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Shape Operators of Orbits of Isotropy Subgroups in Riemannian Symmetric Spaces of the Compact Type *

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Abstract. In the present paper orbits of isotropy subgroups are discussed in symmetric spaces of the compact type. Shape operators of the orbits are described by using the root space decompositions of orthogonal symmetric Lie algebras. Furthermore, some properties of totally geodesic orbits are established.

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1. Introduction

Isometries of symmetric spaces form relatively large Lie groups. In order to study the properties of isometric actions on Riemannian manifolds we can take the actions of the subgroups of these Lie groups as models. It seems to be interesting to consider those so-called polar isometric actions which admit orthogonally transversal submanifolds (in other words, sections). For example, regarding a symmetric space, the action of the isotropy subgroup at a given point admits sections, since the maximal flat totally geodesic submanifolds containing the point intersect all the orbits orthogonally.

The first studies on the subject mentioned above were made by R. Bott and H. Samelson in the famous paper [3]. Later such special actions have been studied in several papers by L. Conlon, J. Szenthe, R. S. Palais and C. L. Terng (see [6], [18] and [16]). It was proved that a section is always a totally geodesic submanifold in the ambient space. Furthermore, it was shown that the principal orbits of such an action are isoparametric submanifolds, that is, their normal bundles have trivial holonomy group and they have constant principal curvatures with respect to parallel normal vector fields (see [18] and [16]). An isometric action is said to be hyperpolar if it admits locally Euclidean sections. Hyperpolar actions on symmetric spaces have been discussed in the papers [8] and [9].

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In the present paper we study orbits of isotropy subgroups in symmetric spaces of the compact type. Concerning basic properties of these symmetric spaces, we refer to the book [10] of S. Helgason. Our purpose is to describe shape operators of these orbits and to select the totally geodesic ones. In the discussions we use the method of root space decompositions of orthogonal symmetric Lie algebras. Clearly, our discussions can be accomplished in symmetric spaces of the noncompact type, too. Since they are diffeomorphic with Euclidean spaces, orbits of isometric actions on them can be handled more easily. Furthermore, it can be seen that in symmetric spaces of the noncompact type isotropy subgroups have no totally geodesic orbit. Considering these facts, we discuss the compact case in this paper.

First we shall prove some statements concerning totally geodesic submanifolds of rank one in symmetric spaces of the compact type. Then the orbits of a fixed isotropy subgroup will be characterized by using the restricted root system of the ambient space. The main result of this paper is the presentation of the principal curvatures of each orbit with respect to any normal direction (see Theorem 1). By this statement we can give a necessary and sufficient condition for an orbit to be a totally geodesic one (see Theorem 2). This means that all the totally geodesic orbits can be found by regarding the weight lattice of the restricted root system of the ambient symmetric space. We acknowledge that this result can be obtained from the criterion given by R. Hermann in [11], too (see Remark 3). We shall show that the totally geodesic orbits are reflective submanifolds (see Theorem 3). Finally, as a corollary of Theorem 1 we obtain that all the orbits are curvature-adapted submanifolds (see Corollary 3).

Throughout this paper N denotes a Riemannian globally symmetric space with the metric tensor $\langle\ ,\ \rangle$ and with the Levi-Civita connection $\tilde{\nabla}$. The exponential map in N defined on the tangent bundle TN will be denoted by Exp and the curvature tensor by R. Concerning submanifolds, the notions and facts used here can be found in the books [14] and [17]. Considering a connected submanifold M in N, we always take the inherited Riemannian metric and the induced connection on it. The normal vector bundle of a given submanifold M will be denoted by $\nu(M)$. Let $X,\ \xi$ be vector fields on M such that X is tangential and ξ is normal to M. Then the corresponding Weingarten equation is given by

$$\tilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi$$

where A_ξ denotes the shape operator of M with respect to ξ and D denotes the induced normal connection in $\nu(M)$. If u is a unit vector in the normal subspace $\nu_p M$ $(p \in M)$, then the eigenvalues of A_u are called principal curvatures of M at p with respect to the direction of u.

