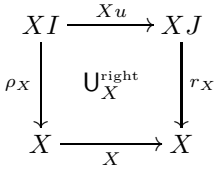
$$\begin{array}{ccc} IX & \xrightarrow{uX} & JX \\ {\scriptstyle \lambda_X}\downarrow & \mathsf{U}^{\text{left}}_X & \downarrow{\scriptstyle \ell_X} \\ X & \xrightarrow[X]{} & X \end{array}$$

such that this compatibility holds:

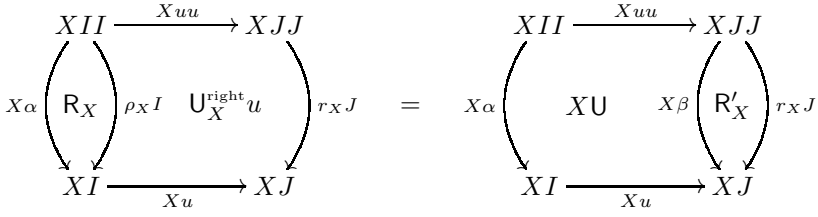


(13)

(iii') For fixed right actions (ρ_X, R_X) for the unit (I, α) and (r_X, R'_X) for the unit (J, β), there is a unique invertible 2-cell $\mathsf{U}_X^{\mathrm{right}}$, natural in X:



such that this compatibility holds:



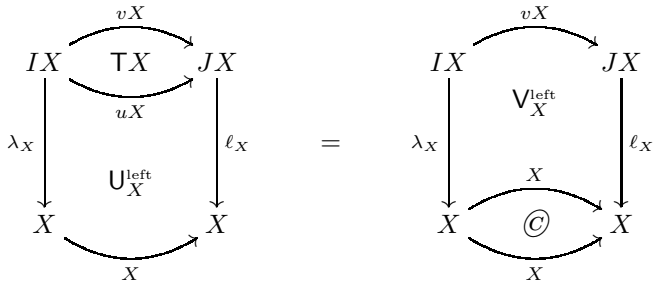
(14)

Proof. (i) implies (ii): 'tensoring with u' can be done in two steps: given an arrow $X \to Y$, first tensor with I to get $IX \to IY$, and then post-compose with uY to get $IX \to JY$. The first step is an equivalence because I is a unit, and the second step is an equivalence because u is an equi-arrow.
(ii) implies (iii): In Equation (13), the 2-cell labelled $u\mathsf{U}_X^{\mathrm{left}}$ is uniquely defined by the three other cells, and it is invertible since the three other cells are. Since tensoring with u on the left is an equivalence, this cell comes from a unique invertible cell $\mathsf{U}_X^{\mathrm{left}}$, justifying the label $u\mathsf{U}_X^{\mathrm{left}}$.
(iii) implies (i): The invertible 2-cell $\mathsf{U}_X^{\mathrm{left}}$ shows that uX is isomorphic to an equi-arrow, and hence is an equi-arrow itself. Now take X to be a right cancellable object (like for example I) and conclude that already u is an equi-arrow.
Finally, the equivalence (i)⇒(ii')⇒(iii')⇒(i) is completely analogous. □

Note that for (u, U) the identity morphism on (I, α), we recover Observation 3.4.

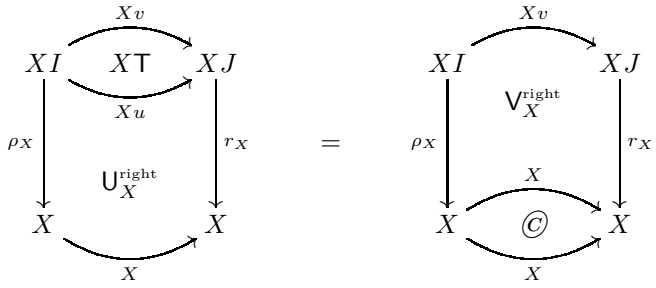
5.2 LEMMA. *Let (I,α) and (J,β) be units; let (u,U) and (v,V) be morphisms of pseudo-idempotents from I to J; and consider a 2-cell $\mathsf{T} : u \Rightarrow v$. Then the following are equivalent.*

(i) T *is an invertible 2-morphism of pseudo-idempotents.*
(ii) T *is a left cancellable 2-morphism of pseudo-idempotents (i.e., induces a bijection on hom sets (of hom cats) by tensoring with* T *from the left).*
(ii') T *is a right cancellable 2-morphism of pseudo-idempotents (i.e., induces a bijection on hom sets (of hom cats) by tensoring with* T *from the right).*
(iii) For fixed left actions $(\lambda_X, \mathsf{L}_X)$ for (I,α) and (ℓ_X, L'_X) for (J,β), with induced canonical 2-cells $\mathsf{U}^{\mathrm{left}}_X$ and $\mathsf{V}^{\mathrm{left}}_X$ as in 5.1, we have:



(15)

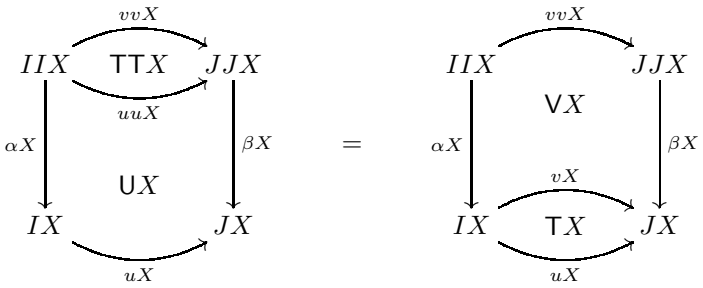
(iii') For fixed right actions (ρ_X, R_X) for (I,α) and (r_X, R'_X) for (J,β), with induced canonical 2-cells $\mathsf{U}^{\mathrm{right}}_X$ and $\mathsf{V}^{\mathrm{right}}_X$ as in 5.1, we have:



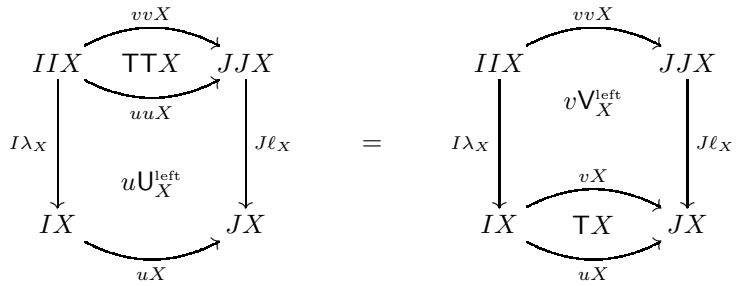
(16)

Proof. It is obvious that (i) implies (ii). Let us prove that (ii) implies (iii), so assume that tensoring with T on the left defines a bijection on the level of 2-cells. Start with the cylinder diagram for compatibility of tensor 2-cells

(cf. 2.5). Tensor this diagram with X on the right to get



On each side of this equation, paste an L_X along αX, apply Equation (13) on each side, and cancel the L'_X that appear on the other side of the square. The resulting diagram



is the tensor product of T with the promised equation (15). Since T is cancellable, we can cancel it away to finish.
(iii) implies (i): the arguments in (ii)⇒(iii) can be reversed: start with (15), tensor with T on the left, and apply (13) to arrive at the axiom for being a 2-morphism of pseudo-idempotents. Since both $\mathsf{U}^{\text{left}}_X$ and $\mathsf{V}^{\text{left}}_X$ are invertible, so is $\mathsf{T}X$. Now take X to be a right cancellable object, and cancel it away to conclude that already T is invertible.
Finally, the equivalence (i)⇒(ii')⇒(iii')⇒(i) is completely analogous. □

5.3 COROLLARY. *Given two parallel morphisms of units, there is a unique unit 2-morphism between them.*

Proof. Choose left actions for (I, α) and (J, β) as in Lemma 5.2 (iii), and take X to be a right cancellable object. For given morphisms of units u and v as in Lemma 5.2, Equation (15) defines the 2-cell T uniquely, since λ_X is an equi-arrow and X is right cancellable. □

Next we aim at proving that there is a unit morphism between any two units. The strategy is to use the left and right constraints to produce a unit morphism

$$I \longrightarrow IJ \longrightarrow J.$$

As a first step towards this goal we have:

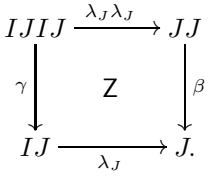
5.4 LEMMA. *Let I and J be units, and pick a left constraint λ for I and a right constraint r for J. Put*

$$\gamma := r_I \lambda_J : IJIJ \to IJ$$

Then (IJ, γ) is a unit.

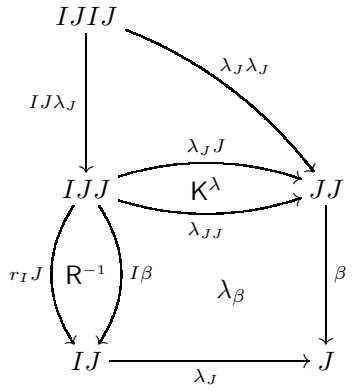
Proof. Since I and J are cancellable, clearly IJ is cancellable too. Since λ_J and r_I are equi-arrows, γ is too. □

5.5 LEMMA. *There is an invertible 2-cell*



Hence (λ_J, Z) is a unit map. (And there is another 2-cell making r_I a unit map.)

Proof. The 2-cell Z is defined like this:



where the 2-cell K^λ is constructed in Lemma 7.2. □

5.6 COROLLARY. *Given two units, there exists a unit morphism between them.*

Proof. Continuing the notation from above, by Lemma 5.4, (IJ, γ) is a unit, and by Lemma 5.5, $\lambda : IJ \to J$ is a morphism of units. Similarly, $r : IJ \to I$ is a unit morphism, and by Lemma 2.6 any chosen pseudo-inverse $r^{-1} : I \to IJ$ is again a unit morphism. Finally we take

$$I \xrightarrow{r^{-1}} IJ \xrightarrow{\lambda} J.$$

□

Theorem C (Contractibility). *The* 2*-category of units in* $\mathscr{C}$ *is contractible, if non-empty. In other words, between any two units there exists a unit morphism, and between any two parallel unit morphisms there is a unique unit* 2*-morphism.*

Proof. By Lemma 5.6 there is a unit morphism between any two units (an equi-arrow by definition), and by Corollary 5.3 there is a unique unit 2-morphism between any two parallel unit morphisms. □

6 Classical units

In this section we review the classical theory of units in a monoidal 2-category, as extracted from the definition of tricategory of Gordon, Power, and Street [2]. In the next section we compare this notion with the cancellable-idempotent approach of this work. The equivalence is stated explicitly in Theorem E.

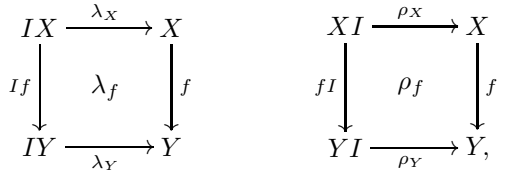
6.1. Tricategories. The notion of tricategory introduced by Gordon, Power, and Street [2] is roughly a weak category structure enriched over bicategories: this means that the structure maps (composition and unit) are weak 2-functors satisfying weak versions of associativity and unit constraints. For the associativity, the pentagon equation is replaced by a specified pentagon 3-cell (TD7), required to satisfy an equation corresponding to the 3-dimensional associahedron. This equation (TA1) is called the nonabelian 4-cocycle condition. For the unit structure, three families of 3-cells are specified (TD8): one corresponding to the Kelly axiom, one left variant, and one right variant (those two being the higher-dimensional analogues of Axioms (2) and (3) of monoidal category). Two axioms are imposed on these three families of 3-cells: one (TA2) relating the left family with the middle family, and one (TA3) relating the right family with the middle family. These are called left and right normalisation. (These two axioms are the higher-dimensional analogues of the first argument in Kelly's lemma 1.6.) It is pointed out in [2] that the middle family together with the axioms (TA2) and (TA3) completely determine the left and right families if they exist.

