Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 42 (2001), No. 2, 419-430.

Quantum Co-Adjoint Orbits of the Group of Affine Transformations of the Complex Line[1]

Do Ngoc Diep **Nguyen Viet Hai**

Institute of Mathematics, National Center for Natural Sciences and Technology, P. O. Box 631, Bo Ho, 10.000, Hanoi, Vietnam
e-mail: dndiep@hn.vnn.vn

Haiphong Teacher's Training College
Haiphong City, Vietnam
e-mail: hainviet@yahoo.com

Abstract. We construct star-products on the co-adjoint orbit of the Lie group $\mathrm{Aff}(\mathbb{C})$ of affine transformations of the complex line and apply them to obtain the irreducible unitary representations of this group. These results show the effectiveness of the Fedosov quantization even for groups which are neither nilpotent nor exponential. Together with the result for the group $\mathrm{Aff}(\mathbb{R})$ (see [5]), we thus have a description of quantum $\overline{\mathrm{MD}}$ co-adjoint orbits.

1. Introduction

The notion of $\star$-products was a few years ago introduced and played a fundamental role in the basic problem of quantization, see e.g. references [1, 2, 6, 7, ...], as a new approach to quantization on arbitrary symplectic manifolds. In [5] we have constructed star-products on upper half-plane, obtained the operator $\hat{\ell}_Z,\ Z \in \mathrm{aff}(\mathbb{R}) = \mathrm{Lie}\,\mathrm{Aff}(\mathbb{R})$ and proved that the representation

$$\exp(\hat{\ell}_Z) = \exp(\alpha\frac{\partial}{\partial s} + i\beta e^s)$$

[1]This work was supported in part by the National Foundation for Fundamental Science research of Vietnam and the Alexander von Humboldt Foundation, Germany

of the group $\mathrm{Aff}_0(\mathbb{R})$ coincides with the representation T_{Ω_+} obtained from the orbit method or Mackey small subgroup method. One of the advantages of this group, with which the computation is rather accessible is the fact that its connected component $\mathrm{Aff}_0(\mathbb{R})$ is exponential. We could use therefore the canonical coordinates for Kirillov form on the orbits. It is natural to consider the same problem for the group $\mathrm{Aff}(\mathbb{C})$. We can expect that the calculations and final expressions could be similar to the corresponding ones in real line case, but this group $\mathrm{Aff}(\mathbb{C})$ is no more the exponential, i.e. the exponential map

$$\exp : \mathrm{aff}(\mathbb{C}) \to \widetilde{\mathrm{Aff}(\mathbb{C})}$$

is no longer a global diffeomorphism and the general theory of D. Arnal and J. Cortet (see [1], [2]) could not be directly applicable. We overcame these difficulties by a rather different way which could give new ideas for more general non-exponential groups: To overcome the main difficulty in applying the deformation quantization to this group, we replace the global diffeomorphism in Arnal-Cortet's setting by a local diffeomorphism. With this replacement, we need to pay attention on the complexity of the symplectic Kirillov form in new coordinates. We then computed the inverse image of the Kirillov form on appropriate local charts. The question raised here is how to choose a good local chart in order to make the calculation as simple as possible. The calculation we propose is realized by using complex analysis on very simple complex domain.

Our main result consists of an explicit star-product formula (Proposition 3.5) on the local charts. This means that the functional algebras on co-adjoint orbits admit a suitable deformation, or in other words, we obtained the quantum co-adjoint orbits of this group as exact models of new quantum objects, called "quantum punctured complex planes" $(\mathbb{C}^2 \setminus L)_q$. Then, by using the Fedosov deformation quantization, it is not hard to obtain the list of all irreducible unitary representations (Theorem 4.2) of the group $\mathrm{Aff}(\mathbb{C})$, although the computation in this case, using the Mackey small subgroup method or the modern orbit method, is rather delicate. The infinitesimal generators of those exact models of infinite dimensional irreducible unitary representations, nevertheless, are given by rather simple formulae. We introduce some preliminary result in §2. The operators $\hat{\ell}_A$ which define the representation of the Lie algebra $\mathrm{aff}(\mathbb{C})$ are found in §3. In particular, we obtain the unitary representations of the Lie group $\widetilde{\mathrm{Aff}(\mathbb{C})}$, the universal covering group of the $\mathrm{Aff}(\mathbb{C})$, in Theorem 4.3 of §4.

2. Preliminary results

Recall that the Lie algebra $\mathfrak{g} = \mathrm{aff}(\mathbb{C})$ of affine transformations of the complex line is described as follows (see [4]). It is well-known that the group $\mathrm{Aff}(\mathbb{C})$ is a real 4-dimensional Lie group which is isomorph to the group of matrices:

$$\mathrm{Aff}(\mathbb{C}) \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a, b \in \mathbb{C}, a \neq 0 \right\}$$

The most easy method is to consider X,Y as complex generators, $X = X_1 + iX_2$ and $Y = Y_1 + iY_2$. Then from the relation $[X, Y] = Y$, we get $[X_1, Y_1] - [X_2, Y_2] + i([X_1 Y_2] + [X_2, Y_1]) = Y_1 + iY_2$. This means that the Lie algebra $\mathrm{aff}(\mathbb{C})$ is a real 4-dimensional Lie algebra, having four generators with the only nonzero Lie brackets: $[X_1,Y_1]-[X_2,Y_2]=Y_1$; $[X_2,Y_1]+[X_1,Y_2]=Y_2$ and we can choose another basis denoted again by the same letters such that:

$$[X_1,Y_1]=Y_1;[X_1,Y_2]=Y_2;[X_2,Y_1]=Y_2;[X_2,Y_2]=-Y_1$$

Remark 2.1. The exponential map

$$\exp:\mathbb{C}\longrightarrow\mathbb{C}^*:=\mathbb{C}\backslash\{0\}$$

given by $z\mapsto e^z$ is just the covering map and therefore the universal covering of $\mathbb{C}^*$: $\widetilde{\mathbb{C}^*}\cong\mathbb{C}$. As a consequence one deduces that