Taking a vector u in the tangent space T_pN, consider the endomorphism $R_u : T_pN \to T_pN$ defined by $R_u(v) = R(v,u)u$ for $v \in T_pN$. This self-adjoint map is called the curvature endomorphism with respect to u. Clearly, if $u,\ v$ are two orthonormal vectors in T_pN, then $\langle R_u(v), v\rangle$ gives the sectional curvature of the plane spanned by u and v. The submanifold M is said to be *curvature-adapted* if T_pM is an invariant subspace of R_u and if the restriction of R_u to T_pM commutes with A_u for any $p \in M$ and $u \in \nu_p M$. This concept was introduced by J. Berndt and L. Vanhecke in the paper [1].

2. Totally geodesic submanifolds of rank one in symmetric spaces of the compact type

In this chapter some auxiliary statements are presented which will be used in the proofs of the main results. First let us recall some basic facts concerning symmetric spaces (see further [10]). Let G be the identity component of the Lie group formed by the isometries of the symmetric space N. Let us fix a point o in N and denote by K the compact isotropy subgroup of G at o. It is well-known that N can be identified with the coset space $G|K$. The identification is given by the diffeomorphism $\pi : G|K \to N$ where $\pi(gK) = g(o)$ for $gK \in G|K$. Considering the canonical left actions of G on $G|K$ and on N, π is an equivariant map since $\pi(h(gK)) = h(\pi(gK))$ holds for $h \in G$ and $gK \in G|K$.

Denoting by r_o the reflection of N on the point o, take the automorphism σ of G defined by the assignment $\sigma(g) = r_o g r_o$ for any $g \in G$. This induces an involutive automorphism σ of the Lie algebra $\mathfrak{g}$ of G. Considering the eigenspaces of σ with respect to the eigenvalues 1 and -1, we obtain the direct sum

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p},$$

where $\mathfrak{k}$ coincides with the Lie algebra of the isotropy subgroup K. Since $\mathfrak{p}$ can be regarded as the tangent space of $G|K$ at K, the tangent linear map $T\pi$ of π presents an isomorphism between $\mathfrak{p}$ and T_oN. Furthermore, denoting by exp the exponential map of $\mathfrak{g}$ into G, the equality

$$exp(Y)(o) = Exp(T\pi(Y))$$

is valid for any $Y \in \mathfrak{p}$. Hereafter T_oN and $\mathfrak{p}$ are considered to be identical. Concerning the curvature tensor at o, it is known that for any vectors $v_1,\ v_2,\ v_3$ in $\mathfrak{p}$

$$R(v_1, v_2)v_3 = -\big[\,[v_1, v_2]\, v_3\big] \tag{1}$$

is valid where $[\ ,\ ]$ denotes the bracket operation in $\mathfrak{g}$.

As usual, let $Ad : G \to Gl(\mathfrak{g})$ denote the adjoint representation of the Lie group G on $\mathfrak{g}$. It is well-known that the subspace $\mathfrak{p}$ is invariant with respect to the subgroup Ad_K.

Using normal coordinate systems and Jacobi vector fields along geodesics, the following well-known assertion can easily be proved.

Remark 1. Let $N^1,\ N^2$ be locally symmetric spaces and $p_1,\ p_2$ be points in $N^1,\ N^2$, respectively. If there is an isometric linear map $\varphi : T_{p_1}N^1 \to T_{p_2}N^2$ such that $\varphi \circ R^1_u = R^2_{\varphi(u)} \circ \varphi$ holds for any $u \in T_{p_1}N^1$, then N^1 and N^2 are locally isometric.

Henceforth we assume that N is a Riemannian symmetric space of the compact type. Therefore the pair $(\mathfrak{g}, \sigma)$ presents an orthogonal symmetric Lie algebra of the compact type (see Chapter VII in [10]). In order to characterize the totally geodesic submanifolds of rank one in N we apply the so-called root space decomposition of $\mathfrak{g}$.

