Bundles, Cohomology and Truncated Symmetric Polynomials

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Abstract. The cohomology of the classifying space $BU(n)$ of the unitary group can be identified with the the ring of symmetric polynomials on n variables by restricting to the cohomology of BT, where $T \subset U(n)$ is a maximal torus. In this paper we explore the situation where $BT = (\mathbb{C}P^\infty)^n$ is replaced by a product of finite dimensional projective spaces $(\mathbb{C}P^d)^n$, fitting into an associated bundle

$$U(n) \times_T (\mathbb{S}^{2d+1})^n \to (\mathbb{C}P^d)^n \to BU(n).$$

We establish a purely algebraic version of this problem by exhibiting an explicit system of generators for the ideal of truncated symmetric polynomials. We use this algebraic result to give a precise descriptions of the kernel of the homomorphism in cohomology induced by the natural map $(\mathbb{C}P^d)^n \to BU(n)$. We also calculate the cohomology of the homotopy fiber of the natural map $E\,\mathrm{S}_n \times_{\mathrm{S}_n} (\mathbb{C}P^d)^n \to BU(n)$.

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1. Introduction

One of the nicest calculations in algebraic topology is that of the cohomology of the classifying space $BU(n)$ of the unitary groups as the ring of symmetric polynomials on n variables (see [3]). In fact the restriction map identifies $H^*(BU(n), \mathbb{Z})$ with the invariants in the cohomology of the classifying space BT of a maximal torus under the action of the Weyl group S_n. This leads to a beautiful description of the cohomology of the flag manifold $U(n)/T$ and more specifically a detailed understanding of the fibration $U(n)/T \to BT \to BU(n)$. In this paper we explore the situation where $BT = (\mathbb{C}P^\infty)^n$ is replaced by a product of finite dimensional projective spaces $(\mathbb{C}P^d)^n$, fitting into an associated bundle

$$U(n) \times_T (\mathbb{S}^{2d+1})^n \to (\mathbb{C}P^d)^n \to BU(n).$$

This requires an analysis of truncated symmetric invariants and in particular a precise description of the kernel $I(n,d)$ of the algebra surjection $H^*(BU(n),\mathbb{F}) \to H^*((\mathbb{C}P^d)^n,\mathbb{F})^{\mathrm{S}_n}$. The purely algebraic version of this problem is studied in §5 and §6. In particular, Theorem 5.1 allows us to exhibit an explicit set of generators for $I(n,d)$ as follows.

Theorem 1.1. *Let $\mathbb{F}$ be a field and $I(n,d)$ be the kernel of the map $H^*(BU(n),\mathbb{F}) \to H^*((\mathbb{C}P^d)^n,\mathbb{F})$.*

(a) If $n!$ is invertible in $\mathbb{F}$ then $I(n,d)$ is generated by the elements $P_{d+1}, P_{d+2}, \ldots, P_{d+n}$

(b) If $n < 2\,\mathrm{char}(\mathbb{F}) - 1$ then $I(n,d)$ is generated by $P_{d+1}, P_{d+2}, \ldots, P_{d+n}$ and $P_{\underbrace{d+1,\ldots,d+1}_{p \text{ times}}}$.

For the definition of P_{d+i} and $P_{\underbrace{d+1,\ldots,d+1}_{p^i \text{ times}}}$, see §5. Note that the degree of P_{d+i} is $2(d+i)$ and the degree of $P_{\underbrace{d+1,\ldots,d+1}_{p \text{ times}}}$ is $2p(d+1)$.

If $n!$ is invertible in a field $\mathbb{F}$, then we show that the elements P_{d+i}, $1 \le i \le n$, form a generating regular sequence for $I(n,d)$. In contrast, using Theorem 6.1 we show that in most other cases $I(n,d)$ cannot be generated by a regular sequence:

Theorem 1.2. *If $n \geq \mathrm{char}(\mathbb{F}) > 0$ and $d > 1$, then $I(n,d)$ cannot be generated by a regular sequence.*

There is a free action of S_n on the fiber space $W(n,d) = U(n) \times_T (\mathbb{S}^{2d+1})^n$ arising from the normalizer of the maximal torus in $U(n)$. The orbit space $X(n,d)$ can be realized as the fiber of the natural map $E\,\mathrm{S}_n \times_{\mathrm{S}_n} (\mathbb{C}P^d)^n \to BU(n)$. Our algebraic results allow us to calculate the cohomology of this space in good characteristic.

Theorem 1.3. *If $\mathbb{F}$ is a field where $n!$ is invertible, then the cohomology of $X(n,d)$ is an exterior algebra on n generators*

$$H^*(X(n,d),\mathbb{F}) \cong \Lambda_{\mathbb{F}}(E_{d+1},\ldots,E_{d+n})$$

where E_j is a cohomology class in dimension $2j-1$.

This has an interesting computational consequence.

Theorem 1.4. *For any field $\mathbb{F}$ of coefficients, the Serre spectral sequence for the fibration $(\mathbb{S}^{2d+1})^n \to W(n,d) \to U(n)/T$ collapses at E_2 if and only if $d \geq n-1$. Consequently, we obtain an additive calculation*

$$H^*(W(n,d),\mathbb{F}) \cong H^*(U(n)/T,\mathbb{F}) \otimes H^*((\mathbb{S}^{2d+1})^n,\mathbb{F})$$

whenever $d \geq n-1$. In particular if $n!$ is invertible in $\mathbb{F}$, then

$$H^*(X(n,d),\mathbb{F}) \cong [H^*(U(n)/T,\mathbb{F}) \otimes H^*((\mathbb{S}^{2d+1})^n,\mathbb{F})]^{\mathrm{S}_n} \cong \Lambda_{\mathbb{F}}(E_{d+1},\ldots,E_{d+n})\,.$$

These results follow from a general theorem about the cohomology of fibrations which, although "classical" in nature, seems to be new.

THEOREM 1.5. *Let $\mathbb{F}$ be a field and let $\pi : E \to B$ denote a fibration with fiber F of finite type such that B is simply connected. Assume*

- *$H^*(B, \mathbb{F})$ is a polynomial algebra on n even dimensional generators,*
- *$\pi^* : H^*(B, \mathbb{F}) \to H^*(E, \mathbb{F})$ is surjective,*
- *the kernel of π^* is generated by a regular sequence $u_1, \ldots, u_n$, where $|u_i| = 2r_i$.*

Then $H^*(F, \mathbb{F})$ is an exterior algebra on n odd dimensional generators $e_1, \ldots, e_n$, where $|e_i| = 2r_i - 1$.

It is natural to ask whether the results of this paper can be extended to compact Lie groups, other than $U(n)$. We thus conclude this introduction with the following open problem.

PROBLEM: Let G be a compact Lie group with maximal torus T of rank n and Weyl group W. Describe generators for the kernel $I_G(n, d)$ of the natural map $H^*(BG, \mathbb{F}) \to H^*((\mathbb{C}P^d)^n, \mathbb{F})$ and use this to describe the cohomology of the homotopy fiber of $(\mathbb{C}P^d)^n \to BG$ when $|W|$ is invertible in $\mathbb{F}$.

Theorems 5.1(a) and 6.1(a) have been independently proved in a recent preprint [4] by A. Conca, C. Krattenthaler, J. Watanabe. We are grateful to J. Weyman for bringing this preprint to our attention.

