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σ-Semisimple Rings

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Abstract. The aim of this paper is to give a complete description of σ-rings. Indeed, we define and study a more general class of rings with involution that we call σ-semisimple rings. In particular, we prove that for the left artinian rings with involution, this new definition coincides with the classical definition of semisimple rings.

An involution on a ring A is a map $\sigma : A \longrightarrow A$ subject to the following conditions: $\sigma(x+y) = \sigma(x) + \sigma(y), \sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$, for each $x, y \in A$. The most common example of involution is the transpose when we consider the matrix algebra $M_n(K)$ over an arbitrary field K.
Rings and algebras with involutions have been the object of many studies since von Neumann remarked the role played by the classical adjoint in the algebra of linear operators on a Hilbert space. Especially, the theory of rings with involution has been developped to investigate Lie algebras, Jordan algebras and rings of operators. It was known, that there is a connection between semisimple algebras with involution and the classical semisimple Lie groups (see [6]). Recently, the book of involutions that appeared in 1998, gives more complete description of the new investigations concerning this topic (see [3]).
Let A be a ring with unity 1 and let σ be an involution on A. For clarity, it is interesting to elucidate some of the terminology to be used in the sequel. Given a subset B of A, $\sigma(B)$ will stand for the subset of all involutive images of elements of B. An ideal I of A is called a *σ-ideal* if $\sigma(I) \subseteq I$. Moreover, I is said to be a *σ-minimal* (resp. *σ-maximal*) *ideal of* A if I is minimal (resp. maximal) in the set of nonzero (resp. proper) σ-ideals of A. Observe that if I is an ideal of A, then $I + \sigma(I), I\sigma(I), \sigma(I)I$ and $I \cap \sigma(I)$ are σ-ideals of A. Moreover, if we denote by $\bar{\sigma}$ the map from A/I to A/I defined by $\bar{\sigma}(a + I) = \sigma(a) + I$, then $\bar{\sigma}$ is a well-defined involution on A/I.
Throughout this paper, if (A, σ) and (B, τ) are rings with involutions, we use the notation

$(A,\sigma) \simeq (B,\tau)$ to express the existence of a ring isomorphism $f\ : A \longrightarrow B$ such that $f\circ\sigma = \tau\circ f$.

1. σ-minimal ideals

Throughout this section, A is a ring with unity and σ is an involution of A. If I and J are ideals of A, we will denote the set of all left A-module homomorphisms from I to J by $Hom_A(I,J)$. Then, $f \in Hom_A(I,J)$ is said to be a *σ-homomorphism* if $f\circ\sigma = \sigma\circ f$. We will write $Hom_A^\sigma(I,J)$ for the set of all σ-homomorphisms from I to J.
In what follows, for a σ-ideal I of A, we denote by $S_\sigma(I)$ the set of all σ-ideals J of A such that $J \subseteq I$. Hence, for a nonzero σ-ideal I of A we have $S_\sigma(I) = I$ if and only if I is σ-minimal.

Lemma 1. *Let I and J be σ-ideals of A. Suppose that $f\ :\ \ I \to J$ is a nonzero σ-homomorphism.*
1) If I is σ-minimal, then f is injective.
2) If J is σ-minimal, then f is surjective.

Proof. 1) Let x be an element of $Ker(f)$ and let $a \in A$. Since f is a left module homomorphism, then $ax \in Ker(f)$. Moreover,

$$f(xa) = f\circ\sigma(\sigma(a)\sigma(x)) = \sigma\circ f(\sigma(a)\sigma(x)) = \sigma[\sigma(a)f(\sigma(x))] = \sigma\circ f\circ\sigma(x)\sigma^2(a) = f(x)a = 0.$$