$$\widetilde{\mathrm{Aff}(\mathbb{C})}\cong\mathbb{C}\ltimes\mathbb{C}\cong\{(z,w)|z,w\in\mathbb{C}\}$$

with the following multiplication law:

$$(z,w)(z',w'):=(z+z',w+e^zw')$$

Remark 2.2. The co-adjoint orbit of $\widetilde{\mathrm{Aff}(\mathbb{C})}$ in $\mathfrak{g}^*$ passing through $F\in\mathfrak{g}^*$ is denoted by

$$\Omega:=K(\widetilde{\mathrm{Aff}(\mathbb{C})})F=\{K(g)F|g\in\widetilde{\mathrm{Aff}(\mathbb{C})}\}.$$

Then, (see [4]):

1. Each point $(\alpha,0,0,\delta)$ is 0-dimensional co-adjoint orbit $\Omega_{(\alpha,0,0,\delta)}$.
2. The open set $\beta^2+\gamma^2\neq0$ is the single 4-dimensional co-adjoint orbit $\Omega=\Omega_{\beta^2+\gamma^2\neq0}$. We shall use Ω in the form $\Omega\cong\mathbb{C}\times\mathbb{C}^*$.

Remark 2.3. Set

$$\mathbb{H}_k=\{w=q_1+iq_2\in\mathbb{C}|-\infty<q_1<+\infty;\ 2k\pi<q_2<2k\pi+2\pi\};\ k=0,\pm1,\ldots$$

$$L=\{\rho e^{i\varphi}\in\mathbb{C}\,|\,0<\rho<+\infty;\ \varphi=0\}$$

and $\mathbb{C}_k=\mathbb{C}\backslash L$ is a univalent sheet of the Riemann surface of the multi-valued complex analytic function $\mathrm{Ln}(w)$, $(k=0,\pm1,\ldots)$. Then there is a natural diffeomorphism $w\in\mathbb{H}_k\longmapsto e^w\in\mathbb{C}_k$ with each $k=0,\pm1,\ldots$. Now consider the map:

$$\mathbb{C}\times\mathbb{C}\longrightarrow\Omega=\mathbb{C}\times\mathbb{C}^*$$

$$(z,w)\longmapsto(z,e^w),$$

with a fixed $k\in\mathbb{Z}$. We have a local diffeomorphism

$$\varphi_k:\mathbb{C}\times\mathbb{H}_k\longrightarrow\mathbb{C}\times\mathbb{C}_k$$

$$(z,w)\longmapsto(z,e^w).$$

This diffeomorphism φ_k will be needed in the sequel.

On $\mathbb{C} \times \mathbb{H}_k$ we have the natural symplectic form

$$\omega_o = \frac{1}{2}[dz \wedge dw + d\overline{z} \wedge d\overline{w}], \tag{1}$$

induced from $\mathbb{C}^2$. Put $z = p_1 + ip_2$, $w = q_1 + iq_2$ and $(x^1, x^2, x^3, x^4) = (p_1, q_1, p_2, q_2) \in \mathbb{R}^4$, then

$$\omega_o = dp_1 \wedge dq_1 - dp_2 \wedge dq_2.$$

The corresponding symplectic matrix of ω_o is

$$\bigwedge = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \text{ and } \bigwedge^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We have therefore the Poisson brackets of functions as follows. With each $f, g \in C^\infty(\Omega)$,

$$\{f, g\} = \wedge^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1} \frac{\partial g}{\partial p_1} - \frac{\partial f}{\partial p_2} \frac{\partial g}{\partial q_2} + \frac{\partial f}{\partial q_2} \frac{\partial g}{\partial p_2} =$$

$$= 2\Big[\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z} + \frac{\partial f}{\partial \overline{z}} \frac{\partial g}{\partial \overline{w}} - \frac{\partial f}{\partial \overline{w}} \frac{\partial g}{\partial \overline{z}}\Big].$$

Proposition 2.4. *Fixing the local diffeomorphism $\varphi_k (k \in \mathbb{Z})$, we have:*

1. *For any element $A \in \mathrm{aff}(\mathbb{C})$, the corresponding Hamiltonian function $\widetilde{A}$ in local coordinates (z, w) of the orbit Ω is of the form*

$$\widetilde{A} \circ \varphi_k(z, w) = \frac{1}{2}[\alpha z + \beta e^w + \overline{\alpha z} + \overline{\beta} e^{\overline{w}}]$$

2. *In local coordinates (z, w) of the orbit Ω, the Kirillov form Ω is just the standard form (1).*

Proof. 1. Each element $F \in \Omega \subset (\mathrm{Aff}(\mathbb{C}))^*$ is of the form

$$F = zX^* + e^w Y^* = \begin{pmatrix} z & 0 \\ e^w & 0 \end{pmatrix}$$

in local Darboux coordinates (z, w). From this it follows that

$$\widetilde{A}(F) = \langle F, A \rangle = \Re\,\mathrm{tr}(F.A) =$$

$$= \Re\,\mathrm{tr}\begin{pmatrix} \alpha z & \beta z \\ \alpha e^w & \beta e^w \end{pmatrix} = \frac{1}{2}[\alpha z + \beta e^w + \overline{\alpha z} + \overline{\beta} e^{\overline{w}}]$$

2. Using the definition of the Poisson brackets$\{,\}$, associated to a symplectic form ω, we have

$$\{\widetilde{A}, f\} = \alpha \frac{\partial f}{\partial w} - \beta e^w \frac{\partial f}{\partial z} - \overline{\beta} e^{\overline{w}} \frac{\partial f}{\partial \overline{z}} + \overline{\alpha} \frac{\partial f}{\partial \overline{w}}. \tag{2}$$