6.2. Monoidal 2-categories. By specialising the definition of tricategory to the one-object case, and requiring everything strict except the units, we arrive at the following notion of monoidal 2-category: a *monoidal* 2*-category* is a semi-monoidal 2-category (cf. 2.2) equipped with an object I, two natural transformations λ and ρ with equi-arrow components

$$\lambda_X : IX \to X$$

$$\rho_X : XI \to X$$

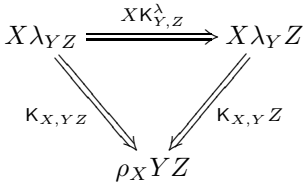
and (invertible) 2-cell data



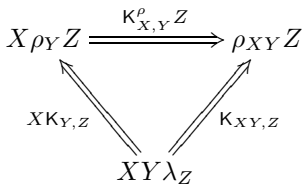
together with three natural modifications K, K^λ, and K^ρ, with invertible components

$$\begin{aligned}\mathsf{K} &: X\lambda_Y \Rightarrow \rho_X Y\\ \mathsf{K}^\lambda &: \lambda_{XY} \Rightarrow \lambda_X Y\\ \mathsf{K}^\rho &: X\rho_Y \Rightarrow \rho_{XY}.\end{aligned}$$

We call K the *Kelly cell*.
These three families are subject to the following two equations:



(17)



(18)

6.3 REMARK. We have made one change compared to [2], namely the direction of the arrow ρ_X: from the viewpoint of α it seems more practical to work with $\rho_X : XI \to X$ rather than with the convention of $\rho_X : X \to XI$ chosen in [2]. Since in any case it is an equi-arrow, the difference is not essential. (Gurski in his thesis [3] has studied a version of tricategory where all the equi-arrows in the definition are equipped with specified pseudo-inverses. This has the advantage that the definition becomes completely algebraic, in a technical sense.)

6.4 LEMMA. *The object I is cancellable (independently of the existence of K, K^λ, and K^ρ.)*

Proof. We need to establish that 'tensoring with I on the left',

$$\mathbb{L} : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(IX, IY),$$

is an equivalence of categories. But this follows since the diagram

$$\begin{array}{ccc} \mathrm{Hom}(X,Y) & \xrightarrow{\mathbb{L}} & \mathrm{Hom}(IX,IY) \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle _ \,\#\, \lambda_Y} \\ \mathrm{Hom}(X,Y) & \xrightarrow[\lambda_X \,\#\, _]{} & \mathrm{Hom}(IX,Y) \end{array}$$

is commutative up to isomorphism: the component at $f : X \to Y$ of this isomorphism is just the naturality square λ_f. Since the functors $\lambda_X \,\#\, _$ and $_ \,\#\, \lambda_Y$ are equivalences, it follows from this isomorphism that $\mathbb{L}$ is too. □

6.5. COHERENCE OF THE KELLY CELL. As remarked in [2], if the K^λ and K^ρ exist, they are determined uniquely from K and the two axioms. Indeed, the two equations

$$\begin{array}{ccc} I\lambda_{YZ} & \overset{I\mathsf{K}^\lambda_{Y,Z}}{\Longrightarrow} & I\lambda_Y Z \\ & {\scriptstyle \mathsf{K}_{I,YZ}}\searrow \quad \swarrow{\scriptstyle \mathsf{K}_{I,Y}Z} & \\ & \rho_I YZ & \end{array} \qquad \begin{array}{ccc} X\rho_Y I & \overset{\mathsf{K}^\rho_{X,Y}I}{\Longrightarrow} & \rho_{XY} I \\ & {\scriptstyle X\mathsf{K}_{Y,I}}\nwarrow \quad \nearrow{\scriptstyle \mathsf{K}_{XY,I}} & \\ & XY\lambda_I & \end{array} \tag{19}$$

which are just special cases of (17) and (18) uniquely determine K^λ and K^ρ, by cancellability of I. But these two special cases of the axioms do not imply the general case.

We shall take the Kelly cell K as the main structure, and say that K is *coherent on the left* (resp. *on the right*) if Axiom (17) (resp. (18)) holds for the induced cell K^λ (resp. K^ρ). We just say *coherent* if both hold. We shall see (7.8) that in fact coherence on the left implies coherence on the right and vice versa.

6.6. NATURALITY. The Kelly cell is a modification. For future reference we spell out the naturality condition satisfied: given arrows $f : X \to X'$ and $g : Y \to Y'$, we have

$$\begin{array}{ccc} XIY & \overset{\rho_X Y}{\Longrightarrow} & XY \\ & {\scriptstyle \mathsf{K}_{X,Y}}\searrow \quad \swarrow{\scriptstyle X\lambda_Y} & \\ {\scriptstyle fIg}\downarrow & & \downarrow{\scriptstyle fg} \\ & f\lambda_g & \\ X'IY' & & X'Y' \end{array} \quad = \quad \begin{array}{ccc} XIY & \overset{\rho_X Y}{\Longrightarrow} & XY \\ {\scriptstyle fIg}\downarrow & \rho_{fg} & \downarrow{\scriptstyle fg} \\ & \overset{\rho_{X'Y'}}{\Longrightarrow} & \\ X'IY' & {\scriptstyle \mathsf{K}_{X',Y'}}\searrow \quad \swarrow{\scriptstyle X'\lambda_{Y'}} & X'Y' \end{array}$$

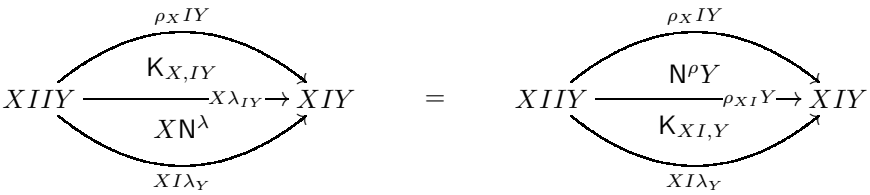
6.7 REMARK. Particularly useful is naturality of λ with respect to λ_X and naturality of ρ with respect to ρ_X. In these cases, since λ_X and ρ_X are equi-arrows, we can cancel them and find the following invertible 2-cells:

$$\mathsf{N}^\lambda : I\lambda_X \Rightarrow \lambda_{IX}$$
$$\mathsf{N}^\rho : \rho_{XI} \Rightarrow X\rho_I,$$

in analogy with Observation (5) of monoidal categories.