2. BUNDLES AND SYMMETRIC INVARIANTS

A classical computation in algebraic topology is that of the cohomology of the classifying space $BU(n)$ where $U(n)$ is the unitary group of $n \times n$ matrices. We briefly recall how that goes; details can be found, e.g., in the survey paper [3] by A. Borel. Let $T = (\mathbb{S}^1)^n \subset U(n)$ denote the maximal torus of diagonal matrices in $U(n)$; its classifying space is $BT = (\mathbb{C}P^\infty)^n$. The inclusion $T \subset U(n)$ induces a map between the cohomology of $BU(n)$ and the cohomology of BT. Note that the normalizer NT of the torus is a wreath product $\mathbb{S}^1 \wr \mathrm{S}_n$, where the symmetric group S_n acts by permuting the n diagonal entries. Thus the Weyl group NT/T is the symmetric group S_n. Recall that $H^*(BT, \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]$, where the $x_1, \ldots, x_n$ are 2-dimensional generators.

THEOREM 2.1. *The inclusion $T \subset U(n)$ induces an inclusion in cohomology with image the ring of symmetric invariants in the graded polynomial algebra*

$$H^*(BU(n), \mathbb{Z}) \cong H^*(BT, \mathbb{Z})^{\mathrm{S}_n} = \mathbb{Z}[x_1, \ldots, x_n]^{\mathrm{S}_n},$$

where the action of S_n arises from that of the Weyl group. □

Now recall that the complex projective space $\mathbb{C}P^d$ is a natural subspace of $\mathbb{C}P^\infty$; this induces a map

$$\tilde{F}(n, d) : (\mathbb{C}P^d)^n \to BT \to BU(n).$$

The permutation matrices $\mathrm{S}_n \subset U(n)$ act via conjugation on $U(n)$; this restricts to an action on the diagonal maximal torus T which permutes the factors. Applying the classifying space functor yields actions of S_n on BT and $BU(n)$ which make the map $\tilde{F}(n,d)$ equivariant. Note however that the conjugation action on $U(n)$ is homotopic to the identity on $BU(n)$. We conclude that $\tilde{F}(n,d)$ induces the natural map

$$\tilde{F}(n,d)^* : H^*(BU(n),\mathbb{Z}) \cong \mathbb{Z}[x_1,\ldots,x_n]^{\mathrm{S}_n} \to \mathbb{Z}[x_1,\ldots,x_n]/(x_1^{d+1},\ldots x_n^{d+1})$$

in integral cohomology whose image is precisely the ring of truncated symmetric invariants. We should also note that the map $\tilde{F}(n,d)$ is (up to homotopy) the classifying map for the n–fold product of the canonical complex line bundle over $\mathbb{C}P^d$.

To make this effective geometrically, we need to describe the map $\tilde{F}(n,d)$ explicitly as a fibration. The space $(\mathbb{C}P^d)^n$ is a quotient of $(\mathbb{S}^{2d+1})^n$ by the free action of the maximal torus T. Using a standard induction construction we can view our map as a fibration which lies over the classical fibration connecting $U(n)/T$, BT and $BU(n)$. Indeed, the following commutative diagram has fibrations in its rows and columns:

$$\begin{tikzcd}
 & (\mathbb{S}^{2d+1})^n \arrow[r, equal] \arrow[d] & (\mathbb{S}^{2d+1})^n \arrow[d] & \\
W(n,d) \arrow[r, equal] & U(n) \times_T (\mathbb{S}^{2d+1})^n \arrow[r] \arrow[d] & (\mathbb{C}P^d)^n \arrow[r, "\tilde{F}(n,d)"] \arrow[d] & BU(n) \arrow[d, equal] \\
 & U(n)/T \arrow[r] & BT \arrow[r] & BU(n)
\end{tikzcd}$$

Note that we also have a bundle

$$U(n) \to U(n) \times_T (\mathbb{S}^{2d+1})^n \to (\mathbb{C}P^d)^n$$

and its classifying map is $\tilde{F}(n,d)$.

In some of our applications it will also make sense to take a quotient by the action of the symmetric group S_n. For technical reasons this requires taking a *homotopy orbit space* which we now define.

Definition 2.2. Let G denote a compact Lie group acting on a space X, its homotopy orbit space X_{hG} is defined as the quotient of the product space $EG \times X$ by the diagonal G–action, where EG is the universal G–space.

Remark 2.3. It should be noted that if G is a finite group, X is a G–space and $|G|$ is invertible in the coefficients, then the natural projection $X_{hG} \to X/G$ induces an isomorphism in cohomology (this follows from the Vietoris-Begle theorem). Hence for example if $|G|$ is invertible in a coefficient field $\mathbb{F}$, then $H^*(X_{hG},\mathbb{F}) \cong H^*(X,\mathbb{F})^G$ (the algebra of invariants).

In our context, the symmetric group S_n acts by permuting the factors in $(\mathbb{C}P^d)^n$ and we can consider the associated homotopy orbit space

$$(\mathbb{C}P^d)^n_{h\,\mathrm{S}_n} = E\,\mathrm{S}_n \times_{\mathrm{S}_n} (\mathbb{C}P^d)^n .$$

More precisely, the map $BT \to BU(n)$ naturally factors through the classifying space of the normalizer NT, as we have $T \subset NT \subset U(n)$. The space BNT can be identified with $BT_{h\,\mathrm{S}_n} = (\mathbb{C}P^\infty)^n_{h\,\mathrm{S}_n}$, where S_n acts by permuting the factors, as before. This homotopy orbit space restricts to the truncated projective spaces, yielding a map

$$F(n,d) : (\mathbb{C}P^d)^n_{h\,\mathrm{S}_n} \to BU(n)\,,$$

which is surjective in rational cohomology. We would also like to describe this map as a fibration.

The map $(\mathbb{C}P^d)^n \to BT$ is an S_n-equivariant fibration, with fiber $(\mathbb{S}^{2d+1})^n$. This arises from the free T–action on the product of spheres, which extends in the usual way to an action of the semidirect product NT. If we take homotopy orbit spaces we obtain a fibration sequence

$$(\mathbb{S}^{2d+1})^n \to (\mathbb{S}^{2d+1})^n_{hNT} \to BNT\,.$$

Dividing out by the free T–action we can identify $(\mathbb{S}^{2d+1})^n_{hNT} \simeq (\mathbb{C}P^d)^n_{h\,\mathrm{S}_n}$. This makes the fiber of the map $(\mathbb{C}P^d)^n_{h\,\mathrm{S}_n} \to BNT$ very explicit. As before, in order to describe the fibration with target $BU(n)$, it suffices to induce up the action on the fiber to a $U(n)$–action by taking the balanced product $Z = U(n) \times_{NT} (\mathbb{S}^{2d+1})^n$. This yields a fibration sequence

$$Z \to Z_{hU(n)} \to BU(n)\,.$$

Note that

$$Z_{hU(n)} \simeq EU(n) \times_{NT} (\mathbb{S}^{2d+1})^n \simeq (\mathbb{S}^{2d+1})^n_{hNT} \simeq (\mathbb{C}P^d)^n_{h\,\mathrm{S}_n}\,,$$

where the last equivalence follows from taking quotients by the free T–action, as before. Our discussion is summarized in the following diagram of fibrations, analogous to the non–equivariant situation:

$$\begin{tikzcd}
 & (\mathbb{S}^{2d+1})^n \arrow[r, equal] \arrow[d] & (\mathbb{S}^{2d+1})^n \arrow[d] & \\
X(n,d) \arrow[r, equal] & U(n) \times_{NT} (\mathbb{S}^{2d+1})^n \arrow[r] \arrow[d] & E\,\mathrm{S}_n \times_{\mathrm{S}_n} (\mathbb{C}P^d)^n \arrow[r, "F(n,d)"] \arrow[d] & BU(n) \arrow[d, equal] \\
 & U(n)/NT \arrow[r] & BNT \arrow[r] & BU(n)
\end{tikzcd}$$

Hence we have

Proposition 2.4. *Up to homotopy the map $\tilde{F}(n,d) : (\mathbb{C}P^d)^n \to BU(n)$ is a fibration with fiber the compact simply connected manifold*

$$W(n,d) = U(n) \times_T (\mathbb{S}^{2d+1})^n$$

of dimension equal to $n(n+2d)$. There is a free S_n–action on this manifold, and its quotient

$$X(n,d) = U(n) \times_{NT} (\mathbb{S}^{2d+1})^n$$

is homotopy equivalent to the fiber of $F(n,d) : (\mathbb{C}P^d)^n_{h\,\mathrm{S}_n} \to BU(n)$. □

Remark 2.5. Note that there are fibrations

$$(\mathbb{S}^{2d+1})^n \to X(n,d) \to U(n)/NT$$

and

$$U(n) \to X(n,d) \to (\mathbb{C}P^d)^n_{h\,\mathrm{S}_n},$$

where the second one is obtained from pulling back the universal $U(n)$ bundle over $BU(n)$ using $F(n,d)$.