Consequently, $xa \in Ker(f)$. Which proves that $Ker(f)$ is an ideal of A. The fact that $f\circ\sigma(x) = \sigma\circ f(x) = 0$ yields $\sigma(x) \in Ker(f)$. Thus $\sigma(Ker(f)) \subset Ker(f)$. Hence $Ker(f) \in S_\sigma(I)$. As $f \neq 0$, the σ-minimality of I implies that $Ker(f) = 0$ and therefore f is injective.
2) Similarly, $Im(f) \in S_\sigma(J)$. In view of the σ-minimality of J, the fact that $f \neq 0$ implies that $Im(f) = J$, proving the surjectivity of f. □

Corollary 1. *If I is a σ-minimal ideal of A, then $End_A^\sigma(I)$ is a division ring.*

To prove the converse of Corollary 1, we need to introduce a new class of σ-ideals. A σ-ideal I of A is said σ-indecomposable if I cannot be written as a direct sum of nonzero σ-ideals: if $I = P \oplus Q$, then $P = (0)$ or $Q = (0)$. Note that every σ-minimal ideal of A is σ-indecomposable.

Proposition 1. *Let I be a σ-ideal of A such that $I = \oplus_{i\in S} I_i$, where each I_i is a σ-minimal ideal of A. Then the following conditions are equivalent:*
1) I is a σ-minimal ideal.
2) $End_A^\sigma(I)$ is a division ring.
3) I is a σ-indecomposable ideal.

Proof. 1) $\Rightarrow$ 2) This follows from Corollary 1.
2) $\Rightarrow$ 3) Suppose that $I = P \oplus Q$, where both P and Q are σ-ideals of A. Let π denote the projection of I on P associated with this decomposition. It is straightforward to check that $\pi \in End_A^\sigma(I)$. Since $\pi(\pi - id_I) = 0$, the assumption that $End_A^\sigma(I)$ is a division ring implies that $P = (0)$ or $Q = (0)$.
3) $\Rightarrow$ 1) This is obvious. □

2. σ-semisimple rings

We introduce now a new class of rings with involution. We say that a ring with involution (A, σ) is *σ-semisimple* if A is a sum of σ-minimal ideals.

Lemma 2. *Let A be a σ-semisimple ring such that $A = \sum_{i \in S} I_i$, where each I_i is a σ-minimal ideal of A. If P is a σ-minimal ideal of A, then there is a subset T of S such that $A = P \oplus (\oplus_{j \in T} I_j)$.*

Proof. Since I_i are σ-minimal and $P \neq A$, then there exists some $i \in S$ such that $I_i + P$ is a direct sum. Indeed, otherwise $I_i \cap P = I_i$ for all $i \in S$, which implies that $P = A$. Applying Zorn's lemma, there is a subset T of S such that the collection $\{I_i : i \in T\} \cup \{P\}$ is maximal with respect to independence: $(\oplus_{i \in T} I_i) + P = (\oplus_{i \in T} I_i) \oplus P$. Setting $B = (\oplus_{i \in T} I_i) + P$, the maximality of T implies that $I_i \cap B \neq (0)$ for all $i \in S$. Then, the σ-minimality of I_i yields that $I_i \cap B = I_i$, hence $I_i \subseteq B$ for all $i \in S$. Consequently, $B = A$. $\square$

Corollary 2. *For a ring with involution (A, σ), the following conditions are equivalent:*
1) *A is σ-semisimple.*
2) *A is a direct sum of σ-minimal ideals.*

Example. Let A_4 be the alternating group on 4 letters. Consider the group algebra $\mathbb{R}[A_4]$ provided with its canonical involution σ defined by $\sigma(\sum_{g \in A_4} r_g g) = \sum_{g \in A_4} r_g g^{-1}$. From [2], the decomposition of the semisimple algebra $\mathbb{R}[A_4]$ into a direct sum of simple components is as follows: $\mathbb{R}[A_4] = B_1 \oplus B_2 \oplus B_3$, where each B_i is invariant under σ. More explicitely, $B_1 \simeq \mathbb{R}$, $B_2 \simeq \mathbb{C}$ and $B_3 \simeq M_3(\mathbb{R})$. In particular, each B_i is a σ-minimal ideal of $\mathbb{R}[A_4]$. Consequently, $\mathbb{R}[A_4]$ is a σ-semisimple ring.