Regarding the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$, let us take the dual orthogonal symmetric Lie algebra $(\tilde{\mathfrak{g}}, \theta)$ of the noncompact type, where $\tilde{\mathfrak{g}} = \mathfrak{k} + i\mathfrak{p}$ and θ is defined by the equality $\theta(Z + iY) = Z - iY$ for $Z \in \mathfrak{k}$, $Y \in \mathfrak{p}$. Let $\mathfrak{p}_0$ be a fixed maximal Abelian subspace in $\mathfrak{p}$.

Clearly, $\tilde{\mathfrak{p}}_0 = i\mathfrak{p}_0$ is also a maximal Abelian subspace in $\tilde{\mathfrak{p}} = i\mathfrak{p}$. Considering a linear form α on the vector space $\tilde{\mathfrak{p}}_0$, take the subspace

$$\tilde{\mathfrak{g}}_\alpha = \{X \in \tilde{\mathfrak{g}} \mid [\tilde{H}, X] = \alpha(\tilde{H})X \quad \textit{for any } \tilde{H} \in \tilde{\mathfrak{p}}_0\}.$$

A dual vector α ($\alpha \neq 0$) in $\tilde{\mathfrak{p}}_0^*$ is said to be a restricted root of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{p}}_0$ if $\tilde{\mathfrak{g}}_\alpha \neq 0$. Let $\triangle$ denote the set of all the restricted roots. It is known that

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0 + \sum_{\alpha \in \triangle} \tilde{\mathfrak{g}}_\alpha \tag{2}$$

gives a decomposition of the real semisimple Lie algebra $\tilde{\mathfrak{g}}$, furthermore, $\tilde{\mathfrak{g}}_0 = \mathfrak{k}_0 + \tilde{\mathfrak{p}}_0$ is a direct sum with $\mathfrak{k}_0 = \tilde{\mathfrak{g}}_0 \cap \mathfrak{k}$ (see [10; pp. 263-264]).

Considering two subpaces $\mathfrak{a}$, $\mathfrak{b}$ in $\tilde{\mathfrak{g}}$, let $[\mathfrak{a}, \mathfrak{b}]$ denote the subspace in $\tilde{\mathfrak{g}}$ generated by the vectors $[X, Y]$ where $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$. Concerning the above decomposition, for any α and β in $\triangle \cup \{0\}$ we have

$$\theta(\tilde{\mathfrak{g}}_\alpha) = \tilde{\mathfrak{g}}_{-\alpha}, \qquad [\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_\beta] \subset \tilde{\mathfrak{g}}_{\alpha+\beta}. \tag{3}$$

Let B denote the Killing form of the complex Lie algebra $\mathfrak{g}^C$. Obviously, the restrictions of B to $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ give the Killing forms of these real semisimple Lie algebras. Let us introduce the symmetric bilinear form D on $\tilde{\mathfrak{g}}$ defined by the equality $D(X_1, X_2) = -2B(X_1, \theta(X_2))$ for $X_1,\ X_2 \in \tilde{\mathfrak{g}}$. It is well-known that the restriction of B to $\tilde{\mathfrak{p}}$ (respectively to $\mathfrak{k}$) is positive (respectively negative) definite. This fact implies that D presents a positive definite bilinear form on $\tilde{\mathfrak{g}}$. Furthermore, it can be seen that (2) gives an orthogonal decomposition of $\tilde{\mathfrak{g}}$ with respect to D.

For each root $\alpha \in \triangle$ we can define the vector $\tilde{A}_\alpha$ in $\tilde{\mathfrak{p}}_0$ which is determined by the equality $B(\tilde{A}_\alpha, \tilde{H}) = \alpha(\tilde{H})$ for $\tilde{H} \in \tilde{\mathfrak{p}}_0$. The number $|\alpha| = \sqrt{B(\tilde{A}_\alpha, \tilde{A}_\alpha)}$ is called the length of the root α.