One of our main results in this paper will be to calculate the cohomology of the fibers $W(n,d)$ and $X(n,d)$ associated to the fibrations $\tilde{F}(n,d)$ and $F(n,d)$ respectively.

3. Cohomology calculations when $n!$ is invertible

Our standing assumption in this section (unless stated otherwise) will be that $\mathbb{F}$ is a field such that $n!$ is invertible in $\mathbb{F}$, and cohomology will be computed with $\mathbb{F}$–coefficients. A good example is the field $\mathbb{Q}$ of rational numbers. In this situation we have $H^*(X(n,d),\mathbb{F}) \cong H^*(W(n,d),\mathbb{F})^{\mathrm{S}_n}$; it is this cohomology algebra that we will be most interested in.
We begin by considering the limit case $d = \infty$. In this case $X(n,\infty) = U(n)/NT$ and we are looking at the classical fibration

$$U(n)/NT \to BNT \to BU(n)$$

Proposition 3.1. *The map $BNT \to BU(n)$ induces an isomorphism in cohomology and $U(n)/NT$ is $\mathbb{F}$–acyclic.*

Proof. Indeed, both maps in the sequence

$$H^*(BU(n),\mathbb{F}) \to H^*(BNT,\mathbb{F}) \to H^*(BT,\mathbb{F})^{\mathrm{S}_n}$$

are isomorphisms. Since $BU(n)$ is simply connected, this can only happen if $U(n)/NT$ is acyclic. □

Note that this computation is very different from what the cohomology of the flag manifold $U(n)/T$ looks like; when we divide out by the action of the symmetric group all the reduced cohomology vanishes.
We now consider the unstable case of this result, namely when d is finite. This is considerably more interesting, as we know that the cohomology must be non–trivial. This calculation will be a special case of a more general result about the cohomology of fibrations.

Theorem 3.2. *Let $\pi : E \to B$ denote a fibration with fiber F of finite type such that B is simply connected and*

- *$H^*(B, \mathbb{F})$ is a polynomial algebra on n even dimensional generators,*
- *$\pi^* : H^*(B, \mathbb{F}) \to H^*(E, \mathbb{F})$ is surjective,*
- *the kernel of π^* is generated by a regular sequence $u_1, \ldots, u_n$, where $|u_i| = 2r_i$.*

Then $H^*(F, \mathbb{F})$ is an exterior algebra on n odd dimensional generators $e_1, \ldots, e_n$, where $|e_i| = 2r_i - 1$.

Proof. The cohomology of the fiber F in a fibration

$$F \to E \to B$$

can be studied using the Eilenberg–Moore spectral sequence. We refer the reader to [8], Chapter VIII for details. It has the form:

$$E_2^{*,*} = Tor_{H^*(B,\mathbb{F})}(\mathbb{F}, H^*(E, \mathbb{F})).$$

On the other hand, the hypotheses imply that

$$H^*(E, \mathbb{F}) \cong H^*(B, \mathbb{F})/(u_1, \ldots u_n)\,,$$

where $u_1, \ldots, u_n$ form a regular sequence of maximal length in $H^*(B, \mathbb{F})$, a polynomial algebra on n even dimensional generators. In other words the cohomology of B is free and finitely generated over $\mathbb{F}[u_1, \ldots, u_n]$. Thus the spectral sequence simplifies to

$$E_2^{*,*} = Tor_{H^*(B,\mathbb{F})}(\mathbb{F}, H^*(B, \mathbb{F}) \otimes_{\mathbb{F}[u_1,\ldots,u_n]} \mathbb{F}) \cong Tor_{\mathbb{F}[u_1,\ldots,u_n]}(\mathbb{F}, \mathbb{F})$$

This can be computed using the standard Koszul complex, yielding

$$E_2 = \Lambda_{\mathbb{F}}(e_1, \ldots, e_n)\,,$$

where the e_i are exterior classes in degree $2r_i - 1$. There are no further differentials, as the algebra generators for $E_2^{*,*}$ represent non–trivial elements in the cohomology of F which by construction must transgress to the regular sequence $\{u_1, \ldots, u_n\}$ in $H^*(B, \mathbb{F})$ in the Serre spectral sequence for the fibration

$$F \to E \to B.$$

Therefore the Eilenberg–Moore spectral sequence collapses at $E_2 = E_\infty$. Now this algebra is a free graded commutative algebra, hence there are no extension problems and it follows that

$$H^*(F, \mathbb{F}) \cong \Lambda_{\mathbb{F}}(e_1, \ldots, e_n)$$

as stated in the theorem. □

We now apply this result to the spaces $X(n, d)$.

Theorem 3.3. *The cohomology of $X(n, d)$ is an exterior algebra on n generators*

$$H^*(X(n, d), \mathbb{F}) \cong \Lambda_{\mathbb{F}}(E_{d+1}, \ldots, E_{d+n})\,,$$

where E_j is a cohomology class in dimension $2j - 1$.

Proof. As observed previously we have a fibration

$$X(n,d) \to (\mathbb{C}P^d)^n_{h\,\mathrm{S}_n} \to BU(n)\,.$$

The Eilenberg–Moore spectral sequence can therefore be applied to compute the cohomology of $X(n,d)$. The map $F(n,d) : (\mathbb{C}P^d)^n_{h\,\mathrm{S}_n} \to BU(n)$ induces a surjection of algebras

$$H^*(BU(n),\mathbb{F}) \to H^*((\mathbb{C}P^d)^n_{h\,\mathrm{S}_n},\mathbb{F}) \to 0$$

which can be identified with the natural map

$$\mathbb{F}[x_1,\dots,x_n]^{\mathrm{S}_n} \to (\mathbb{F}[x_1,\dots,x_n]/(x_1^{d+1},\dots x_n^{d+1}))^{\mathrm{S}_n}.$$

The kernel of this map is precisely the ideal

$$I_{n,d} = (x_1^{d+1},\dots,x_n^{d+1}) \cap \mathbb{F}[x_1,\dots,x_n]^{\mathrm{S}_n}.$$

By Theorem 6.1(a), $I_{n,d}$ is generated by a regular sequence of elements $P_{d+1},\dots,P_{d+n}$. Here each P_j is a homogeneous polynomial in $x_1,\dots,x_n$ of degree j; its degree as a cohomology class is $2j$. These classes form a regular sequence of maximal length in the polynomial algebra $H^*(BU(n),\mathbb{F})$. Thus the hypotheses of Theorem 3.2 hold, and the proof is complete. $\square$

Corollary 3.4. *If $d < \infty$, then $X(n,d)$ is a compact, connected, orientable manifold.*

Proof. According to our calculation, for $m = n(n+2d)$ we have $H^m(X(n,d),\mathbb{Q}) \cong \mathbb{Q}$. This is precisely the dimension of the compact manifold $X(n,d) = U(n) \times_{NT} (\mathbb{S}^{2d+1})^n$, whence the result follows. $\square$

Remark 3.5. Note that as d gets large, the connectivity of the space $X(n,d)$ increases; this is consistent with the stable calculation, namely the acyclicity of $U(n)/NT$. Also note that the manifold $U(n)/NT$ is not orientable, as it is $\mathbb{Q}$–acyclic.