Now, let A be a σ-semisimple ring. Since A is finitely generated (indeed, 1 generates A), then A has finite lenght. Thus $A = \oplus_{i=1}^{l} I_i$, where each I_i is a σ-minimal ideal of A. It is easy to verify that each I_i is generated by a central symmetric idempotent element $e_i \in A$ (i.e. $e_i^2 = e_i$ and $\sigma(e_i) = e_i$), where $1 = \sum_{i=1}^{l} e_i$. Moreover, $e_i e_j = 0$ for all $i \neq j$. In what follows, we denote by S the set of central symmetric orthogonal idempotents of A, i.e. $S = \{e_1, \ldots, e_l\}$ such that $I_i = Ae_i$.

We say that a central symmetric idempotent $e \in A$ is a σ-primitive idempotent of A if e cannot be written as a sum of two orthogonal central symmetric idempotents of A.

Proposition 2. *For a σ-semisimple ring A, the following statements hold:*
1) *For each $e_i \in S$, e_i is a σ-primitive idempotent.*
2) *$I_1, \ldots, I_l$ are the only σ-minimal ideals of A.*
3) *Every nonzero σ-ideal of A is a direct sum of σ-minimal ideals of A.*
4) *Every σ-ideal of A is generated by a central symmetric idempotent.*

Proof. 1) Suppose $e_i = f_1 + f_2$, where f_1 and f_2 are orthogonal central symmetric idempotents. As $f_1 f_2 = 0$ then $I_i = Ae_i = Af_1 \oplus Af_2$. Since Af_1 and Af_2 are σ-ideals of A, the σ-minimality of I_i yields that $Af_1 = (0)$ or $Af_2 = (0)$. Hence $f_1 = 0$ or $f_2 = 0$.
2) Let T be a σ-minimal ideal of A. For all $1 \leq i \leq l$, it is clear that $TI_i = Te_i = e_i T =$

$e_iAT = I_iT$. Thus TI_i is a σ-ideal of A contained in I_i. The fact that I_i is σ-minimal, implies that either $TI_i = (0)$ or $TI_i = I_i$. If $TI_i = (0)$ for all $1 \le i \le l$ then $TA = T = (0)$ which is impossible. Consequently, there exists some $1 \le j \le l$ such that $TI_j = I_j$. As TI_j is a nonzero σ-ideal contained in T, the σ-minimality of T yields $TI_j = T$. Therefore, $T = I_j$.
3) Let I be a nonzero σ-ideal of A. As $II_i = I_iI$ is a σ-ideal of A contained in I_i, for all $1 \le i \le l$, then either $II_i = (0)$ or $II_i = I_i$. Since $I \ne (0)$, the fact that $I = IA$ assures the existence of some $1 \le t \le l$ such that $I = I_1 \oplus \cdots \oplus I_t$ (one can arrange the indices to have this equality).
4) Let I be a nonzero σ-ideal of A. From 3), there exists some $1 \le t \le l$ such that $I = I_1 \oplus \cdots \oplus I_t$. Setting $e = e_1 + \cdots + e_t$, it is clear that e is a central idempotent element generating I. As $\sigma(e_i) = e_i$ for all $1 \le i \le t$, it follows that e is symmetric i.e. $\sigma(e) = e$. □

Remark. From Proposition 2, it follows that the decomposition of a σ-semisimple ring into a direct sum of σ-minimal ideals is unique.

Now, recall that a ring with involution (A, σ) is said to be a σ-simple ring if (0) and A are the only σ-ideals of A. Let $A = \oplus_{i=1}^{l} I_i$ be a σ-semisimple ring, we have already seen that each I_i is generated by a central symmetric idempotent e_i such that $1 = \sum_{i=1}^{l} e_i$. Hence, I_i is a subring of A with unity e_i. Moreover, I_i is a σ-simple ring for all $1 \le i \le l$. Consequently, every σ-semisimple ring is a direct sum of σ-simple rings.