Regarding an element α in $\triangle$, let m_α denote the dimension of the subspace $\tilde{\mathfrak{g}}_\alpha$ which is called the multiplicity of the root α. Let us take an orthonormal basis $X_{\alpha,1}, \dots, X_{\alpha,m_\alpha}$ in $\tilde{\mathfrak{g}}_\alpha$ with respect to D. Further on we use the vectors

$$Z_{\alpha,j} = X_{\alpha,j} + \theta(X_{\alpha,j}), \qquad \tilde{Y}_{\alpha,j} = X_{\alpha,j} - \theta(X_{\alpha,j}) \qquad (j = 1, \dots, m_\alpha).$$

Denoting by $\mathbf{R}$ the set of real numbers, let us consider the subspaces

$$\mathfrak{k}_\alpha = \sum_{j=1}^{m_\alpha} \mathbf{R} Z_{\alpha,j}, \qquad \tilde{\mathfrak{p}}_\alpha = \sum_{j=1}^{m_\alpha} \mathbf{R} \tilde{Y}_{\alpha,j}.$$

It is obvious that the equalities $\mathfrak{k}_\alpha = \mathfrak{k}_{-\alpha}$, $\tilde{\mathfrak{p}}_\alpha = \tilde{\mathfrak{p}}_{-\alpha}$, $\tilde{\mathfrak{g}}_\alpha + \tilde{\mathfrak{g}}_{-\alpha} = \mathfrak{k}_\alpha + \tilde{\mathfrak{p}}_\alpha$ $(\alpha \in \triangle)$ are valid.

Now we can return to the original orthogonal symmetric Lie algebra $(\mathfrak{g}, \sigma)$ of the compact type. Regarding an element v in $\mathfrak{g}$, $ad\ v$ will denote the endomorphism defined by $ad\ v(w) = [v, w]$ for $w \in \mathfrak{g}$. In this case we shall use the vectors $A_\alpha = i\tilde{A}_\alpha$, $Y_{\alpha,j} =$

$i\tilde{Y}_{\alpha,j}$ and the subspaces $\mathfrak{p}_\alpha = i\tilde{\mathfrak{p}}_\alpha \quad (\alpha \in \triangle)$ in the calculations. It is well-known that the Killing form of $\mathfrak{g}$ is negative definite. The choice of the vectors $X_{\alpha,j}$ implies that $Y_{\alpha,1}, \ldots Y_{\alpha,m_\alpha}$ (respectively $Z_{\alpha,1}, \ldots Z_{\alpha,m_\alpha}$) form an orthonormal basis in $\mathfrak{p}_\alpha$ (respectively in $\mathfrak{k}_\alpha$) with respect to $-B$. Obviously, the elements of $\triangle$ obtain purely imaginary values on $\mathfrak{p}_0$. Considering the above vectors, it can easily be seen that the equalities

$$[H, Z_{\alpha,j}] = -\alpha(iH)Y_{\alpha,j}, \qquad [H, Y_{\alpha,j}] = \alpha(iH)Z_{\alpha,j} \tag{4}$$

hold for any $H \in \mathfrak{p}_0$.

Endow the dual space $\tilde{\mathfrak{p}}_0^*$ with an arbitrary lexicographic ordering and denote by $\triangle^+$ the set of positive roots with respect to it. Using the equalities (2) and (3) and making computations with the basis vectors $Z_{\alpha,j},\ Y_{\alpha,j}\ (\alpha \in \triangle^+)$, the following statement can be proved (see [10; pp. 335-336]).

Lemma 1. *The equalities*

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \triangle^+} \mathfrak{k}_\alpha, \qquad \mathfrak{p} = \mathfrak{p}_0 + \sum_{\alpha \in \triangle^+} \mathfrak{p}_\alpha \tag{5}$$

present orthogonal decompositions with respect to B. Furthermore, the relations

$$[\mathfrak{k}_\alpha, \mathfrak{k}_\beta] \subset \mathfrak{k}_{\alpha+\beta} + \mathfrak{k}_{\alpha-\beta}, \quad [\mathfrak{k}_\alpha, \mathfrak{p}_\beta] \subset \mathfrak{p}_{\alpha+\beta} + \mathfrak{p}_{\alpha-\beta}, \quad [\mathfrak{p}_\alpha, \mathfrak{p}_\beta] \subset \mathfrak{k}_{\alpha+\beta} + \mathfrak{k}_{\alpha-\beta} \tag{6}$$

hold for any $\alpha,\ \beta \in \triangle \cup \{0\}$.