For the case of $W(n,d)$ we offer the following general result:

Theorem 3.6. *For any field $\mathbb{F}$ of coefficients, the Serre spectral sequence for the fibration $(\mathbb{S}^{2d+1})^n \to W(n,d) \to U(n)/T$ collapses at E_2 if and only if $d \geq n-1$, from which we obtain an additive calculation*

$$H^*(W(n,d),\mathbb{F}) \cong H^*(U(n)/T,\mathbb{F}) \otimes H^*((\mathbb{S}^{2d+1})^n,\mathbb{F}).$$

In particular if $n!$ is invertible in $\mathbb{F}$, then

$$H^*(X(n,d),\mathbb{F}) \cong [H^*(U(n)/T,\mathbb{F})\otimes H^*((\mathbb{S}^{2d+1})^n,\mathbb{F})]^{\mathrm{S}_n} \cong \Lambda_{\mathbb{F}}(E_{d+1},\dots,E_{d+n})\,.$$

Proof. Consider the Serre spectral sequence with $\mathbb{F}$ coefficients for the fibration $(\mathbb{S}^{2d+1})^n \to W(n,d) \to U(n)/T$. The base is simply connected and the cohomology of the fiber is generated by the natural generators for the $2d+1$–dimensional cohomology of each sphere. The first differential in the spectral sequence can be computed as follows: if $e_i \in H^{2d+1}((\mathbb{S}^{2d+1})^n,\mathbb{F})$ is a natural generator then

$$d_{2d+2}(e_i) = [x_i^{d+1}] \in H^*(U(n)/T,\mathbb{F}) \cong H^*(BT,\mathbb{F})/(s_1,s_2,\dots,s_n)\,,$$

where the $s_1, s_2, \ldots, s_n$ are the symmetric polynomials. This follows from the diagram of fibrations in the previous section and the well-known calculation of the cohomology of $(\mathbb{C}P^d)^n$ and $U(n)/T$ as quotients of $H^*(BT, \mathbb{F})$. We now need the following algebraic lemma.

Lemma 3.7. *Let $\mathbb{F}$ be a commutative ring and I be the ideal of $\mathbb{F}[x_1, ..., x_n]$ generated by the elementary symmetric polynomials $s_1, ..., s_n$ in $x_1, ..., x_n$. Then (a) $x_1^n \in I$ but (b) $x_1^{n-1} \notin I$.*

Suppose Lemma 3.7 is established (we only need it in the case where $\mathbb{F}$ is a field). Then we conclude that $d_{2d+2}(e_i) = [x_i^{d+1}] = 0$ in $H^*(U(n)/T, \mathbb{F})$ for all $i = 1, \ldots, n$ if and only if $d \geq n-1$. This implies that all the differentials in the spectral sequence are zero and so it collapses at E_2. The assertions of Theorem 3.6 follow from this and Theorem 3.3.

It thus remains to prove Lemma 3.7.

(a) Recall that $x_1, \ldots, x_n$ are, by definition, the roots of the polynomial

$$x^n - x^{n-1}s_1 + x^{n-2}s_2 - ... + (-1)^n s_n = 0\,.$$

Thus $x_1^n = x_1^{n-1}s_1 - x^{n-2}s_2 + ... - (-1)^n s_n$, and since every term in the right hand side lies in I, part (a) follows.

(b) Assume, to the contrary, that

$$x_1^{n-1} = f_1 s_1 + ... + f_n s_n \tag{1}$$

for some polynomials $f_1, ..., f_n \in \mathbb{F}[x_1, \ldots, x_n]$. If such an identity is possible over $\mathbb{F}$, and $\alpha \colon \mathbb{F} \to L$ is a ring homomorphism then, applying α to each of the coefficients of $f_1, \ldots, f_n$, we obtain an identity of the same form over L. Thus, for the purpose of showing that (1) is not possible, we may, without loss of generality, replace $\mathbb{F}$ by L. In particular, we may take L to be the algebraic closure of the field $\mathbb{F}/M$, where M is a maximal ideal of $\mathbb{F}$. After replacing $\mathbb{F}$ by this L, we may assume that $\mathbb{F}$ is an algebraically closed field.

Equating the homogeneous terms of degree $n-1$ on both sides, we see that after replacing $f_1, f_2, \ldots, f_{n-1}$ by their homogeneous parts of degrees $n-2, n-3, \ldots, 0$, respectively, we may assume that $f_n = 0$.

Since $\mathbb{F}$ is an algebraically closed field, $x^n - 1$ factors into a product of linear terms

$$x^n - 1 = (x-\zeta_1)(x-\zeta_2)\cdot\ldots\cdot(x-\zeta_n)\,. \tag{2}$$

for some $\zeta_1, \ldots, \zeta_n \in \mathbb{F}$. (As an aside, we remark that $\zeta_1, \ldots, \zeta_n \in \mathbb{F}$ are distinct if $p = \mathrm{char}(\mathbb{F})$ does not divide n but not in general; at the other extreme, if n is a power of p then $\zeta_1 = \cdots = \zeta_n = 1$.) By (2)

$$s_i(\zeta_1, \ldots, \zeta_n) = (-1)^i \text{ (coefficient of } x^{n-i} \text{ in } x^n - 1) = 0$$

for every $i = 1, \ldots, n-1$. Hence, substituting ζ_i for x_i in (1), and remembering that $f_n = 0$, we obtain $\zeta_1^{n-1} = 0$, i.e., $\zeta_1 = 0$. Since ζ_1 is a root of $x^n - 1 = 0$, we have arrived at a contradiction. This shows that (1) is impossible. The proof of Lemma 3.7 and thus of Theorem 3.6 is now complete. □

Calculations with field coefficients can be pieced together to provide information on the integral cohomology of $X(n,d)$.

Proposition 3.8. *The cohomology ring $H^*(X(n,d),\mathbb{Z})$ has no p–torsion if $p>n$.*

Proof. By our previous results if $p>n$ then

$$\dim_{\mathbb{F}_p} \mathrm{H}^*(\mathrm{X}(\mathrm{n},\mathrm{d}),\mathbb{F}_\mathrm{p}) = \dim_{\mathbb{Q}} \mathrm{H}^*(\mathrm{X}(\mathrm{n},\mathrm{d}),\mathbb{Q}) = 2^\mathrm{n}.$$

Hence by the universal coefficient theorem, there can be no p–torsion in the integral cohomology of $X(n,d)$. □

The situation is more complicated if $n \geq p = \mathrm{char}(\mathbb{F})$. In particular, we will show that in this case the kernel $I(n,d)$ of the map $H^*(BU(p),\mathbb{F}_p) \to H^*((\mathbb{C}P^d)^p,\mathbb{F}_p)$ cannot be generated by a regular sequence for any $d \geq 2$ (and, in most cases for $d=1$ as well); see Theorem 6.1(b). We now provide an explicit calculation in the case where $n=d=p=2$.