Let us from now on denote by ξ_A the Hamiltonian vector field (symplectic gradient) corresponding to the Hamiltonian function $\tilde{A}$, $A \in \mathrm{aff}(\mathbb{C})$. Now we consider two vector fields:

$$\xi_A = \alpha_1 \frac{\partial}{\partial w} - \beta_1 e^w \frac{\partial}{\partial z} - \overline{\beta_1} e^{\overline{w}} \frac{\partial}{\partial \overline{z}} + \overline{\alpha_1} \frac{\partial}{\partial \overline{w}};\ \xi_B = \alpha_2 \frac{\partial}{\partial w} - \beta_2 e^w \frac{\partial}{\partial z} - \overline{\beta_2} e^{\overline{w}} \frac{\partial}{\partial \overline{z}} + \overline{\alpha_2} \frac{\partial}{\partial \overline{w}},$$

where $A = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & 0 \end{pmatrix}$; $B = \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & 0 \end{pmatrix} \in \mathrm{aff}(\mathbb{C})$. It is easy to check that

$$\begin{aligned} \xi_A \otimes \xi_B = {} & \beta_1\beta_2 e^{2w} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial z} + \alpha_1\alpha_2 \frac{\partial}{\partial w} \otimes \frac{\partial}{\partial w} + \overline{\beta_1\beta_2} e^{2\overline{w}} \frac{\partial}{\partial \overline{z}} \otimes \frac{\partial}{\partial \overline{z}} + \overline{\alpha_1\alpha_2} \frac{\partial}{\partial \overline{w}} \otimes \frac{\partial}{\partial \overline{w}} + \\ & + (\alpha_1\beta_2 - \alpha_2\beta_1) e^w \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial w} + (\overline{\alpha_1}\overline{\beta_2} - \overline{\alpha_2}\overline{\beta_1}) e^{\overline{w}} \frac{\partial}{\partial \overline{z}} \otimes \frac{\partial}{\partial \overline{w}} + (\alpha_1\overline{\beta_2} - \alpha_2\overline{\beta_1}) e^{\overline{w}} \frac{\partial}{\partial \overline{z}} \otimes \frac{\partial}{\partial w} + \\ & + (\overline{\alpha_1}\beta_2 - \overline{\alpha_2}\beta_1) e^w \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \overline{w}} + (\beta_1\overline{\beta_2} - \overline{\beta_1}\beta_2) e^{w+\overline{w}} \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \overline{z}} + (\alpha_1\overline{\alpha_2} - \overline{\alpha_1}\alpha_2) \frac{\partial}{\partial w} \otimes \frac{\partial}{\partial \overline{w}}. \end{aligned}$$

Thus,

$$\langle \omega, \xi_A \otimes \xi_B \rangle = \frac{1}{2} \Big[(\alpha_1\beta_2 - \alpha_2\beta_1) e^w + (\overline{\alpha_1}\overline{\beta_2} - \overline{\alpha_2}\overline{\beta_1}) e^{\overline{w}} \Big] = \Re\, \mathrm{tr}(F.[A,B]) = \langle F, [A,B] \rangle.$$

Thus the proposition is proved. □

3. Computation of operators $\hat{\ell}_A^{(k)}$

Proposition 3.1. *With $A, B \in \mathrm{aff}(\mathbb{C})$, the Moyal $\star$-product satisfies the relation:*

$$i\widetilde{A} \star i\widetilde{B} - i\widetilde{B} \star i\widetilde{A} = i\widetilde{[A,B]}. \tag{3}$$

Proof. Consider two arbitrary elements $A = \alpha_1 X + \beta_1 Y$; $B = \alpha_2 X + \beta_2 Y$. Then the corresponding Hamiltonian functions are:

$$\widetilde{A} = \frac{1}{2}[\alpha_1 z + \beta_1 e^w + \overline{\alpha_1 z} + \overline{\beta_1} e^{\overline{w}}];\ \widetilde{B} = \frac{1}{2}[\alpha_2 z + \beta_2 e^w + \overline{\alpha_2 z} + \overline{\beta_2} e^{\overline{w}}].$$

It is easy, then, to see that:

$$P^0(\widetilde{A}, \widetilde{B}) = \widetilde{A}.\widetilde{B}$$

$$\begin{aligned} P^1(\widetilde{A}, \widetilde{B}) = \{\widetilde{A}.\widetilde{B}\} &= 2\Big[\frac{\partial \tilde{A}}{\partial z}\frac{\partial \tilde{B}}{\partial w} - \frac{\partial \tilde{A}}{\partial w}\frac{\partial \tilde{B}}{\partial z} + \frac{\partial \tilde{A}}{\partial \overline{z}}\frac{\partial \tilde{B}}{\partial \overline{w}} - \frac{\partial \tilde{A}}{\partial \overline{w}}\frac{\partial \tilde{B}}{\partial \overline{z}}\Big] \\ &= \frac{1}{2}\Big[(\alpha_1\beta_2 - \alpha_2\beta_1)e^w + (\overline{\alpha_1}\overline{\beta_2} - \overline{\alpha_2}\overline{\beta_1})e^{\overline{w}}\Big] \end{aligned}$$

and $P^r(\tilde{A}, \tilde{B}) = 0,\ \forall r \geq 2$.