Hereafter we assume that α is an element of $\triangle^+$. Now let us take the following simple lemma which will also be used later.

Lemma 2. *The equality $[Z_{\alpha,j}, Y_{\alpha,j}] = A_\alpha$ holds. Furthermore, $[Z_{\alpha,j}, Y_{\alpha,l}] \in \mathfrak{p}_{2\alpha}$ and $[Z_{\alpha,j}, Y_{\alpha,l}] + [Z_{\alpha,l}, Y_{\alpha,j}] = 0$ are valid provided that $j \neq l$.*

Proof. By definition the equality

$$[Z_{\alpha,j}, Y_{\alpha,j}] = i([\theta(X_{\alpha,j}), X_{\alpha,j}] - \theta\,[\theta(X_{\alpha,j}), X_{\alpha,j}])$$

holds, therefore the relations (3) imply that $[Z_{\alpha,j}, Y_{\alpha,j}]$ is an element of $\mathfrak{p}_0$. Considering the equalities (4), we obtain that

$$B(H, [Z_{\alpha,j}, Y_{\alpha,j}]) = B([H, Z_{\alpha,j}], Y_{\alpha,j}) = \alpha(iH) = B(H, A_\alpha)$$

is valid for any $H \in \mathfrak{p}_0$. It follows from this the validity of $[Z_{\alpha,j}, Y_{\alpha,j}] = A_\alpha$. The second assertion can be proved in analogous way. $\square$

Let Z_G denote the centre of the Lie group G. It is known that an element g of G is contained in Z_G if and only if $Ad_g = id$ holds where id denotes the identity map of $\mathfrak{g}$. The central elements in the subgroup $exp(\mathfrak{p}_0)$ can be found by the following statement.

Lemma 3. *Let H be an element of $\mathfrak{p}_0$. Then $Ad_{exp(H)} = id$ is valid if and only if $\alpha(H) \in 2\mathbf{Z}\pi i$ for any $\alpha \in \triangle$.*

Proof. Clearly, the map $Ad_{exp(H)} = e^{ad\ H}$ $(H \in \mathfrak{p}_0)$ leaves the subspace $\mathfrak{k}_0 + \mathfrak{p}_0$ pointwise fixed. Hence, consider the proof of Lemma 6.5 of Chapter VII in [10] and use the pairs of the vectors $Z_{\alpha,j}$, $Y_{\alpha,j}$ $(\alpha \in \triangle^+)$ with the equalities (4). $\square$

By Lemma 3 the central elements in $exp(\mathfrak{p}_0)$ define a lattice in $\mathfrak{p}_0$. In the following chapter we shall see that all the totally geodesic orbits of K can be found by using this lattice.

Let us take now the Riemannian metric $\langle\ ,\ \rangle$ on the tangent space $T_oN = \mathfrak{p}$. Clearly, $\triangle$ uniquely decomposes as the union of irreducible root systems $\triangle^i$ $(i = 1, \dots s)$. By these component root systems we obtain a decomposition $\mathfrak{p}_0 = \sum_{i=1}^{s} \mathfrak{p}_0^i$ where the subspace $\mathfrak{p}_0^i$ is generated by the vectors A_α $(\alpha \in \triangle^i)$. Considering the subspaces $\mathfrak{p}^i = \mathfrak{p}_0^i + \sum_{\alpha\in\triangle^i} \mathfrak{p}_\alpha$, it is clear that

$$\mathfrak{p} = \sum_{i=1}^{s} \mathfrak{p}^i$$

presents an orthogonal decomposition with respect to B and $\langle\ ,\ \rangle$ (see Chapter VIII, Proposition 5.2 in [10]). Since the action of Ad_K on $\mathfrak{p}^i$ is irreducible, the equality

$$\langle\ ,\ \rangle|_{\mathfrak{p}^i \times \mathfrak{p}^i} = -c_i^{\,2}\ B|_{\mathfrak{p}^i \times \mathfrak{p}^i}$$

holds for the restricted forms with a positive number c_i. Further on we use the notation c_α $(\alpha \in \triangle^+)$ where $c_\alpha = c_i$ if $\alpha \in \triangle^i$.