Example 3.9. Consider the map $\tilde{F}(2,1) : \mathbb{S}^2 \times \mathbb{S}^2 \to BU(2)$. Its fiber is

$$W(2,1) = U(2) \times_T (\mathbb{S}^3 \times \mathbb{S}^3)$$

which itself fibers over $U(2)/T = \mathbb{S}^2$ with fiber $\mathbb{S}^3 \times \mathbb{S}^3$. Hence for dimensional reasons $H^*(W(2,1),\mathbb{Z}) \cong H^*(\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2,\mathbb{Z})$. The S_2–action on this space exchanges the two 3-spheres and applies the antipodal map on $\mathbb{S}^2$. Thus the orbit space $X(2,1)$ will be rationally cohomologous to $\mathbb{S}^3 \times \mathbb{S}^5$, as predicted by Theorem 3.3. However, it can be shown that $H^*(X(2,1),\mathbb{F}_2)$ has Poincaré series

$$p(t) = 1 + t + t^2 + t^3 + t^5 + t^6 + t^7 + t^8 .$$

On the other hand, the corresponding Poincaré series for rational cohomology is

$$q(t) = 1 + t^3 + t^5 + t^8$$

which accounts for the torsion free classes in the integral cohomology. This example illustrates the presence of 2–torsion in the cohomology of $X(2,1)$. Of course in this case we have $\pi_1(X(2,1)) = \mathbb{Z}/2$, which accounts for the classes in degrees one and two in mod 2 cohomology, and by Poincaré duality for the classes in degrees six and seven.
On the other hand, recall that if $H^*(BU(2),\mathbb{F}_2) \cong \mathbb{F}_2[c_2,c_4]$ and $H^*(\mathbb{S}^2 \times \mathbb{S}^2,\mathbb{F}_2) \cong \Lambda(u_2,v_2)$ then $\tilde{F}(2,1)^*(c_2) = u_2 + v_2$ and $\tilde{F}(2,1)^*(c_4) = u_2v_2$. Thus we see that $\tilde{F}(2,1)^*$ is not surjective and that its kernel is generated by the classes $c_2^2, c_2^3 + c_2c_4, c_4^2$. These classes correspond to the symmetric polynomials $P_2 = x_1^2 + x_2^2$, $P_3 = x_1^3 + x_2^3$ and $P_{2,2} = x_1^2x_2^2$. Note that if 2 is invertible in the coefficients then

$$P_{2,2} = \frac{P_2^2 - (x_1+x_2)P_3 + (x_1x_2)P_2}{2} ,$$

and the third generator is redundant.

More generally, using the algebraic calculations in Theorem 5.1, Theorem 6.1 and Corollary 6.3 we obtain the following.

Theorem 3.10. *Assume that $p \leq n \leq 2p-1$ and $d \geq 2$. Then the kernel of the map induced by $\tilde{F}(n,d)$ in cohomology*

$$\tilde{F}(n,d)^* : H^*(BU(n), \mathbb{F}_p) \to H^*((\mathbb{C}P^d)^n, \mathbb{F}_p)$$

is generated by the following $n+1$ elements:

- P_{d+i}, *where $1 \leq i \leq n$ and $|P_j| = 2j$*
- $P_{\underbrace{d+1,\ldots,d+1}_{p \text{ times}}}$ *and* $|P_{\underbrace{d+1,\ldots,d+1}_{p \text{ times}}}| = 2p(d+1)$

Moreover this kernel cannot be generated by a regular sequence or by fewer than $n+1$ elements. □

4. The orthogonal groups and more calculations at $p = 2$

The situation for $p = 2$ is somewhat different, as there are specific geometric models which are special to this characteristic. Here we consider the standard diagonal inclusion $V = (\mathbb{Z}/2)^n \hookrightarrow O(n)$ into the group of orthogonal $n \times n$ matrices. The group V is self-centralizing in $O(n)$; its normalizer NV is the wreath product $NV = \mathbb{Z}/2 \wr \mathrm{S}_n$. The Weyl group $W = NV/V$ of V in $O(n)$ is thus isomorphic to S_n; it acts on $V = (\mathbb{Z}/2)^n$ by permuting the n factors of $\mathbb{Z}/2$. The classifying space for V is $BV = (\mathbb{R}P^\infty)^n$, its mod 2 cohomology is a polynomial algebra on n one dimensional generators $\mathbb{F}_2[x_1, \ldots, x_n]$. The inclusion induces a map from the cohomology of $BO(n)$ to this algebra, which gives rise to an isomorphism onto the symmetric invariants. As before, the truncated projective space $\mathbb{R}P^d$ is a natural subspace of $\mathbb{R}P^\infty$, and Theorem 5.1 provides a description of the kernel of the homomorphism induced by the map $H(n,d) : (\mathbb{R}P^d)^n \to BO(n)$ for $n = 1, 2, 3$.

The classifying space for $NV = \mathbb{Z}/2 \wr \mathrm{S}_n$ is $BNV = (\mathbb{R}P^\infty)^n_{h\,\mathrm{S}_n}$. However, as our calculations are at $p = 2$ and $|\,\mathrm{S}_n\,|$ is even, the homotopy orbit space has a lot more cohomology than just the truncated symmetric invariants (for example, it contains a copy of $H^*(\mathrm{S}_n, \mathbb{F}_2)$). The wreath product NV acts on $(\mathbb{S}^d)^n$ extending the coordinatewise antipodal action of V. Thus we have a fiber bundle $(\mathbb{S}^d)^n \to (\mathbb{R}P^d)^n_{h\,\mathrm{S}_n} \to BNV$, where we identify $(\mathbb{S}^d)^n_{hNV} \simeq (\mathbb{R}P^d)^n_{h\,\mathrm{S}_n}$.

Example 4.1. For $n = 2$ we can identify NV with the dihedral group D_8 and its cohomology has generators e, u v in degrees $1, 1, 2$ respectively with the single relation $e \cdot u = 0$ (see [1]). The elements u, v can be identified with the standard symmetric generators $x_1 + x_2$ and $x_1 x_2$ in $H^*(V, \mathbb{F}_2)^{\mathrm{S}_2}$ via the restriction map. In fact we have isomorphisms (see [1], page 118) $H^*(BD_8, \mathbb{F}_2) \cong H^*(\mathrm{S}_2, H^*(V, \mathbb{F}_2))$ and $H^*((\mathbb{S}^d)^2_{hD_8}, \mathbb{F}_2) \cong H^*(\mathrm{S}_2, H^*((\mathbb{R}P^d)^2, \mathbb{F}_2)$. Using these descriptions and Theorem 5.1 it can be shown that the the kernel of the homomorphism $H^*(BD_8, \mathbb{F}_2) \to H^*((\mathbb{S}^d)^2_{hD_8}, \mathbb{F}_2)$ is the ideal generated by the three elements $P_{d+1} = x_1^{d+1} + x_2^{d+2}$, $P_{d+2} = x_1^{d+2} + x_2^{d+2}$ and $P_{d+1,d+1} = x_1^{d+1} x_2^{d+1}$. This ideal is called the Fadell–Husseini index (see [6]) of the D_8–space $\mathbb{S}^d \times \mathbb{S}^d$;

it has some interesting applications in topology and it has been fully calculated in [2].