Thus,

$$\begin{aligned} i\widetilde{A} \star i\widetilde{B} - i\widetilde{B} \star i\widetilde{A} &= \frac{1}{2i}\Big[P^1(i\widetilde{A}, i\widetilde{B}) - P^1(i\widetilde{B}, i\widetilde{A})\Big] = \\ &= \frac{i}{2}\Big[(\alpha_1\beta_2 - \alpha_2\beta_1)e^w + (\overline{\alpha_1}\overline{\beta_2} - \overline{\alpha_2}\overline{\beta_1})e^{\overline{w}}\Big] \end{aligned}$$

on one hand. On the other hand, because of $[A,B]=(\alpha_1\beta_2-\alpha_2\beta_1)Y$ we have

$$i\widetilde{[A,B]}=i\langle F,[A,B]\rangle=\frac{i}{2}\Big[(\alpha_1\beta_2-\alpha_2\beta_1)e^{w}+(\overline{\alpha_1}\overline{\beta_2}-\overline{\alpha_2}\overline{\beta_1})e^{\overline{w}}\Big]$$

The proposition is hence proved. □

For each $A\in\mathfrak{aff}(\mathbb{C})$, the corresponding Hamiltonian function is

$$\widetilde{A}=\frac{1}{2}[\alpha z+\beta e^{w}+\overline{\alpha z}+\overline{\beta}e^{\overline{w}}]$$

and we can consider the operator $\ell_A^{(k)}$ acting on the dense subspace of smooth functions $\mathrm{L}^2(\mathbb{R}^2\times(\mathbb{R}^2)^*,dp_1dq_1dp_2dq_2/(2\pi)^2)^\infty$ by left $\star$-multiplication by $i\widetilde{A}$, i.e. $\ell_A^{(k)}(f)=i\widetilde{A}\star f$. Because of the relation in Proposition 3.1, we have

Corollary 3.2.

$$\ell^{(k)}_{[A,B]}=\ell^{(k)}_A\circ\ell^{(k)}_B-\ell^{(k)}_B\circ\ell^{(k)}_A:=\left[\ell^{(k)}_A,\ell^{(k)}_B\right] \tag{4}$$

From this it is easy to see that the correspondence $A\in\mathfrak{aff}(\mathbb{C})\longmapsto\ell_A^{(k)}=i\tilde{A}\star$. is a representation of the Lie algebra $\mathfrak{aff}(\mathbb{C})$ on the space $\mathrm{C}^\infty(\Omega)\big[[\frac{i}{2}]\big]$ of formal power series, see [7] for more detail.

Now, let us denote $\mathcal{F}_z(f)$ the partial Fourier transform of the function f from the variable $z=p_1+ip_2$ to the variable $\xi=\xi_1+i\xi_2$, i.e.

$$\mathcal{F}_z(f)(\xi,w)=\frac{1}{2\pi}\iint_{\mathbb{R}^2}e^{-i\Re(\xi\overline{z})}f(z,w)dp_1dp_2.$$

Let us denote by

$$\mathcal{F}_z^{-1}(f)(z,w)=\frac{1}{2\pi}\iint_{\mathbb{R}^2}e^{i\Re(\xi\overline{z})}f(\xi,w)d\xi_1d\xi_2$$

the inverse Fourier transform.

Lemma 3.3. *Putting $g=g(z,w)=\mathcal{F}_z^{-1}(f)(z,w)$ we obtain:*

1. $\partial_z g=\frac{i}{2}\overline{\xi}g\ ;\partial_z^r g=\left(\frac{i}{2}\overline{\xi}\right)^r g, r=2,3,\dots$
2. $\partial_{\overline{z}} g=\frac{i}{2}\xi g\ ;\partial_{\overline{z}}^r g=\left(\frac{i}{2}\xi\right)^r g, r=2,3,\dots$
3. $\mathcal{F}_z(zg)=2i\partial_{\overline{\xi}}\mathcal{F}_z(g)=2i\partial_{\overline{\xi}}f\ ;\mathcal{F}_z(\overline{z}g)=2i\partial_\xi\mathcal{F}_z(g)=2i\partial_\xi f$
4. $\partial_w g=\partial_w(\mathcal{F}_z^{-1}(f))=\mathcal{F}_z^{-1}(\partial_w f);\ \partial_{\overline{w}}g=\partial_{\overline{w}}(\mathcal{F}_z^{-1}(f)=\mathcal{F}_z^{-1}(\partial_{\overline{w}}f)$

Proof. As $\partial_z=\frac{1}{2}(\partial_{p_1}-i\partial_{p_2});\quad\partial_{\overline{z}}=\frac{1}{2}(\partial_{p_1}+i\partial_{p_2})$ we obtain 1. and 2.

3. We have $\mathcal{F}_z(zg)=$

$$=\frac{1}{2\pi}\iint e^{-i(p_1\xi_1+p_2\xi_2)}p_1g(z,w)dp_1dp_2+i\frac{1}{2\pi}\iint e^{-i(p_1\xi_1+p_2\xi_2)}p_2g(z,w)dp_1dp_2=$$

$$=i\partial_{\xi_1}\mathcal{F}_z(g)+i^2\partial_{\xi_2}\mathcal{F}_z(g)=(i\partial_{\xi_1}-\partial_{\xi_2})\mathcal{F}_z(g)=2i\partial_{\overline{\xi}}\mathcal{F}_z(g)=2i\partial_{\overline{\xi}}f$$

and $\mathcal{F}_z(\overline{z}g) =$

$$= \frac{1}{2\pi}\iint e^{-i(p_1\xi_1+p_2\xi_2)}p_1 g(z,w)dp_1dp_2 - i\frac{1}{2\pi}\iint e^{-i(p_1\xi_1+p_2\xi_2)}p_2 g(z,w)dp_1dp_2 = 2i\partial_\xi f.$$

4. The proof is straightforward. The Lemma 3.3 is therefore proved. □

We also need another lemma which will be used in the sequel.