Theorem 1. *Let (A, σ) be a ring with involution. The following conditions are equivalent:*
1) *A is σ-semisimple.*
2) *Every σ-ideal of A is generated by a unique central symmetric idempotent.*
3) *Every σ-ideal I of A has a complement in $S_\sigma(A)$, i.e. there is a σ-ideal J such that $A = I \oplus J$.*

Proof. 1) $\Rightarrow$ 2) Let I be a σ-ideal of A. The existence of a central symmetric idempotent e generating I, follows from Proposition 2. To prove the uniqueness of e, suppose f a central symmetric idempotent such that $I = Af$. Then $f = fe$ since $f \in Ae$. Samely, $e \in Af$ then $e = ef$. Since both e and f are central, these two equalities yield $e = f$.
2) $\Rightarrow$ 3) Let I be a σ-ideal of A, by hypotheses there exists a central symmetric idempotent e generating I, i.e. $I = Ae$. Set $J = A(1-e)$, it is clear that J is a σ-ideal of A such that $I \oplus J = A$.
3) $\Rightarrow$ 1) If A is a σ-simple ring then A is σ-semi-simple. Now, suppose that A is not a σ-simple ring and let I be a nonzero proper σ-ideal of A. Consider the set $\mathcal{F} = \{J \mid J$ proper σ-ideal of $A,\ I \subseteq J\}$. Since $\mathcal{F}$ is a non-empty set, then Zorn's lemma assures the existence of a maximal element L of $\mathcal{F}$. It is clear that L is a σ-maximal ideal of A. As L has a complement in $S_\sigma(A)$, then there exists a σ-ideal N of A such that $N \oplus L = A$. Hence $(N, \sigma) \simeq (A/L, \bar{\sigma})$, which proves that N is a σ-minimal ideal of A. Consider $P := \sum_{N \in \mathcal{M}} N$, where $\mathcal{M}$ is the set of all σ-minimal ideals of A. As P is a σ-ideal of A, then $P \oplus Q = A$ for some σ-ideal Q of A. Suppose $Q \ne (0)$ and write $1_A = e_1 + e_2$, where $e_1 \in P$ and $e_2 \in Q$ (e_1 and e_2 being central symmetric elements), it is easy to verify that (Q, σ) is a σ-ring with unity e_2. Q cannot be a σ-simple ring. On the other hand, it is clear that every σ-ideal of Q admits a complement. Reasoning as above, we conclude that Q has a σ-minimal ideal which

is also a σ-minimal ideal of A. But this again contradicts the fact that $P \cap Q = (0)$. Hence $Q = (0)$ and therefore $A = P$. Consequently, A is a σ-semisimple ring. □

Corollary 3. *Let $f : (A, \sigma) \longrightarrow (B, \tau)$ be a surjective homomorphism of rings with involution. If A is σ-semisimple then B is τ-semisimple.*

Proof. From Theorem 1, it suffices to show that every τ-ideal of B is generated by a central symmetric idempotent. Let J be a τ-ideal of B. It is straightforward to check that $I = f^{-1}(J)$ is a σ-ideal of A. The σ-semisimplicity of A implies that $I = Ae$ for some central idempotent $e \in A$. Let $\mu = f(e)$, then μ is a central idempotent of B such that $J = B\mu$. Moreover, $\tau(\mu) = \tau \circ f(e) = f \circ \sigma(e) = \mu$, which ends our proof. □

Corollary 4. *If I is a nonzero σ-ideal of a σ-semisimlpe ring A, then $(\frac{A}{I}, \bar{\sigma})$ is a $\bar{\sigma}$-semisimple ring.*

Proposition 3. *Let e be a central symmetric idempotent of a σ-semisimple ring A. Then the following conditions are equivalent :*

1) *Ae is a σ-minimal ideal.*
2) *e is a σ-primitive idempotent.*
3) *$Ae^+ := \{x \in Z(Ae) \mid \sigma(x) = x\}$ is a field, where $Z(Ae)$ denotes the center of the subring Ae of A.*

Proof. 1) $\Rightarrow$ 2) is clear.