By the orthonormal bases $Y_{\alpha,j}$ and $Z_{\alpha,j}$ $(j = 1, \dots, m_\alpha)$ with respect to $-B$ we can define the isomorphism $\mu : \mathfrak{p}_\alpha \to \mathfrak{k}_\alpha$ where $\mu(Y_{\alpha,j}) = Z_{\alpha,j}$. Obviously, the equality $\mu(i(X_\alpha - \theta(X_\alpha))) = X_\alpha + \theta(X_\alpha)$ is valid for any $X_\alpha \in \tilde{\mathfrak{g}}_\alpha$.

In the following we need the unit vectors $a_\alpha = \frac{1}{c_\alpha\,|\alpha|} A_\alpha$ $(\alpha \in \triangle^+)$. If y_α is a unit vector in $\mathfrak{p}_\alpha$ with respect to $\langle\ ,\ \rangle$, then it can be shown by Lemma 2 that the equalities

$$[Z_\alpha, a_\alpha] = -|\alpha|\, y_\alpha, \qquad [Z_\alpha, y_\alpha] = |\alpha|\, a_\alpha \tag{7}$$

are valid with $Z_\alpha = c_\alpha \cdot \mu(y_\alpha)$.

It is known that if $l\alpha$ is also a restricted root for an element α in $\triangle$ with a number l, then l may be equal only to $\pm\frac{1}{2}$, ± 1, ± 2. A root $\alpha \in \triangle$ is said to be unmultipliable if $2\alpha \notin \triangle$, that is, if $m_{2\alpha} = 0$ holds.

Lemma 4. *Let K^α be the connected Lie subgroup in K the Lie algebra of which coincides with $\mathfrak{k}^\alpha = \mathfrak{k}_0 + \mathfrak{k}_\alpha + \mathfrak{k}_{2\alpha}$. Then the orbit $Ad_{K^\alpha}(a_\alpha) = \cup_{k\in K^\alpha} Ad_k(a_\alpha)$ coincides with the unit sphere around 0 in the subspace $\mathfrak{q}_\alpha = \mathbf{R}A_\alpha + \mathfrak{p}_\alpha + \mathfrak{p}_{2\alpha}$.*

Proof. First remark that by Lemma 1 $\mathfrak{k}^\alpha$ is a subalgebra in k and $\mathfrak{q}_\alpha$ is an invariant subspace of the subgroup Ad_{K^α} in Ad_G. The elements of Ad_{K^α} are automorphisms of $\mathfrak{g}$, therefore $Ad_{K^\alpha}(a_\alpha)$ is contained in the unit sphere around 0 in $\mathfrak{q}_\alpha$. Considering the equalities (7), it can be shown that

$$Ad_{exp(tZ_\alpha)}(a_\alpha) = e^{ad\ tZ_\alpha}(a_\alpha) = \cos(|\alpha| t)\, a_\alpha - \sin(|\alpha| t)\, y_\alpha \tag{8}$$

is valid for any unit vector $y_\alpha \in \mathfrak{p}_\alpha$ with $t \in \mathbf{R}$ and $Z_\alpha = c_\alpha \cdot \mu(y_\alpha)$.