The following lemma holds for K independently of Axioms (17) and (18):

6.8 LEMMA. *The Kelly cell* K *satisfies the equation*



Proof. It is enough to establish this equation after post-whiskering with $X\lambda_Y$. The rest is a routine calculation, using on one side the definition of the cell N^λ, then naturality of K with respect to $f = X$ and $g = \lambda_Y$. On the other side, use the definition of N^ρ and then naturality of K with respect to $f = \rho_X$ and $g = Y$. In the end, two K-cells cancel. □

Combining the 2-cells described so far we get

$$\rho_I I \overset{\mathsf{K}^{-1}}{\Rightarrow} I\lambda_I \overset{\mathsf{N}^\lambda}{\Rightarrow} \lambda_{II} \overset{\mathsf{K}^\lambda}{\Rightarrow} \lambda_I I$$

and hence, by cancelling I on the right, an invertible 2-cell

$$\mathsf{P} : \rho_I \Rightarrow \lambda_I.$$

Now we could also define $\mathsf{Q} : \rho_I \Rightarrow \lambda_I$ in terms of

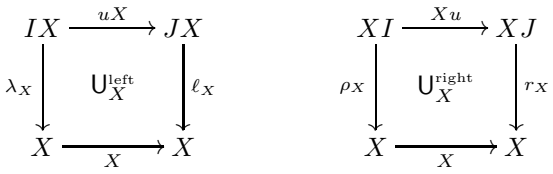
$$I\rho_I \overset{\mathsf{K}^\rho}{\Rightarrow} \rho_{II} \overset{\mathsf{N}^\rho}{\Rightarrow} \rho_I I \overset{\mathsf{K}^{-1}}{\Rightarrow} I\lambda_I.$$

Finally, in analogy with Axiom (1) for monoidal categories:

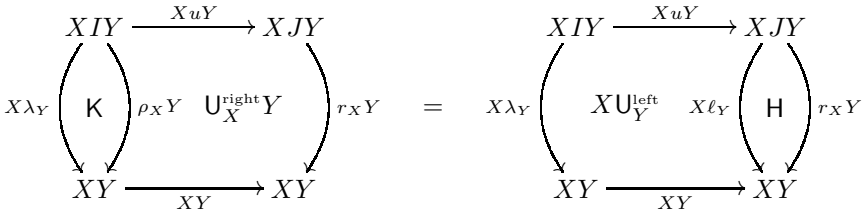
6.9 LEMMA. *We have* $\mathsf{P} = \mathsf{Q}$. *(This is true independently of Axioms* (17) *and* (18)*.)*

Proof. Since I is cancellable, it is enough to show $I\mathsf{P}I = I\mathsf{Q}I$. To establish this equation, use the constructions of P and Q, then substitute the characterising Equations (19) for the auxiliary cells K^λ and K^ρ, and finally use Lemma 6.8. □

6.10. THE 2-CATEGORY OF GPS UNITS. For short we shall say *GPS unit* for the notion of unit just introduced. In summary, a GPS unit is a quadruple $(I, \lambda, \rho, \mathsf{K})$ where I is an object, λ_X and ρ_X are natural transformations with equi-arrow components, and $\mathsf{K} : X\lambda_Y \Rightarrow \rho_X Y$ is a coherent Kelly cell (natural in X and Y, of course).
A *morphism of GPS units* from $(I, \lambda, \rho, \mathsf{K})$ to (J, ℓ, r, H) is an arrow $u : I \to J$ equipped with natural families of invertible 2-cells



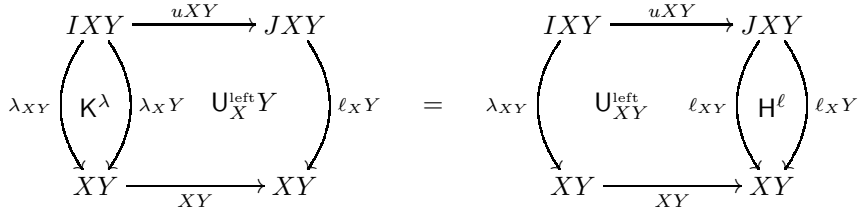
satisfying the equation



(20)

Finally, a 2-*morphism of GPS unit maps* is a 2-cell $\mathsf{T} : u \Rightarrow v$ satisfying the compatibility conditions (15) and (16) of Lemma 5.2.

6.11. REMARKS. Note first that u is automatically an equi-arrow. Observe also that U^{left} and $\mathsf{U}^{\text{right}}$ completely determine each other by Equation (20), as is easily seen by taking on the one hand X to be a left cancellable object and on the other hand Y to be a right cancellable object. Finally note that there are two further equations, expressing compatibility with K^λ and K^ρ, but they can be deduced from Equation (20), independently of the coherence Axioms (17) and (18). Here is the one for K^λ for future reference:



(21)

7 COMPARISON WITH CLASSICAL THEORY (THEOREM E)

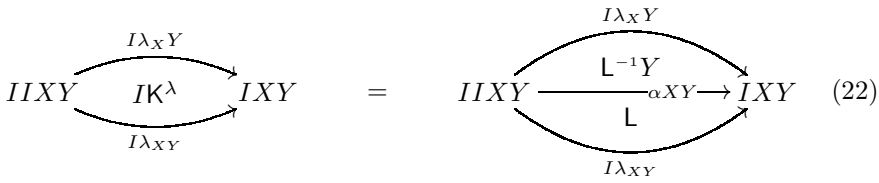
In this section we prove the equivalence between the two notions of unit.