2) $\Rightarrow$ 1) Since $\sigma(e) = e$, then Ae is a σ-ideal of A. Writing $A = \oplus_{i=1}^{l} I_i$, from Proposition 2 there exists some $1 \leq r \leq l$ such that $Ae = I_1 \oplus \cdots \oplus I_r$. Consequently, $e = ee_1 + \cdots + ee_r = ee_1 + (ee_2 + \cdots + ee_r)$. As ee_1 and $ee_2 + \cdots + ee_r$ are orthogonal central symmetric idempotents of A, the σ-primitivity of e yields that $e = ee_i$ for some unique $1 \leq i \leq r$. Hence Ae is a nonzero σ-ideal of A contained in I_i. Accordingly, $Ae = I_i$, since I_i is σ-minimal.

3) $\Rightarrow$ 1) Writing $1 = \sum_{i=1}^{l} e_i$, it follows that $e = ee_1 + \cdots + ee_l$. Since ee_i is an idempotent element of the field Ae^+, we then deduce that for $1 \leq i \leq l$, we have either $ee_i = 0$ or $ee_i = e$. As $e \neq 0$, there exists necessarily a unique $1 \leq i \leq l$ such that $ee_i = e$. This implies that $Ae \subseteq Ae_i = I_i$. The σ-minimality of I_i implies that $Ae = I_i$.

1) $\Rightarrow$ 3) Assume Ae is a σ-minimal ideal of A. According to Corollary 1, $End_A^\sigma(Ae)$ is a division ring. Let $f \in End_A^\sigma(Ae)$, the fact that $\sigma(f(e)) = f \circ \sigma(e) = f(e)$ implies that $f(e)$ is a symmetric element of Ae. If a is any element of A, then

$$f(ea) = f \circ \sigma(\sigma(a)e) = \sigma \circ f(\sigma(a)e) = \sigma[\sigma(a)f(e)] = \sigma \circ f(e)a = f(e)a$$

Thus $f(ea) = f(e)a$ for all $a \in A$. Since

$$aef(e) = af(e) = f(ae) = f(ea) = f(e)a = f(e)ea = f(e)ae$$

it follows that $f(e)$ is a central element of Ae and therefore $f(e) \in Ae^+$. Consequently, the map $\Psi : End_A^\sigma(Ae) \longrightarrow Ae^+$ defined by $\Psi(f) = f(e)$ is a well-defined injective map. Moreover, if $f, g \in End_A^\sigma(Ae)$ then

$$\Psi(f \circ g) = f(g(e)) = f(eg(e)) = f(e)g(e) = \Psi(f)\Psi(g).$$

To prove the surjectivity of Ψ, let ae be any element of Ae^+. Define $g \in End_A(Ae)$ by $g(e) = ae$. On one hand $g \circ \sigma(be) = g(\sigma(b)e) = \sigma(b)ae$, for all $b \in A$. On the other hand, since ae is central and e is the unit element of Ae then

$$\sigma \circ g(be) = \sigma(bae) = ae\sigma(b) = ae\sigma(b)e = \sigma(b)eae = \sigma(b)ae$$

hence $\sigma \circ g = g \circ \sigma$. This proves that Ψ is a ring isomorphism. □

Remark. It follows from Proposition 3 and the fact that A has only a finite number of σ-minimal ideals that A has a finite number of σ-primitive idempotents, namely $e_1, \dots, e_l$.

Recall that the σ-Socle $Soc_\sigma(A)$ of A is defined to be the sum of all σ-minimal ideals of A. It is clear that $Soc_\sigma(A)$ is a σ-ideal of A. Now, using $Soc_\sigma(A)$, we give a σ-semisimplicity criterion for a ring with involution as follows.