Lemma 3.4. *With $g = \mathcal{F}_z^{-1}(f)(z,w)$, we have:*

1. $\mathcal{F}_z(P^0(\widetilde{A},g)) = i(\alpha\partial_{\overline{\xi}} + \overline{\alpha}\partial_\xi)f + \frac{1}{2}\beta e^w f + \frac{1}{2}\overline{\beta}e^{\overline{w}}f.$
2. $\mathcal{F}_z(P^1(\widetilde{A},g)) = \overline{\alpha}\partial_{\overline{w}}f + \alpha\partial_w f - \overline{\beta}e^{\overline{w}}(\frac{i}{2}\xi)f - \beta e^w(\frac{i}{2}\overline{\xi})f.$
3. $\mathcal{F}_z(P^r(\widetilde{A},g)) = (-1)^r.2^{r-1}[\overline{\beta}e^{\overline{w}}(\frac{i}{2}\xi)^r + \beta e^w(\frac{i}{2}\overline{\xi})^r]f \qquad \forall r \geq 2.$

Proof. 1. Applying Lemma 3.3 we obtain $P^0(\widetilde{A},g) = \widetilde{A}.g = \frac{1}{2}[\alpha zg + \beta e^w g + \overline{\alpha z}g + \overline{\beta}e^{\overline{w}}g]$. Thus,

$$\mathcal{F}_z(P^0(\widetilde{A},g)) = \frac{1}{2}[\alpha\mathcal{F}_z(zg) + \beta e^w\mathcal{F}_z(g) + \overline{\alpha}\mathcal{F}_z(\overline{z}g) + \overline{\beta}e^{\overline{w}}\mathcal{F}_z(g)] =$$

$$\frac{1}{2}[2i\alpha\partial_{\overline{\xi}}\mathcal{F}_z(g) + 2i\overline{\alpha}\partial_\xi\mathcal{F}_z(g) + \beta e^w\mathcal{F}_z(g) + \overline{\beta}e^{\overline{w}}\mathcal{F}_z(g)] = i(\alpha\partial_{\overline{\xi}} + \overline{\alpha}\partial_\xi)f + \frac{1}{2}\beta e^w f + \frac{1}{2}\overline{\beta}e^{\overline{w}}f.$$

2.
$$(P^1(\widetilde{A},g)) = \wedge^{12}\partial_{p_1}\widetilde{A}\partial_{q_1}g + \wedge^{21}\partial_{q_1}\widetilde{A}\partial_{p_1}g + \wedge^{34}\partial_{p_2}\widetilde{A}\partial_{q_2}g + \wedge^{43}\partial_{q_2}\widetilde{A}\partial_{p_2}g =$$
$$= \overline{\alpha}\partial_{\overline{w}}g + \alpha\partial_w g - \overline{\beta}e^{\overline{w}}\partial_{\overline{z}}g - \beta e^w\partial_z g.$$

This implies that: $\mathcal{F}_z(P^1(\widetilde{A},g)) =$

$$= \overline{\alpha}\partial_{\overline{w}}\mathcal{F}_z(g) + \alpha\partial_w\mathcal{F}_z(g) - \overline{\beta}e^{\overline{w}}\partial_{\overline{z}}\mathcal{F}_z(g) - \beta e^w\partial_z\mathcal{F}_z(g) =$$
$$= \overline{\alpha}\partial_{\overline{w}}f + \alpha\partial_w f - \overline{\beta}e^{\overline{w}}(\frac{i}{2}\xi)f - \beta e^w(\frac{i}{2}\overline{\xi})f.$$

3.
$$P^2(\widetilde{A},g) = \wedge^{21}\wedge^{21}\partial_{q_1q_1}\widetilde{A}\partial_{p_1p_1}g + \wedge^{21}\wedge^{43}\partial_{q_1q_2}\widetilde{A}\partial_{p_1p_2}g + \wedge^{43}\wedge^{21}\partial_{q_2q_1}\widetilde{A}\partial_{p_2p_1}g +$$
$$+ \wedge^{43}\wedge^{43}\partial_{q_2q_2}\widetilde{A}\partial_{p_2p_2}g = \frac{1}{2}\big[(\overline{\beta}e^{\overline{w}} + \beta e^w - \beta e^w + \overline{\beta}e^{\overline{w}} + \overline{\beta}e^{\overline{w}} - \beta e^w + \beta e^w + \overline{\beta}e^{\overline{w}})\partial^2_{\overline{z}}g +$$
$$+ (\overline{\beta}e^{\overline{w}} + \beta e^w + \beta e^w - \overline{\beta}e^{\overline{w}} - \overline{\beta}e^{\overline{w}} + \beta e^w + \beta e^w + \overline{\beta}e^{\overline{w}})\partial^2_z g\big] = 2\overline{\beta}e^{\overline{w}}\partial^2_{\overline{z}}g + 2\beta e^w\partial^2_z g.$$

This implies also that:

$$\mathcal{F}_z(P^2(\widetilde{A},g)) = 2\overline{\beta}e^{\overline{w}}\mathcal{F}_z(\partial^2_{\overline{z}}g) + 2\beta e^w\mathcal{F}_z(\partial^2_z g) = 2\overline{\beta}e^{\overline{w}}(\frac{i}{2}\xi)^2 f + 2\beta e^w(\frac{i}{2}\overline{\xi})^2 f.$$

By analogy,

$$P^3(\widetilde{A},g) = (-1)^3[4\overline{\beta}e^{\overline{w}}\partial^3_{\overline{z}}g + 4\beta e^w\partial^3_z g],$$
$$\mathcal{F}_z(P^3(\widetilde{A},g)) = (-1)^3.2^2[\overline{\beta}e^{\overline{w}}(\frac{i}{2}\xi)^3 f + \beta e^w(\frac{i}{2}\overline{\xi})^3 f],$$

and with $r \geq 4$

$$P^r(\widetilde{A},g) = (-1)^r.2^{r-1}[\overline{\beta}e^{\overline{w}}\partial^r_{\overline{z}}g + \beta e^w\partial^r_z g],$$

$$\mathcal{F}_z(P^r(\widetilde{A},g)) = (-1)^r.2^{r-1}[\overline{\beta}e^{\overline{w}}(\frac{i}{2}\xi)^r + \beta e^w(\frac{i}{2}\overline{\xi})^r]f.$$