7.1. FROM CANCELLABLE-IDEMPOTENT UNITS TO GPS UNITS. We fix a unit object (I, α). We also assume chosen a left constraint $\lambda_X : IX \to X$ with $\mathsf{L}_X : I\lambda_X \Rightarrow \alpha X$, and a right constraint $\rho_X : XI \to X$ with $\mathsf{R}_X : X\alpha \Rightarrow \rho_X I$. First of all, in analogy with Axioms (2) and (3) of monoidal categories we have:

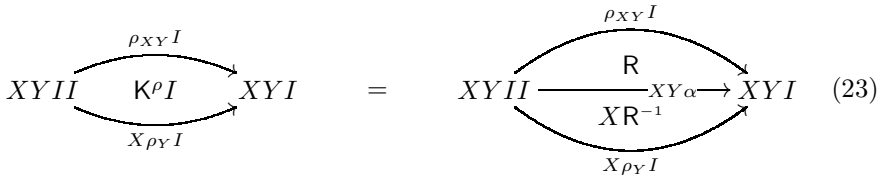
7.2 LEMMA. *In the situation of 7.1, there are unique natural invertible* 2*-cells*

$$\mathsf{K}^\lambda : \lambda_{XY} \Rightarrow \lambda_X Y$$
$$\mathsf{K}^\rho : X\rho_Y \Rightarrow \rho_{XY}$$

satisfying



(22)



(23)

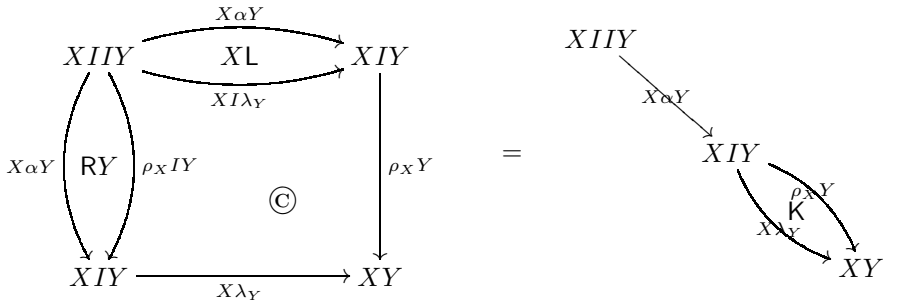
Proof. The conditions precisely define the 2-cells, since I is cancellable. □

7.3 LEMMA. *In the situation of 7.1, there is a canonical family of invertible* 2*-cells (the Kelly cell)*

$$\mathsf{K} : X\lambda_Y \Rightarrow \rho_X Y,$$

natural in X *and* Y*.*

Proof. This is analogous to the construction of the associator: K is defined as the unique 2-cell $\mathsf{K} : X\lambda_Y \Rightarrow \rho_X Y$ satisfying the equation

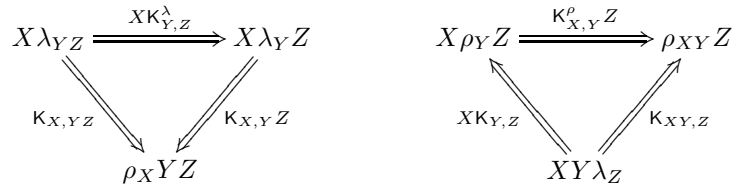


(24)

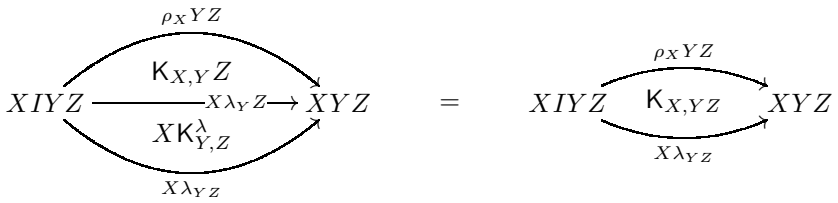
This makes sense since $X\alpha Y$ is an equi-arrow, so we can cancel it away. Clearly K is invertible since L and R are. □

We constructed K^λ and K^ρ directly from L, and R. Meanwhile we also constructed K, and we know from classical theory (6.5) that this cell determines the two others. The following proposition shows that all these constructions match up, and in particular that the constructed Kelly cell is coherent on both sides:

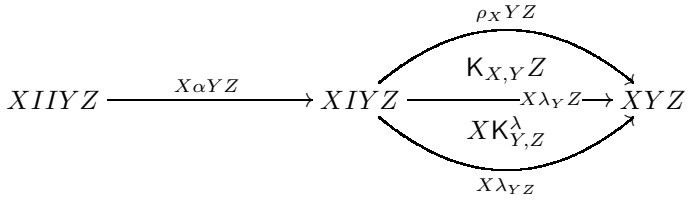
7.4 Proposition. *In the situation of 7.1, the families of* 2*-cells* K*,* K^λ *and* K^ρ *(constructed in 7.2 and 7.3) satisfy the GPS unit axioms* (17) *and* (18)*:*



Proof. We treat the left constraint (the right constraint being completely analogous). We need to establish

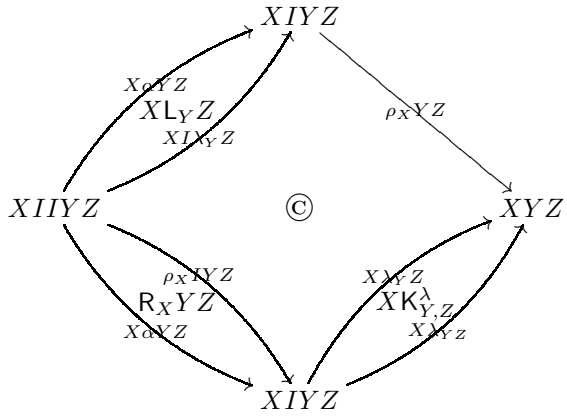


and it is enough to establish this equation pre-whiskered with $X\alpha YZ$. In the diagram resulting from the left-hand side:

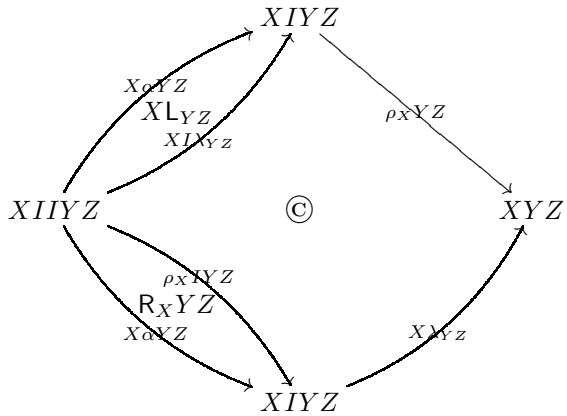


we can replace $(X\alpha YZ)\#(\mathsf{K}_{X,Y}Z)$ by the expression that defined $\mathsf{K}_{X,Y}Z$

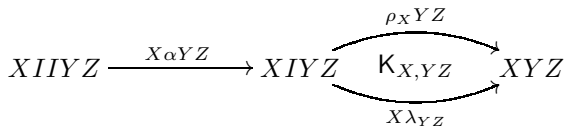
(cf. (24)), yielding altogether



Here we can move the cell $X\mathsf{K}^{\lambda}_{Y,Z}$ across the square, where it becomes $XI\mathsf{K}^{\lambda}_{Y,Z}$ and combines with $X\mathsf{L}_YZ$ to give altogether $X\mathsf{L}_{YZ}$ (cf. (22)). The resulting diagram



is nothing but

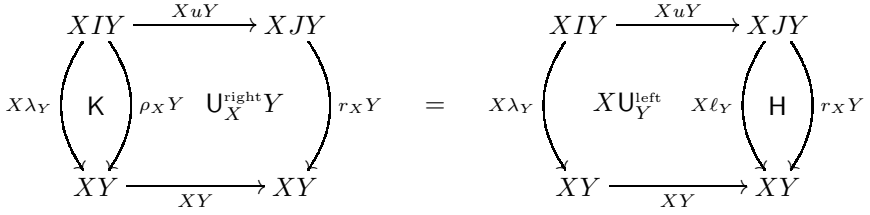


(by Equation (24) again) which is what we wanted to establish. □

Hereby we have concluded the construction of a GPS unit from (I, α). We will also need a result for morphisms:

7.5 PROPOSITION. *Let* $(u, \mathsf{U}) : (I, \alpha) \to (J, \beta)$ *be a morphism of units in the sense of 2.9, and consider the two canonical* 2*-cells* U^{left} *and* $\mathsf{U}^{\text{right}}$ *constructed*

in Lemma 5.1. Then Equation (20) *holds:*



(Hence $(u, \mathsf{U}^{\text{left}}, \mathsf{U}^{\text{right}})$ *is a morphism of GPS units.)*

Proof. It is enough to prove the equation obtained by pasting the 2-cell $X\mathsf{U}Y$ on top of each side of the equation. This enables us to use the characterising equation for K and H. After this rewriting, we are in position to apply Equations (13) and (14), and after cancelling R and L cells, the resulting equation amounts to a cube, where it is easy to see that each side is just $\mathsf{U}^{\text{right}}_X \mathsf{U}^{\text{left}}_Y$. □

7.6. FROM GPS UNITS TO CANCELLABLE-IDEMPOTENT UNITS. Given a GPS unit $(I, \lambda, \rho, \mathsf{K})$, just put

$$\alpha := \lambda_I,$$

then (I, α) is a unit object in the sense of 2.9. Indeed, we already observed that I is cancellable (6.4), and from the outset λ_I is an equi-arrow. That's all! To construct it we didn't even need the Kelly cell, or any of the auxiliary cells or their axioms.

7.7. LEFT AND RIGHT ACTIONS FROM THE KELLY CELL. Start with natural left and right constraints λ and ρ and a Kelly cell $\mathsf{K} : X\lambda_Y \Rightarrow \rho_X Y$ (not required to be coherent on either side). Construct K^λ as in 6.5, put $\alpha := \lambda_I$, and define left and right actions as follows. We define L_X as

$$I\lambda_X \overset{\mathsf{N}^\lambda}{\Rightarrow} \lambda_{IX} \overset{\mathsf{K}^\lambda}{\Rightarrow} \lambda_I X = \alpha X,$$

while we define R_X simply as

$$X\alpha = X\lambda_I \overset{\mathsf{K}_{X,I}}{\Rightarrow} \rho_X I.$$

7.8 PROPOSITION. *For fixed* $(I, \lambda, \rho, \mathsf{K})$*, the following are equivalent:*

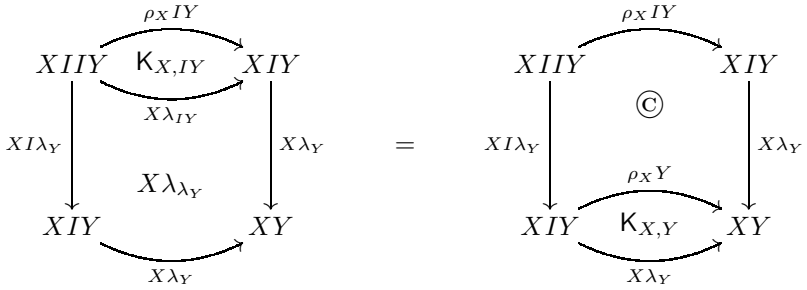
(i) The left and right 2*-cells* L *and* R *just constructed in 7.7 are compatible with the Kelly cell in the sense of Equation* (24)*.*