Proposition 4. *The following conditions are equivalent:*
1) *A is σ-semisimple.*
2) *$Soc_\sigma(A) = A$.*

Proof. Suppose that A is σ-semisimple. Writing $A = \oplus_{i=1}^{l} I_i$ where each I_i is a σ-minimal ideal of A. It follows from 2) of Proposition 2, that $I_1, \dots, I_l$ are the only σ-minimal ideals of A. Consequently $Soc_\sigma(A) = A$. The converse is immediate by the definition of a σ-semisimple ring. □

We will say that a ring A with involution σ is a *σ-artinian ring* if $S_\sigma(A)$ satisfies the descending condition. That is, there are no infinite decreasing sequences of elements of $S_\sigma(A)$. Equivalently, A is σ-artinian if every nonempty subset of $S_\sigma(A)$ contains a minimal element.

Remark. It is straightforward to verify that every σ-semisimple ring is σ-artinian.

Let $\mathcal{M}_\sigma$ denote the set of all σ-maximal ideals of A, and set:

$$Rad_\sigma(A) = \begin{cases} A & \text{if } \mathcal{M}_\sigma = \emptyset \\ \cap_{L \in \mathcal{M}_\sigma} L & \text{otherwise} \end{cases}$$

Proposition 5. *The following statements are equivalent:*
1) *A is σ-semisimple.*
2) *A is σ-artinian and $Rad_\sigma(A) = (0)$.*

Proof. 1) $\Rightarrow$ 2) Writing $A = \oplus_{i=1}^{l} I_i$ where each I_i is a σ-minimal ideal of A and setting $L_i = \oplus_{j \neq i} I_j$, then plainly L_i is a σ-maximal ideal of A. Since $Rad_\sigma(A) \subset \cap_{i=1}^{l} L_i$ and $\cap_{i=1}^{l} L_i = (0)$, then $Rad_\sigma(A) = (0)$.
2) $\Rightarrow$ 1) Since $Rad_\sigma(A) = (0)$, then $\mathcal{M}_\sigma$ is a non-empty set. Let us consider $\mathcal{P} = \{L_{i_1} \cap \cdots \cap L_{i_r}$, where $r \in \mathbb{N}$ and $L_{i_j} \in \mathcal{M}_\sigma\}$. The fact that A is σ-artinian implies that $\mathcal{P}$ has a minimal element, say $L_1 \cap \cdots \cap L_r$ and denote it by I. We claim that $I = (0)$. Indeed, otherwise there exists some $L_j \in \mathcal{M}_\sigma$ such that $I \cap L_j \subset I$ and $I \cap L_j \neq I$, and this contradicts the minimality of I. Since $(A, \sigma) \simeq (\prod_{i=1}^{r} A/L_i, \bar{\sigma})$, the $\bar{\sigma}$-simplicity of A/L_i implies that A is a σ-semisimple ring. □

Since a σ-semisimple ring is a direct sum of σ-simple subrings, it is worthwhile to give some properties of σ-simple rings. For this, observe that every simple ring with involution (A, σ) is a σ-simple ring. The following counterexample shows that the converse is not true.

Counterexample. Let B be a simple ring. We denote by B^o the opposite ring of B and by σ the *exchange involution* defined on $A = B \oplus B^o$ by $\sigma(x, y) = (y, x)$. It is clear that the ring A is *not simple*, since the ideals of A are (0), A, $\{0\} \times B^o$ and $B \times \{0\}$. But A is σ-simple. Indeed, the only σ-ideals of A are 0 and A.

Now we give a sufficient condition for a σ-simple ring to be simple. For this, we use the following terminology: we say that σ is *anisotropic* if

$$\sigma(a)a = 0 \;\Rightarrow\; a = 0 \;:\; \text{for all } a \in A.$$

Proposition 6. *Let (A, σ) be a σ-simple ring. If the involution σ is anisotropic, then A is a simple ring.*

Proof. Let I be an ideal of A. Using the fact that $I \cap \sigma(I)$ is a σ-ideal of A, it follows that either $I \cap \sigma(I) = (0)$ or $I \cap \sigma(I) = A$. If $I \cap \sigma(I) = (0)$, then $\sigma(x)x = 0$ for all $x \in I$. As σ is anisotropic, we then deduce that $x = 0$, and therefore $I = (0)$. If $I \cap \sigma(I) = A$, then $I = A$. Consequently, A is a simple ring. □