Let us take an arbitrary unit vector $y = e_0 a_\alpha + e_1 y_\alpha + e_2 y_{2\alpha}$ in $\mathfrak{q}_\alpha$ $(e_0,\ e_1,\ e_2 \in \mathbf{R})$. It is not difficult to show that y is included in $Ad_{K^\alpha}(a_\alpha)$. Obviously, there are numbers $t_1,\ t_2$ such that $\cos(2\,|\alpha| t_2) \cdot \cos(|\alpha| t_1) = e_0$, $-\sin(|\alpha| t_1) = e_1$ and $-\sin(2\,|\alpha| t_2) \cdot \cos(|\alpha| t_1) = e_2$ hold. The relation $[\mathfrak{k}_{2\alpha}, \mathfrak{p}_\alpha] \subset \mathfrak{p}_\alpha$ implies that $\mathfrak{p}_\alpha$ is an invariant subspace with respect to $Ad_{K^{2\alpha}}$. Let us regard the vectors $Z_{2\alpha} = c_\alpha \cdot \mu(y_{2\alpha})$, $\hat{y}_\alpha = Ad_{exp(-t_2 Z_{2\alpha})}(y_\alpha)$ and $\hat{Z}_\alpha = c_\alpha \cdot \mu(\hat{y}_\alpha)$. Using the equality (8), we obtain that $Ad_{k_2 k_1}(a_\alpha) = y$ holds where $k_1 = exp(t_1 \hat{Z}_\alpha)$ and $k_2 = exp(t_2 Z_{2\alpha})$. □

Regarding a subspace $\mathfrak{a}$ in $\mathfrak{p}$, $Exp(\mathfrak{a})$ is a totally geodesic submanifold in N if and only if $\mathfrak{a}$ is a Lie triple system in $\mathfrak{p}$. It is known that N contains totally geodesic submanifolds of constant curvature (see Chapter VII, Theorem 11.1 in [10]).

Proposition 1. *The subspace $\mathfrak{q}_\alpha = \mathbf{R}A_\alpha + \mathfrak{p}_\alpha + \mathfrak{p}_{2\alpha}$ is a Lie triple system in $\mathfrak{p}$. The corresponding totally geodesic submanifold $Q_\alpha = Exp(\mathfrak{q}_\alpha)$ is a compact symmetric space of rank one. Furthermore, if $2\alpha \notin \triangle$ and G is simply connected, then Q_α coincides with the sphere of constant curvature $\kappa = \frac{|\alpha|^2}{{c_\alpha}^2} = \alpha(ia_\alpha)^2$.*

Proof. Using Lemma 2 and the relations (6), it can be seen that $\mathfrak{q}_\alpha$ is a Lie triple system in $\mathfrak{p}$. Since Q_α is totally geodesic in N, Q_α is also a compact symmetric space. Hence, the sectional curvatures in Q_α can be expressed by using the equality (1). Considering (4), we obtain that the curvature endomorphism $R_{a_\alpha} = -(ad\ a_\alpha)^2$ on $\mathfrak{q}_\alpha$ has the eigenvalues $0,\ \frac{|\alpha|^2}{{c_\alpha}^2},\ 4\frac{|\alpha|^2}{{c_\alpha}^2}$ with the eigenspaces $\mathbf{R}A_\alpha,\ \mathfrak{p}_\alpha,\ \mathfrak{p}_{2\alpha}$. Regarding a unit vector y in $\mathfrak{q}_\alpha$, by Lemma 4 there exists an element $k \in K^\alpha$ such that $Ad_k(\mathfrak{q}_\alpha) \subset \mathfrak{q}_\alpha$ and $Ad_k(a_\alpha) = y$ hold for the automorphism Ad_k on $\mathfrak{g}$. This fact implies that the restriction of the endomorphism R_y to $\mathfrak{q}_\alpha$ has the same eigenvalues with the same multiplicities. Therefore all the flat totally geodesic submanifolds in Q_α are one-dimensional.

Suppose that $2\alpha \notin \triangle$ is valid. Then Q_α is a space of constant curvature $\kappa = \frac{|\alpha|^2}{{c_\alpha}^2}$ (see Remark 1). This means that Q_α is a sphere or a real projective space. It is known that the geodesic $\gamma : \mathbf{R} \to Q_\alpha$ defined by $\gamma(t) = Exp(tA_\alpha) = exp(tA_\alpha)(o)$ is periodic. If in addition the Lie group G is simply connected, then $exp(tA_\alpha) \in K$ holds if and only if $t \in \frac{2\pi}{|\alpha|^2}\mathbf{Z}$ where $\mathbf{Z}$ denotes the set of integers (see Chapter VII, Theorem 8.5 in [10]). Therefore the length of the closed geodesic γ is equal to $2\pi \frac{c_\alpha}{|\alpha|}$. This immediately implies that Q_α is a sphere, which completes the proof. □