Lemma 3.4 is therefore proved. □

Proposition 3.5. *For each* $A = \begin{pmatrix}\alpha & \beta\\ 0 & 0\end{pmatrix} \in \mathrm{aff}(\mathbb{C})$ *and for each compactly supported* C^∞-*function* $f \in C^\infty_c(\mathbb{C}\times\mathbb{H}_k)$, *we have:* $\ell^{(k)}_A f := \mathcal{F}_z \circ \ell^{(k)}_A \circ \mathcal{F}^{-1}_z(f) =$

$$= [\alpha(\frac{1}{2}\partial_w - \partial_{\overline{\xi}})f + \overline{\alpha}(\frac{1}{2}\partial_{\overline{w}} - \partial_\xi)f + \frac{i}{2}(\beta e^{w-\frac{1}{2}\overline{\xi}} + \overline{\beta}e^{\overline{w}-\frac{1}{2}\xi})f]. \tag{5}$$

Proof. Applying Lemma 3.4, we have:

$$\begin{aligned}
\ell^{(k)}_A(f) &:= \mathcal{F}_Z(i\widetilde{A}\star\mathcal{F}^{-1}_z(f)) = i\sum_{r\geq 0}\frac{1}{r!}(\frac{1}{2i})^r\mathcal{F}_z\Big(P^r(\widetilde{A},\mathcal{F}^{-1}_z(f))\Big) =\\
&= i\Big\{[i(\alpha\partial_{\overline{\xi}} + \overline{\alpha}\partial_\xi)f + \frac{1}{2}\beta e^w f + \frac{1}{2}\overline{\beta}e^{\overline{w}}f] + \frac{1}{1!}(\frac{1}{2i})[\overline{\alpha}\partial_{\overline{w}}f + \alpha\partial_w f - \overline{\beta}e^{\overline{w}}(\frac{i}{2}\xi)f -\\
&\quad -\beta e^w(\frac{i}{2}\overline{\xi})f] + \frac{1}{2!}(\frac{-1}{2i})^2 2[\overline{\beta}e^{\overline{w}}(\frac{i}{2}\xi)^2 f + \beta e^w(\frac{i}{2}\overline{\xi})^2 f] + \dots +\\
&\quad +\frac{1}{r!}(\frac{-1}{2i})^r 2^{r-1}[\overline{\beta}e^{\overline{w}}(\frac{i}{2}\xi)^r f + \beta e^w(\frac{i}{2}\overline{\xi})^r f] + \dots\Big\}\\
&= -(\alpha\partial_{\overline{\xi}} - \overline{\alpha}\partial_\xi)f + \frac{1}{2}(\overline{\alpha}\partial_{\overline{w}} + \alpha\partial_w)f + i\Big\{\Big[\frac{1}{2}\beta e^w + \frac{1}{2}\overline{\beta}e^{\overline{w}} - \frac{1}{2}\overline{\beta}e^{\overline{w}}(\frac{1}{2}\xi) - \frac{1}{2}\beta e^w(\frac{1}{2}\overline{\xi})\Big]f +\\
&\quad +\frac{1}{2}.\frac{1}{2!}\Big[\overline{\beta}e^{\overline{w}}(\frac{-1}{2}\overline{\xi})^2 + \beta e^w(\frac{-1}{2}\xi)^2\Big]f + \dots + \frac{1}{2}\frac{1}{k!}\Big[\overline{\beta}e^{\overline{w}}(\frac{-1}{2}\xi)^r + \beta e^w(\frac{-1}{2}\overline{\xi})^r\Big]f + \dots\Big\}\\
&= \Big[\alpha(\frac{1}{2}\partial_w - \partial_{\overline{\xi}}) + \overline{\alpha}(\frac{1}{2}\partial_{\overline{w}} - \partial_\xi) + \frac{i}{2}\beta e^w e^{-\frac{1}{2}\overline{\xi}} + \frac{i}{2}\overline{\beta}e^{\overline{w}}e^{-\frac{1}{2}\xi}\Big]f\\
&= \Big[\alpha(\frac{1}{2}\partial_w - \partial_{\overline{\xi}}) + \overline{\alpha}(\frac{1}{2}\partial_{\overline{w}} - \partial_\xi) + \frac{i}{2}(\beta e^{w-\frac{1}{2}\overline{\xi}} + \overline{\beta}e^{\overline{w}-\frac{1}{2}\xi})\Big]f
\end{aligned}$$

The proposition is therefore proved. □

Remark 3.6. Set u=$w - \frac{1}{2}\overline{\xi}$;$v = w + \frac{1}{2}\overline{\xi}$ we obtain

$$\hat{\ell}^{(k)}_A(f) = \alpha\frac{\partial f}{\partial u} + \overline{\alpha}\frac{\partial f}{\partial\overline{u}} + \frac{i}{2}(\beta e^u + \overline{\beta}e^{\overline{u}})f|_{(u,v)}, \quad \text{i.e.} \tag{6}$$

$$\hat{\ell}^{(k)}_A = \alpha\frac{\partial}{\partial u} + \overline{\alpha}\frac{\partial}{\partial\overline{u}} + \frac{i}{2}(\beta e^u + \overline{\beta}e^{\overline{u}}),$$

which provides a (local) representation of the Lie algebra $\mathrm{aff}(\mathbb{C})$.