Geometrically, the fibration which our mod 2 calculations can be applied to is described by the diagram:

$$
\begin{tikzcd}
 & (\mathbb{S}^d)^n \arrow[r, equal] \arrow[d] & (\mathbb{S}^d)^n \arrow[d] & \\
Y(n,d) \arrow[r, equal] & O(n)\times_V (\mathbb{S}^d)^n \arrow[r] \arrow[d] & (\mathbb{R}P^d)^n \arrow[r, "H(n,d)"] \arrow[d] & BO(n) \arrow[d, equal] \\
 & O(n)/V \arrow[r] & BV \arrow[r] & BO(n)
\end{tikzcd}
$$

Here we recall some classical results. First, from the homotopy long exact sequence of the fibration we see that $O(n)/V$ is path–connected because $\pi_1(BV) \to \pi_1(BO(n)) \cong \mathbb{Z}/2$ is surjective (the dual map in mod 2 cohomology is injective). Its fundamental group acts homologically trivially on $H^*((\mathbb{S}^d)^n, \mathbb{F}_2)$, as it acts through its image in V. Therefore the Serre spectral sequence for the fibration $(\mathbb{S}^d)^n \to Y(n,d) \to O(n)/V$ has the form

$$E_2^{*,*} = H^*(O(n)/V) \otimes H^*((\mathbb{S}^d)^n, \mathbb{F}_2) \implies H^*(Y(n,d), \mathbb{F}_2)\,.$$

Using Lemma 3.7, we see that this spectral sequence collapses at E_2 if and only if $d \geq n-1$.

Theorem 4.2. *If $d \geq n-1$ then we have an additive isomorphism*

$$H^*(Y(n,d), \mathbb{F}_2) \cong H^*(O(n)/V) \otimes H^*((\mathbb{S}^d)^n, \mathbb{F}_2).$$

□

5. Truncated symmetric polynomials

The remainder of this paper will be devoted to the algebraic results used in the previous sections. Let $\mathbb{F}$ be a field. We begin by recalling some standard notational conventions and facts concerning the ring

$$R_n := \mathbb{F}[x_1, \ldots, x_n]^{\mathrm{S}_n}$$

of symmetric polynomials in n variables. For details we refer the reader to [7, Chapter I.2].

If $a_1, \ldots, a_n$ are non-negative integers, we will write $P_{a_1,\ldots,a_n}$ for the sum of monomials $x^{a_1'} \ldots x_n^{a_n'}$, as $a_1', \ldots, a_n'$ range over all possible permutations of $a_1, \ldots, a_n$. A sum of this form is called a *monomial symmetric function.* It has $\dfrac{n!}{\lambda_1! \cdots \lambda_m!}$ terms, where $\lambda_1, \ldots, \lambda_m$ is the partition of n associated to $a_1, \ldots, a_n$. (Recall that this means that that there are m distinct integers among $a_1, \ldots, a_n$, occurring with multiplicities $\lambda_1, \ldots, \lambda_m$, respectively.)

Permuting $a_1, \dots, a_n$ does not change $P_{a_1,\dots,a_n}$, so we will always assume that $a_1 \geq \dots \geq a_n$. With this convention, the monomial symmetric functions $P_{a_1,\dots,a_n}$ form a basis of $R_n := \mathbb{F}[x_1, \dots, x_n]^{\mathrm{S}_n}$ as an $\mathbb{F}$-module. One easily checks that the multiplication rule in this basis is given by

$$P_{a_1,\dots,a_n} \cdot P_{b_1,\dots,b_n} = \sum k_{c_1,\dots,c_n} P_{c_1,\dots,c_n}\,, \tag{3}$$

where $c_1 \geq \ldots \geq c_n$ and there are exactly $k_{c_1,\dots,c_n}$ different ways to write

$$(c_1, \dots, c_n) = (a'_1, \dots, a'_n) + (b'_1, \dots, b'_n)$$

for some permutation $a'_1, \dots, a'_n$ of $a_1, \dots, a_n$ and some permutation $b'_1, \dots, b'_n$ of $b_1, \dots, b_n$.
To make our formulas less cumbersome, we will often abbreviate $P_{a_1,\dots,a_r,0,\dots,0}$ as $P_{a_1,\dots,a_r}$. As long as the number of variables n is fixed, this will not lead to any confusion. For example, in this notation,

$$P_i = x_1^i + \dots + x_n^i$$

is the usual power sum of degree i and

$$\begin{aligned} &P_1 = x_1 + \dots + x_n, \\ &P_{1,1} = x_1x_2 + \dots + x_{n-1}x_n, \\ &\dots \\ &P_{\underbrace{1,\,\dots,\,1}_{n \text{ times}}} = x_1x_2 \dots x_n \end{aligned} \tag{4}$$

are the elementary symmetric polynomials.
The main result of this section is the following theorem.

Theorem 5.1. *Let $\mathbb{F}$ be a field of characteristic $p \geq 0$.*
(a) If $p = 0$ or $n < p$ then the ideal $I_{n,d} := (x_1^{d+1}, \dots, x_n^{d+1}) \cap \mathbb{F}[x_1, \dots, x_n]^{\mathrm{S}_n}$ of $R_n := \mathbb{F}[x_1, \dots, x_n]^{\mathrm{S}_n}$ is generated by $P_{d+1}, \dots, P_{d+n}$.
(b) If $n \leq 2p - 1$ then $I_{n,d}$ is generated by $P_{d+1}, \dots, P_{d+n}$ and $P_{\underbrace{d+1,\,\dots,\,d+1}_{p \text{ times}}}$.

The rest of this section will be devoted to proving Theorem 5.1. Let I be the ideal of $R_n = \mathbb{F}[x_1, \dots, x_n]^{\mathrm{S}_n}$ generated by the polynomials listed in the statement of Theorem 5.1. Clearly, $I \subset I_{n,d}$; we want to prove the opposite inclusion. First we note that every element of $I_{n,d}$ is an $\mathbb{F}$-linear combination of monomial symmetric functions $P_{a_1,\dots,a_n}$, where $a_1 \geq d+1$. Thus in order to prove Theorem 5.1 it suffices to show that every $P_{a_1,\dots,a_n}$ with $a_1 \geq d+1$ lies in I. Our first step in this direction is the following lemma.
We define the *weight* of the monomial symmetric function $P_{a_1,\dots,a_n}$ as the largest integer $r \leq n$ such that $a_r \geq 1$. As mentioned above, we will abbreviate $P_{a_1,\dots,a_n}$ of weight $\leq r$ as $P_{a_1,\dots,a_r}$.
We define the *leading multiplicity* of $P_{a_1,\dots,a_n}$ as the largest integer $s \leq n$ such that $a_1 = \dots = a_s$. Here, as always, we are assuming that $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$.

LEMMA 5.2. *Let $\mathbb{F}$ be a field and $J_{n,d}$ be the ideal of $R_n = \mathbb{F}[x_1, \ldots, x_n]^{\mathrm{S}_n}$ generated by $P_{d+1}, \ldots, P_{d+n}$. Then $J_{n,d}$ contains every monomial symmetric function $P_{a_1,\ldots,a_n}$ with $a_1 \geq d+1$, whose leading multiplicity is invertible in $\mathbb{F}$.*

The leading multiplicity of $P_{a_1,\ldots,a_n}$ is, by definition, an integer between 1 and n. Theorem 5.1(a) is thus an immediate consequence of this lemma.