(ii) The Kelly cell K *is coherent on the left (i.e. satisfies Axiom* (17)*).*

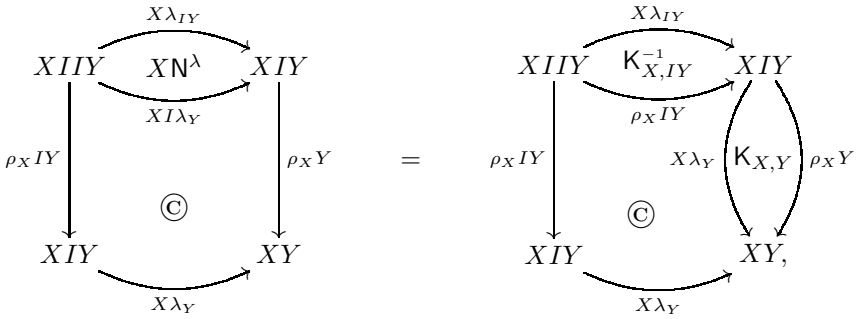
(ii') The Kelly cell K *is coherent on the right (i.e. satisfies Axiom* (18)*).*

Proof. Proposition 7.4 already says that (i) implies both (ii) and (ii'). To prove (ii)⇒(i), we start with an auxiliary observation: by massaging the naturality

equation



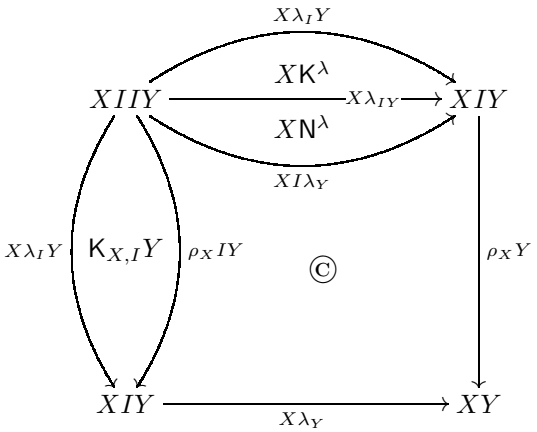
we find the equation



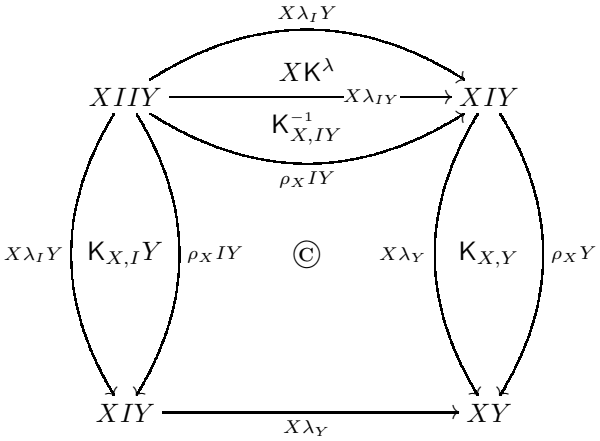
(25)

tailor-made to a substitution we shall perform in a moment.

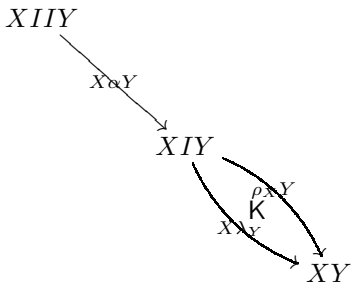
Now for the main computation, assuming first that K is coherent on the left, i.e. that Axiom (17) holds. Start with the left-hand side of Equation (24), and insert the definitions we made for L and R to arrive at



in which we can now substitute (25) to get



Here finally the three 2-cells incident to the $XIIY$ vertex cancel each other out, thanks to Axiom (17), and in the end, remembering $\alpha = \lambda_I$, we get



as required to establish that K satisfies Equation (24). Hence we have proved (ii)⇒(i), and therefore altogether (ii)⇒(ii'). The converse, (ii')⇒(ii) follows now by left-right symmetry of the statements. (But note that the proof via (i) is not symmetric, since it relies on the definition $\alpha = \lambda_I$. To spell out a proof of (ii')⇒(ii), use rather $\alpha = \rho_I$, observing that the intermediate result (i) would refer to different L and R.) □

We have now given a construction in each direction, but both constructions involved choices. With careful choices, applying one construction after the other in either way gets us back where we started. It is clear that this should constitute an equivalence of 2-categories. However, the involved choices make it awkward to make the correspondence functorial directly. (In technical terms, the constructions are ana-2-functors.) We circumvent this by introducing an intermediate 2-category dominating both. With this auxiliary 2-category, the results we already proved readily imply the equivalence.

7.9. A CORRESPONDENCE OF 2-CATEGORIES OF UNITS. Let $\mathscr{U}$ be following 2-category. Its objects are septuples

$$(I, \alpha, \lambda, \rho, \mathsf{L}, \mathsf{R}, \mathsf{K}),$$

with equi-arrows

$$\alpha : II \to I, \quad \lambda_X : IX \to X, \quad \rho_X : XI \to X,$$

(and accompanying naturality 2-cell data), and natural invertible 2-cells

$$\mathsf{L} : I\lambda_X \Rightarrow \alpha X, \quad \mathsf{R} : X\alpha \Rightarrow \rho_X I, \quad \mathsf{K} : X\lambda_Y \Rightarrow \rho_X Y.$$