4. The irreducible representations of $\widetilde{\mathbf{Aff}}(\mathbb{C})$

Since $\hat{\ell}_A^{(k)}$ is a representation of the Lie algebra $\mathrm{aff}(\mathbb{C})$, we have:

$$\exp(\hat{\ell}_A^{(k)}) = \exp\big(\alpha\frac{\partial}{\partial u} + \overline{\alpha}\frac{\partial}{\partial \overline{u}} + \frac{i}{2}(\beta e^u + \overline{\beta}e^{\overline{u}})\big)$$

is just the corresponding representation of the corresponding connected and simply connected Lie group $\widetilde{\mathrm{Aff}(\mathbb{C})}$. Let us first recall the well-known list of all the irreducible unitary representations of the group of affine transformations of the complex line, see [4] for more details.

Theorem 4.1. *Up to unitary equivalence, every irreducible unitary representation of* $\widetilde{\mathrm{Aff}(\mathbb{C})}$ *is unitarily equivalent to one of the following one-to-another nonequivalent irreducible unitary representations:*

1. *The unitary characters of the group, i.e. the one-dimensional unitary representation* $U_\lambda, \lambda \in \mathbb{C}$, *acting in* $\mathbb{C}$ *following the formula*

$$U_\lambda(z, w) = e^{i\Re(z\overline{\lambda})}, \forall (z, w) \in \widetilde{\mathrm{Aff}(\mathbb{C})}, \lambda \in \mathbb{C}.$$

2. *The infinite dimensional irreducible representations* $T_\theta, \theta \in \mathbf{S}^1$, *acting on the Hilbert space* $L^2(\mathbb{R} \times \mathbf{S}^1)$ *following the formula:*

$$\Big[T_\theta(z, w)f\Big](x) = \exp\Big(i(\Re(wx) + 2\pi\theta[\frac{\Im(x + z)}{2\pi}])\Big)f(x \oplus z), \tag{7}$$

where $(z, w) \in \widetilde{\mathrm{Aff}(\mathbb{C})}$; $x \in \mathbb{R} \times \mathbf{S}^1 = \mathbb{C}\backslash\{0\}$; $f \in \mathrm{L}^2(\mathbb{R} \times \mathbf{S}^1)$;

$$x \oplus z = \Re(x + z) + 2\pi i\{\frac{\Im(x + z)}{2\pi}\}.$$

In this section we will prove the following important theorem which is of interest for us both in theory and practice.

Theorem 4.2. *The representation* $\exp(\hat{\ell}_A^{(k)})$ *of the group* $\widetilde{\mathrm{Aff}(\mathbb{C})}$ *is the irreducible unitary representation* T_θ *of the group* $\widetilde{\mathrm{Aff}(\mathbb{C})}$ *associated to* Ω *by the orbit method, i.e.*

$$\exp(\hat{\ell}_A^{(k)})f(x) = [T_\theta(\exp A)f](x),$$

where $f \in \mathrm{L}^2(\mathbb{R} \times \mathbf{S}^1)$; $A = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \in \mathrm{aff}(\mathbb{C}); \theta \in \mathbf{S}^1; k = 0, \pm 1, \ldots$

Proof. Putting $x = e^u \in \mathbb{C}\backslash\{0\} = \mathbb{R} \times \mathbf{S}^1$ and recall that

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \exp(A) = \exp\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix},$$

we can rewrite (7) as following:

$$[T_\theta(\exp A)f](e^u) = \exp\Big(i(\Re(\frac{e^\alpha - 1}{\alpha}\beta e^u) + 2\pi\theta[\frac{\Im e^{u+\alpha}}{2\pi}])\Big)f(e^{u\oplus\alpha}),$$

where

$$u \oplus \alpha = \Re(u+\alpha) + 2\pi i\{\frac{\Im(u+\alpha)}{2\pi}\} = u + \alpha - 2\pi i[\frac{\Im(u+\alpha)}{2\pi}].$$

Therefore, for the one-parameter subgroup $\exp tA$, $t \in \mathbb{R}$, we have the action formula:

$$\big[T_\theta(\exp tA)f\big](e^u) = \exp\Big(i(\Re\frac{e^{t\alpha} - 1}{\alpha}\beta e^u + 2\pi\theta[\frac{\Im e^{u+t\alpha}}{2\pi}])\Big)f(e^{u\oplus t\alpha}).$$

By a direct computation:

$$\frac{\partial}{\partial t}\big([T_\theta(\exp tA)f](e^u)\big) = \tag{8}$$

$$= \frac{\partial}{\partial t}\Big(\exp\Big(\frac{i}{2}(\frac{e^{t\alpha}-1}{\alpha}\beta e^u + \frac{e^{t\overline{\alpha}}-1}{\overline{\alpha}}\overline{\beta}e^{\overline{u}}) + 2\pi\theta i[\frac{\Im e^{u+t\alpha}}{2\pi}]\Big)\Big) + f(e^{u+t\alpha-2\pi i[\frac{\Im(u+t\alpha)}{2\pi}]})$$
$$+\exp\big(\frac{i}{2}(\frac{e^{t\alpha}-1}{\alpha}\beta e^u + \frac{e^{t\overline{\alpha}}-1}{\overline{\alpha}}\overline{\beta}e^{\overline{u}}) + 2\pi\theta i[\frac{\Im e^{u+t\alpha}}{2\pi}]\big)\frac{\partial}{\partial t}f(e^{u+t\alpha-2\pi i[\frac{\Im(u+t\alpha)}{2\pi}]}) =$$
$$= \frac{i}{2}(\beta e^{u+t\alpha} + \overline{\beta}e^{\overline{u}+t\overline{\alpha}})\big[T_\theta(\exp tA)f\big](e^u) + \exp\Big(i(\Re(\frac{e^{t\alpha}-1}{\alpha}\beta e^u) + 2\pi\theta i[\frac{\Im e^{u+t\alpha}}{2\pi}]\Big)\alpha e^{u\oplus t\alpha}\frac{\partial f}{\partial u}$$