Proof. We will argue by induction on the weight r of $P_{a_1,\ldots,a_n}$. For the base case, let $r = 1$. That is, we claim that $P_i \in J_{n,d}$ for every $i \geq d+1$. For $i = d+1, \ldots, d+n$ this is given. Applying Newton's identities (cf., e.g., [7, pp. 23-24])

$$P_{m+n+1} = P_1 \cdot P_{m+n} - P_{1,1} \cdot P_{m+n-1} + \cdots + (-1)^{n+1} P_{\underbrace{1,\,\ldots\,,1}_{n \text{ times}}} \cdot P_{m+1}$$

recursively, with $m = d, d+1, d+2$, etc., we see that $P_{m+n+1} \in J_{n,d}$ for every $m \geq d$. This settles the base case.
For the induction step assume that $r \geq 2$. By (3),

$$\begin{aligned} P_{a_1} \cdot P_{a_2,\ldots,a_r} = sP_{a_1,a_2,\ldots,a_r} + P_{a_1+a_2,a_3,\ldots,a_r} + P_{a_1+a_3,a_2,a_4,\ldots,a_r} + \\ \ldots + P_{a_1+a_r,a_2,a_3,\ldots,a_{r-1}}\,, \end{aligned} \tag{5}$$

where s is the leading multiplicity of $P_{a_1,\ldots,a_n}$. Each of the terms

$$P_{a_1+a_2,a_3,\ldots,a_r},\ P_{a_1+a_3,a_2,a_4,\ldots,a_r},\ \ldots,\ P_{a_1+a_r,a_2,a_3,\ldots,a_{r-1}}$$

is a monomial symmetric function of leading multiplicity 1 and weight $r-1$. By the induction assumption each of them lies in $J_{n,d}$. Since we also know that $P_{a_1} \in J_{n,d}$, equation (5) tells us that $P_{a_1,\ldots,a_r} \in J_{n,d}$ whenever s is invertible in $\mathbb{F}$. □

We now turn to the proof of Theorem 5.1(b). Recall that it suffices to show that

$$P_{a_1,\ldots,a_n} \in I \text{ whenever } a_1 \geq d+1. \tag{6}$$

Here I be the ideal of $R_n = \mathbb{F}[x_1, \ldots, x_n]^{\mathrm{S}_n}$ generated by the polynomials listed in the statement of Theorem 5.1(b). Denote the leading multiplicity of $P_{a_1,\ldots,a_n}$ by s. We will now consider three cases.

CASE 1. $s \neq p$. Since we are assuming that $n \leq 2p-1$, this is equivalent to s being invertible in $\mathbb{F}$. Clearly, $J_{n,d} \subset I$; Lemma 5.2 thus tells us that (6) holds.

CASE 2. $s = p$ and $P_{a_1,\ldots,a_n}$ has weight p. In other words, we want to show that

$$P_{\underbrace{a,\,\ldots\,,a}_{p \text{ times}}} \in I\,. \tag{7}$$

Let $e = a - (d+1)$. By (3) we see that

$$P_{\underbrace{d+1,\,\ldots\,,d+1}_{p \text{ times}}} \cdot P_{\underbrace{e,\,\ldots\,,e}_{p \text{ times}}} = P_{\underbrace{a,\,\ldots\,,a}_{p \text{ times}}} + \Gamma\,, \tag{8}$$

where Γ is a positive integer linear combination of monomial symmetric functions of leading multiplicity $\leq p-1$. Thus $\Gamma \in I$ by Case 1. Since by definition, $P_{\underbrace{d+1,\ldots,d+1}_{p \text{ times}}}$ lies in I, the left hand side also lies in I. This shows that (7) holds.

Note that the above argument depends, in a crucial way, on our assumption that $n \leq 2p-1$. For $n \geq 2p$ the sum Γ in (8) would contain a term of the form $P_{d+1,\ldots,d+1,e,\ldots,e}$ (or $P_{e,\ldots,e,d+1,\ldots,d+1}$, if $e > d+1$), with each e and $d+1$ repeating exactly p times. This monomial symmetric function has leading multiplicity p, and in the case we cannot conclude that $\Gamma \in I$.

Case 3. $s = p$, general case. Denote $a_1 = \cdots = a_p$ by a. Using formula (3) once again, we see that

$$P_{a_1,\ldots,a_n} = P_{\underbrace{a,\ldots,a}_{p \text{ times}}} \cdot P_{a_{p+1},\ldots,a_n} + \Delta\,,$$

where Δ is an integer linear combination of orbit sums $P_{c_1,\ldots,c_n}$ of leading multiplicity $\leq p-1$. Note that $P_{\underbrace{a,\ldots,a}_{p \text{ times}}} \in I$ by Case 2 and $\Delta \in I$ by Case 1. We thus conclude that $P_{a_1,\ldots,a_n} \in I$ as well. This completes the proof of Theorem 5.1. $\square$

6. Regular sequences

We now turn to the question of whether or not the ideal $I_{n,d} = (x_1^{d+1},\ldots,x_n^{d+1}) \cap R_n$ of $R_n = \mathbb{F}[x_1,\ldots,x_n]^{\mathrm{S}_n}$ can be generated by a regular sequence. In the sequel we will sometimes use the same symbol for an element of R_n and its coset in $R_n/I_{n,d}$; we hope that this slight abuse of notation will make our formulas more transparent and will not lead to any confusion.

Our goal is to prove the following theorem.

Theorem 6.1. *Let $\mathbb{F}$ be a field of characteristic $p \geq 0$.*

(a) If $n!$ is not divisible by p then $I_{n,d}$ is generated by the regular sequence $P_{d+1},\ldots,P_{d+n}$ in R_n.

(b) Assume that $0 < p \leq n$ and either (i) $n \not\equiv -1$ (mod p) and $d \geq 1$ or (ii) $n \equiv -1$ (mod p) and $d \geq 2$. Then $I_{n,d}$ is not generated by any regular sequence in R_n.

The assumptions on d in part (b) cannot be dropped; see Remark 6.4. Our proof of Theorem 6.1 will rely on the following elementary lemma.

Lemma 6.2. *(a) The elements $P_{a_1,\ldots,a_n}$, with $d \geq a_1 \geq \cdots \geq a_n \geq 0$ form a basis for $R_n/I_{n,d}$ as an $\mathbb{F}$-vector space.*

(b) The Krull dimension of $R_n/I_{n,d}$ is 0.

(c) Suppose $I_{n,d}$ is generated by $r_1,\ldots,r_m \in R_n$, as an ideal of R_n. Then $m \geq n$. Moreover, $r_1,\ldots,r_m$ form a regular sequence in R_n if and only if $m = n$.

Proof. (a) The power sums $P_{a_1,\dots,a_n}$ with $a_1 \geq \cdots \geq a_n \geq 0$ form an $\mathbb{F}$-basis of R_n. The power sums $P_{a_1,\dots,a_n}$ with $a_1 \geq \cdots \geq a_n \geq 0$ and $a_1 \geq d+1$ form an $\mathbb{F}$-basis of $I_{n,d}$, and part (a) follows.
(b) By part (a) $R_n/I_{n,d}$ is a finite-dimensional $\mathbb{F}$-vector space.
(c) Recall that R_n is a polynomial ring over $\mathbb{F}$ generated by the elementary symmetric polynomials in $x_1, \dots, x_n$. In particular, R_n is a Cohen-Macauley ring. Part (c) now follows from part (b). □

Proof of Theorem 6.1. (a) If $p = \mathrm{char}(\mathbb{F})$ does not divide $n!$ then Theorem 5.1(a) tells us that $I_{n,d}$ is generated, as an ideal of R_n, by the n elements $P_{d+1}, \dots, P_{d+n}$. By Lemma 6.2(c) these elements form a regular sequence in R_n.
(b) If $I_{n,d}$ is generated by a regular sequence then $\mathrm{Socle}(R_n/I_{n,d})$ is a 1-dimensional $\mathbb{F}$-vector space; see, e.g. [9, p. 144] or [5, Section 21.2]. It is an immediate consequence of the multiplication formula (3) that

$$P_{\underbrace{d,\,\dots\,,d}_{n \text{ times}}} \in \mathrm{Socle}(R_n/I_{n,d})$$

for any $\mathbb{F}$, d and n.
Thus in order to show that $I_{n,d}$ is not generated by a regular sequence it suffices to exhibit a monomial symmetric function $P_{a_1,\dots,a_n} \in \mathrm{Socle}(R_n/I_{n,d})$, with $(a_1, \dots, a_n) \neq (d, \dots, d)$. Note that $P_{a_1,\dots,a_n}$ and $P_{\underbrace{d,\,\dots\,,d}_{n \text{ times}}}$ are $\mathbb{F}$-linearly independent in $R_n/I_{n,d}$ by Lemma 6.2(a).
(i) Suppose $d \geq 1$ and $n = pq + r$, where $q \geq 1$ and $r \in \{0, 1, \dots, p-2\}$. We claim that in this case $P_{a_1,\dots,a_n}$ lies in $\mathrm{Socle}(R_n/I_{n,d})$, if

$$a_1 = \cdots = a_{pq-1} = d \text{ and } a_{pq} = a_{pq+1} = \cdots = a_n = d-1.$$