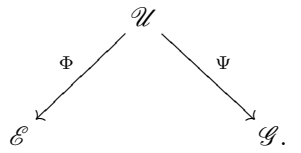
These data are required to satisfy Equation (24) (compatibility of K with L and R).
The arrows in $\mathscr{U}$ from $(I, \alpha, \lambda, \rho, \mathsf{L}, \mathsf{R}, \mathsf{K})$ to $(J, \beta, \ell, r, \mathsf{L}', \mathsf{R}', \mathsf{H})$ are quadruples

$$(u, \mathsf{U}^{\text{left}}, \mathsf{U}^{\text{right}}, \mathsf{U}),$$

where $u : I \to J$ is an arrow in $\mathscr{C}$, U^{left} and $\mathsf{U}^{\text{right}}$ are as in 6.10, and U is a morphism of pseudo-idempotents from (I, α) to (J, β). These data are required to satisfy Equation (20) (compatibility with Kelly cells) as well as Equations (13) and (14) in Lemma 5.1 (compatibility with the left and right 2-cells).
Finally a 2-cell from $(u, \mathsf{U}^{\text{left}}, \mathsf{U}^{\text{right}}, \mathsf{U})$ to $(v, \mathsf{V}^{\text{left}}, \mathsf{V}^{\text{right}}, \mathsf{V})$ is a 2-cell

$$\mathsf{T} : u \Rightarrow v$$

required to be a 2-morphism of pseudo-idempotents (compatibility with U and V as in 2.5), and to satisfy Equation (15) (compatibility with U^{left} and V^{left}) as well as Equation (16) (compatibility with $\mathsf{U}^{\text{right}}$ and $\mathsf{V}^{\text{right}}$).
Let $\mathscr{E}$ denote the 2-category of cancellable-idempotent units introduced in 2.9, and let $\mathscr{G}$ denote the 2-category of GPS units of 6.10. We have evident forgetful (strict) 2-functors



THEOREM E (EQUIVALENCE). *The* 2*-functors* Φ *and* Ψ *are* 2*-equivalences. More precisely they are surjective on objects and strongly fully faithful (i.e. isomorphisms on hom categories).*

Proof. The 2-functor Φ is surjective on objects by Lemma 3.1 and Proposition 7.4. Given an arrow (u, U) in $\mathscr{E}$ and overlying objects in $\mathscr{U}$, Lemma 5.1 says there are unique U^{left} and $\mathsf{U}^{\text{right}}$, and Proposition 7.5 ensures the required compatibility with Kelly cells (Equation (20)). Hence Φ induces a bijection on

objects in the hom categories. Lemma 5.2 says we also have a bijection on the level of 2-cells, hence Φ is an isomorphism on hom categories. On the other hand, Ψ is surjective on objects by 7.7 and Proposition 7.8. Given an arrow $(u, \mathsf{U}^{\mathrm{left}}, \mathsf{U}^{\mathrm{right}})$ in $\mathscr{G}$, Lemma 7.10 below says that for fixed overlying objects in $\mathscr{U}$ there is a unique associated U, hence Ψ induces a bijection on objects in the hom categories. Finally, Lemma 5.2 gives also a bijection of 2-cells, hence Ψ is strongly fully faithful. □

7.10 LEMMA. *Given a morphism of GPS units*

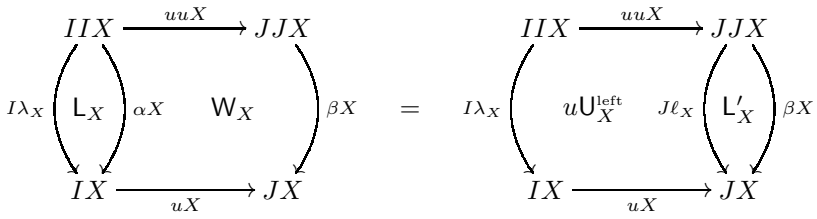
$$(I, \lambda, \rho, \mathsf{K}) \xrightarrow{(u, \mathsf{U}^{\mathrm{left}}, \mathsf{U}^{\mathrm{right}})} (J, \ell, r, \mathsf{H})$$

fix an equi-arrow $\alpha : II \xrightarrow{\sim} I$ with natural families $\mathsf{L}_X : I\lambda_X \Rightarrow \alpha X$ and $\mathsf{R}_X : \alpha X \Rightarrow \rho_X I$ satisfying Equation (24) *(compatibility with K), and fix an equi-arrow $\beta : JJ \xrightarrow{\sim} J$ with natural families $\mathsf{L}'_X : I\ell_X \Rightarrow \beta X$ and $\mathsf{R}'_X : \beta X \Rightarrow r_X I$ also satisfying Equation* (24) *(compatibility with H). Then there is a unique* 2*-cell*

$$\begin{array}{ccc} II & \xrightarrow{uu} & JJ \\ {\scriptstyle\alpha}\downarrow & \mathsf{U} & \downarrow{\scriptstyle\beta} \\ I & \xrightarrow[u]{} & J \end{array}$$

satisfying Equations (13) *and* (14) *(compatibility with $\mathsf{U}^{\mathrm{left}}$ and the left* 2*-cells, as well as compatibility with $\mathsf{U}^{\mathrm{right}}$ and the right* 2*-cells).*

Proof. Working first with left 2-cells, define a family W_X by the equation



It follows readily from Equation (21) that the family has the property

$$\mathsf{W}_{XY} = \mathsf{W}_X Y$$

for all X, Y, and it is a standard argument that since a unit object exists, for example (I, λ_I), this implies that

$$\mathsf{W}_X = \mathsf{U}X$$

for a unique 2-cell

$$\begin{array}{ccc} II & \xrightarrow{uu} & JJ \\ {\scriptstyle\alpha}\downarrow & \mathsf{U} & \downarrow{\scriptstyle\beta} \\ I & \xrightarrow[u]{} & J, \end{array}$$

and by construction this 2-cell has the required compatibility with U^{left} and the left constraints. To see that this U is also compatible with $\mathsf{U}^{\text{right}}$ and the right constraints we reason backwards: (u, U) is now a morphisms of units from (I, α) to (J, β) to which we apply the right-hand version of Lemma 5.1 to construct a new $\mathsf{U}^{\text{right}}$, characterised by the compatibility condition. By Proposition 7.5 this new $\mathsf{U}^{\text{right}}$ is compatible with U^{left} and the Kelly cells K and H (Equation (20)), and hence it must in fact be the original $\mathsf{U}^{\text{right}}$ (remembering from 6.10 that U^{left} and $\mathsf{U}^{\text{right}}$ determine each other via (20)). So the 2-cell U does satisfy both the required compatibilities. □

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