on one hand. On the other hand, we have:

$$\hat{\ell}_A^{(k)}([T_\theta(\exp tA)f](e^u) = \tag{9}$$

$$= \alpha\frac{\partial}{\partial u}\big([T_\theta(\exp tA)f](e^u)\big) + \overline{\alpha}\frac{\partial}{\partial\overline{u}}\big([T_\theta(\exp tA)f](e^u)\big) + \frac{i}{2}(\beta e^u + \overline{\beta}e^{\overline{u}})\big[T_\theta(\exp tA)f](e^u)\big] =$$
$$= \alpha\frac{i}{2}(\frac{e^{t\alpha}-1}{\alpha}\beta e^u)\exp\Big(i(\Re(\frac{e^{t\alpha}-1}{\alpha}\beta e^u) + 2\pi\theta[\frac{\Im e^{u+t\alpha}}{2\pi}])\Big)f(e^{u\oplus t\alpha}) +$$
$$+\alpha\exp\Big(i(\Re(\frac{e^{t\alpha}-1}{\alpha}\beta e^u) + 2\pi\theta[\frac{\Im e^{u+t\alpha}}{2\pi}])\Big)e^{u\oplus t\alpha}\frac{\partial f}{\partial u} +$$
$$+\overline{\alpha}\frac{i}{2}(\frac{e^{t\overline{\alpha}}-1}{\overline{\alpha}}\overline{\beta}e^{\overline{u}})\exp\Big(i(\Re(\frac{e^{t\alpha}-1}{\alpha}\beta e^u) + 2\pi\theta[\frac{\Im e^{u+t\alpha}}{2\pi}])\Big)f(e^{u\oplus t\alpha}) +$$
$$+\frac{i}{2}(\overline{\beta}e^{\overline{u}} + \beta e^u)[T_\theta(\exp tA)f](e^u) = \frac{i}{2}(\beta e^{u+t\alpha} + \overline{\beta}e^{\overline{u}+t\overline{\alpha}})[T_\theta(\exp tA)f](e^u) +$$
$$+\exp\Big(i(\Re(\frac{e^{t\alpha}-1}{\alpha}\beta e^u) + 2\pi\theta[\frac{\Im e^{u+t\alpha}}{2\pi}])\Big)\alpha e^{u\oplus\alpha t}\frac{\partial f}{\partial u}$$

(8) and (9) imply that

$$\frac{\partial}{\partial t}[T_\theta(\exp tA)f](x) = \hat{\ell}_A^{(k)}\big([T_\theta(\exp tA)f](x)\big) \qquad \forall x \in \mathbb{R}\times\mathbf{S}^1.$$

Furthermore,

$$T_\theta(\exp tA)f](e^u)|_{t=0} = \exp(2\pi i[\frac{\Im e^u}{2\pi}]\theta)f(e^{u-2\pi i[\frac{\Im u}{2\pi}]}) = f(e^u).$$

This means that: $\exp(\hat{\ell}_A^{(k)})f(x)$ and $[T_\theta(\exp tA)f](x)$ together are the solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) &= \hat{\ell}_A^{(k)}u(t,x); \\ u(0,x) &= id. \end{cases}$$

The operator $\hat{\ell}_A^{(k)}$ is sufficiently well-behaved, so that the Cauchy problem has a unique solution. From this uniqueness we deduce that

$$\exp(\hat{\ell}_A^{(k)})f(x) \equiv [T_\theta(\exp tA)f](x) \qquad \forall x \in \mathbb{R} \times \mathbf{S}^1.$$

The theorem is hence proved. □

Remark 4.3. We say that a real Lie algebra $\mathfrak{g}$ is in the class $\overline{\text{MD}}$ iff every K-orbit is of dimension 0 or dim $\mathfrak{g}$. Furthermore, it was proven in [4, Theorem 4.4] that, up to isomorphism, every Lie algebra of class $\overline{\text{MD}}$ is one of the following:

1. commutative Lie algebras,
2. the Lie algebra aff($\mathbb{R}$) of affine transformations of the real line,
3. the Lie algebra of affine transformations of the complex line.

Thus, by calculation for the group of affine transformations of the real line Aff($\mathbb{R}$)) in [5] and here for the group of affine transformations of the complex line Aff($\mathbb{C}$) we obtained a description of the quantum $\overline{\text{MD}}$ co-adjoint orbits.

References

[1] Arnal, D.; Cortet, J. C.: *$\star$-product and representations of nilpotent Lie groups.* J. Geom. Phys. **2** (1985)(2), 86–116.

[2] Arnal, D.; Cortet, J. C.: *Représentations $\star$ des groupes exponentiels.* J. Funct. Anal. **92** (1990), 103–135.

[3] Arnold, V. I.: *Mathematical Methods of Classical Mechanics.* Springer Verlag, Berlin - New York - Heidelberg 1984.

[4] Do Ngoc Diep: *Noncommutative Geometry Methods for Group C*-Algebras.* Chapman & Hall/CRC. Press #LM 2003, 1999.

[5] Do Ngoc Diep; Nguyen Viet Hai: *Quantum half-plane via Deformation Quantization.* Math. QA/9905002, 2 May 1999.

[6] Fedosov, B.: *Deformation Quantization and Index Theory.* Akademie Verlag, Berlin 1996.

[7] Gutt, S.: *Deformation quantization.* ICTP Workshop on Representation Theory of Lie groups. SMR 686/14, 1993.

[8] Gel'fand, I. M.; Naimark, M. A.: *Unitary representations of the group of affine transformations of the straight line.* Dokl. AN SSSR **55** (1947)(7), 571–574.

[9] Kirillov, A. A.: *Elements of the Theory of Representations.* Springer Verlag, Berlin - New York - Heidelberg 1976.

Received October 10, 1999; revised version December 12, 2000