To establish this claim, we need to check that for these values of $a_1, \dots, a_n$,

$$P_{a_1,\dots,a_n} \cdot f \in I_{n,d}$$

for every $f \in R_n$. Since R_n is generated by the elementary symmetric polynomials P_1, $P_{1,1}$, etc., it suffices to show that

$$P_{a_1,\dots,a_n} \cdot P_{b_1,\dots,b_n} \in I_{n,d}, \tag{9}$$

where

$$(b_1, \dots, b_n) = (\underbrace{1, \dots, 1}_{s \text{ times}}, 0, \dots, 0)\,. \tag{10}$$

We want to prove (9) for each $s = 1, \dots, n$.
Let us examine the product $P_{a_1,\dots,a_n} \cdot P_{b_1,\dots,b_n}$ using the multiplication formula (3). First of all, note that we may assume without loss of generality that $1 \leq s \leq r+1$. Indeed, if $s > r+1$ then every term $P_{c_1,\dots,c_n}$ appearing in the right hand side of the formula (3) will have $c_1 \geq d+1$ and thus will lie in $I_{n,d}$ (for any base field $\mathbb{F}$).

If $1 \leq s \leq r+1$, the only monomial symmetric functions $P_{c_1,\dots,c_n}$, with $c_1 \leq d$, appearing in the right hand side of (3), will have $c_1 = \cdots = c_{pq+s-1} = d$ and $c_{pq+s} = c_{pq+s+1} = \cdots = c_n = d-1$. This sum will appear with coefficient $k_{c_1,\dots,c_n}$ = number of ways to write $(c_1,\dots,c_n)$ as $(a'_1,\dots,a'_n) + (b'_1,\dots,b'_n)$, where $(a'_1,\dots,a'_n)$ is a permutation of $(a_1,\dots,a_n)$ and $(b'_1,\dots,b'_n)$ is a permutation of $(b_1,\dots,b_n)$. We claim that $k_{c_1,\dots,c_n}$ is divisible by p and hence, is 0 in $\mathbb{F}$; this will immediately imply (9). Indeed, in this case $k_{c_1,\dots,c_n}$ is simply the number of ways to specify which s of the elements $b'_1,\dots,b'_{pq+s-1}$ should be equal to 1 (the remaining ones will be 0). Thus

$$k_{c_1,\dots,c_n} = \binom{pq+s-1}{s}.$$

Since $q \geq 1$ and $1 \leq s \leq r+1 \leq p-1$, this number is divisible by p, as claimed.

(ii) Now suppose $d \geq 2$ and $n = pq+p-1$, where $q \geq 1$. We claim that in this case $P_{a_1,\dots,a_n}$ lies in $\mathrm{Socle}(R_n/I_{n,d})$, if

$$a_1 = \cdots = a_{pq-1} = d,\ a_{pq} = a_{pq+1} = \cdots = a_{pq+p-2} = d-1$$

and $a_{pq+p-1} = d-2$. Once again, it suffices to show that (9) holds for every $s = 1,\dots,n$, where $(b_1,\dots,b_n)$ is as in (10). The analysis of the product $P_{a_1,\dots,a_n} \cdot P_{b_1,\dots,b_n}$, based on formula (3), is similar to part (i) but a bit more involved.

First of all, we may assume without loss of generality that $1 \leq s \leq p$. Indeed, if $s \geq p+1$, then every monomial symmetric function $P_{c_1,\dots,c_n}$ appearing in the right hand side of (3) will lie in $I_{n,d}$, so that (9) will hold over any base field $\mathbb{F}$.

If $1 \leq s \leq p$ then only two monomial symmetric functions $P_{c_1,\dots,c_n}$ with $c_1 \leq d$ will appear in the right hand side of (3), namely

$$P_{\underbrace{d,\dots,d,}_{pq+s-2}\ \underbrace{d-1,\dots,d-1}_{p-s+1}}$$

and

$$P_{\underbrace{d,\dots,d,}_{pq+s-1}\ \underbrace{d-1,\dots,d-1,}_{p-s-1}\ d-2}$$

with coefficients

$$k_{\underbrace{d,\dots,d,}_{pq+s-2}\ \underbrace{d-1,\dots,d-1}_{p-s+1}} = \binom{pq+s-2}{s-1}(p-s+1)$$

and

$$k_{\underbrace{d,\dots,d,}_{pq+s-1}\ \underbrace{d-1,\dots,d-1,}_{p-s-1}\ d-2} = \binom{pq+s-1}{s},$$

respectively. (The second monomial symmetric function does not occur if $s = p$.) Both of these coefficients are divisible by p and hence, are 0 in $\mathbb{F}$. This completes the proof of Theorem 6.1. $\square$

Corollary 6.3. *Suppose (i) $p \leq n \leq 2p-2$ and $d \geq 1$ or (ii) $n = 2p-1$ and $d \geq 2$. Then the ideal $I_{n,d}$ can be generated by $n+1$ elements of R_n but cannot be generated by n elements.*

Proof. Theorem 5.1(b) tells us that $I_{n,d}$ is generated by $n+1$ elements. If $I_{n,d}$ could be generated by n elements then by Lemma 6.2(c) these n elements would form a regular sequence in R_n, contradicting Theorem 6.1(b). □

Remark 6.4. The conditions that $d \geq 1$ and $d \geq 2$ in parts (i) and (ii) of Theorem 6.1(b) respectively, cannot be dropped. The same goes for conditions (i) and (ii) in Corollary 6.3.
Indeed, suppose $d = 0$. Recall that $R_n = \mathbb{F}[x_1, \ldots, x_n]^{\mathrm{S}_n}$ is a polynomial algebra $\mathbb{F}[s_1, \ldots, s_n]$, where $s_1 = P_1$, $s_2 = P_{1,1}$, etc., are the elementary symmetric polynomials in $x_1, \ldots, x_n$. $I_{n,0}$ is clearly the maximal ideal of R_n generated by the regular sequence $s_1, \ldots, s_n$. Thus Theorem 6.1(b) fails if $d = 0$.
Now suppose $d = 1$ and $n = 2p-1$, where $\mathrm{char}(\mathbb{F}) = p$. By Theorem 5.1(b), $I_{n,1}$ is generated by the $n+1$ elements $P_2, \ldots, P_{n-1}, P_{n+1}$ and $P_{\underbrace{2,\ldots,2}_{p \text{ times}}}$.

Since we are in characteristic p, $P_{n+1} = P_{2p} = P_2^p$, is a redundant generator. In other words, $I_{n,1}$ is generated by the n elements $P_2, \ldots, P_{n-1}, P_n$ and $P_{\underbrace{2,\ldots,2}_{p \text{ times}}}$.

By Lemma 6.2(c) these elements form a regular sequence in R_n. This shows that Theorem 6.1(b) fails for $d = 1$ and $n = 2p-1$. □

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