Proposition 7. *Let A be a σ-simple ring and let u be an invertible element of A. If $\sigma(u) = \lambda u$ for some element $\lambda \in Z(A)$ satisfying $\sigma(\lambda)\lambda = 1$, then A is $int(u) \circ \sigma$-simple.*

Proof. Let $\tau = int(u) \circ \sigma$. It is readily verified that τ is a well-defined involution on A. For every ideal I of A, it is easy to show that I is a τ-ideal if and only if I is a σ-ideal, which completes the proof. □

Proposition 8. *Let A be a σ-simple ring which is not simple. Then there exists a simple subring B of A such that $A = B \oplus \sigma(B)$.*

Proof. Let I be a nonzero proper ideal of A. Since $I \cap \sigma(I)$ is a σ-ideal of A, then necessarily $I \cap \sigma(I) = (0)$. The fact that $I + \sigma(I)$ is a σ-ideal of A, yields that $I + \sigma(I) = A$. Indeed, otherwise $I = \sigma(I)$ and the σ-semisimplicity of A implies that either $I = (0)$ or $I = A$, which contradicts our assumption. Accordingly, $I \oplus \sigma(I) = A$. Let J be a nonzero ideal of A that is contained in I. A similar reasoning gives $J \oplus \sigma(J) = A$. Choose any element $i \in I$, then there exist $j, j' \in J$ such that $i = j + \sigma(j')$. Hence $i - j = \sigma(j') \in I \cap \sigma(I)$, proving $i = j$. Therefore, $I = J$. Consequently, I is a minimal ideal of A. Moreover, it is readily verified that there is an idempotent element $e \in A$ satisfying $e + \sigma(e) = 1$ such that $I = Ae$. Hence, I is a simple subring, with unity, of A, proving our proposition. □

Remark. It follows from Proposition 8 that if A is a σ-simple ring which is not simple, then there is an idempotent element $e \in A$ satisfying $1 = e + \sigma(e)$.

Proposition 9. *Let (A, σ) be a semisimple ring with involution. Then A is σ-semisimple.*

Proof. According to Theorem 1, it suffices to show that every σ-ideal of A is generated by a central symmetric idempotent. Let I be a σ-ideal of A, the semi-simplicity of A assures the existence of a central idempotent $e \in A$ generating I, i.e. $I = Ae$. Since $\sigma(I) = I$, it follows that $\sigma(e) = e$, proving our proposition. □

As shown in the following counterexample, the converse of Proposition 9 is not true.

Counterexample. Let B be a simple ring which is not semisimple, such a ring exists for it suffices to choose B not left artinian. Consider the ring $A = B \times B^o$ provided with the exchange involution σ defined by $\sigma(x, y) = (y, x)$. We have already seen that A is a σ-semisimple ring. Since B is not semisimple, then necessarily A is not semisimple too.

In the following proposition, we show that for the category of left artinian rings the notions of semisimplicity and σ-semisimplicity are the same.

Proposition 10. *Let A be a left artinian ring and let σ be an involution of A. Then the following conditions are equivalent:*
1) *A is semisimple.*
2) *A is σ-semisimple.*

Proof. 1) $\Rightarrow$ 2) immediate from Proposition 9.
2) $\Rightarrow$ 1) Write $A = \oplus_{i=1}^{l} B_i$, where B_i is a σ-simple subring of A. Since A is left artinian, then B_i is left artinian too, for all $1 \leq i \leq l$. According to Proposition 8, we have to distinguish to cases :
i) B_i is a simple ring. The fact that B_i is left artinian implies that B_i is semisimple.
ii) $B_i = C_i \oplus \sigma(C_i)$ for some simple subring C_i of B_i. Since C_i is left artinian, then C_i is a semisimple ring. Accordingly, B_i is a semisimple ring too.
As a finite direct sum of semisimple rings is a semisimple ring, we then deduce that also A is a semisimple ring. □

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