Regarding an element α of $\triangle^+$, let us take the sphere $S^t = \{\ w \in \mathfrak{q}_\alpha \mid |w| = t\ \}$ of radius t in $\mathfrak{q}_\alpha$ $(t > 0)$. It is important for us to describe the shape operators of the geodesic sphere $Exp(S^t)$ around o in Q_α. In the proof of Proposition 1 we have seen that if y is a unit vector in $\mathfrak{q}_\alpha$, then the eigenvalues of R_y restricted to $\mathfrak{q}_\alpha$ are $0,\ \kappa,\ 4\kappa$ where $\kappa = \alpha(ia_\alpha)^2$. Denote by V_1 and V_2 the eigenspaces corresponding to $\kappa_1 = \kappa$ and $\kappa_2 = 4\kappa$, respectively. Remark that the multiplicity of $\kappa_2 = 4\kappa$ may be equal to 0, 1, 3, 7. Clearly, this "eigenvalue" should be omitted if $2\alpha \notin \triangle$, that is, if $V_2 = 0$ holds.

For brevity, denote by p the point $Exp(ty)$. Regarding the geodesic $\gamma : \mathbf{R} \to Q_\alpha$ defined by $\gamma(\tau) = Exp(\tau y)$, let $\hat{V}_1,\ \hat{V}_2$ be the subspaces in T_pQ_α obtained by parallel displacements of $V_1,\ V_2$ along γ. Let us denote by $\dot{\gamma}(t)$ the tangent vector of γ at t and by $\hat{A}_{\dot\gamma(t)}$ the shape operator of $Exp(S^t)$ in Q_α with respect to $\dot\gamma(t)$. It can easily be seen that the following statement is valid for each symmetric space of the compact type with rank one (for details see [20]).

Lemma 5. *If $t \notin \frac{1}{2\sqrt{\kappa}}\mathbf{Z}\pi$ holds, then $Exp(S^t)$ is a hypersurface in Q_α and its shape operator $\hat{A}_{\dot\gamma(t)}$ has the eigenvalues*

$$\lambda_1(t) = -\sqrt{\kappa}\,\cot(\sqrt{\kappa}\,t), \qquad \lambda_2(t) = -2\sqrt{\kappa}\,\cot(2\sqrt{\kappa}\,t)$$

with the corresponding eigenspaces $\hat{V}_1$ and $\hat{V}_2$. Furthermore, if $t \in \frac{1}{2\sqrt{\kappa}}\mathbf{Z}\pi$ is valid, then $Exp(S^t)$ is a totally geodesic submanifold in Q_α.

3. Orbits of isotropy subgroups in symmetric spaces of the compact type

In this chapter N always denotes a symmetric space of the compact type. Our purpose is to present the shape operators of the orbits of the isotropy subgroup K at a fixed point o. It is known that the compact Lie group K is not necessarily connected, therefore K may have disconnected orbits. However, it is trivial that the connected components of each disconnected orbit of K are isometric to one another. All the statements of this chapter remain valid if instead of the orbits of K we consider the orbits of its identity component. On the other hand, we have to remark that K is connected if N coincides with a compact semisimple Lie group or if G is simply connected (see Chapter VII, Theorem 8.2 in [10]).

Henceforth, for simplicity we assume that the orbits of K are connected. In the following let $\rho : K \times N \to N$ denote the canonical isometric action of K on N. Consider an arbitrary vector w in $\mathfrak{p}$ and the point $p = Exp(w)$. Regarding the trivial equality $(k\,exp(w)\,k^{-1})K = exp(Ad_k(w))\,K$ and the identification π, we obtain that

$$k(p) = Exp(Ad_k(w)) \tag{9}$$

holds for any $k \in K$.

Concerning orbits of differentiable actions, basic notions and facts we shall use in the following can be found in the book [13]. Let us take take the flat totally geodesic submanifold $P_0 = Exp(\mathfrak{p}_0)$ in N. It can be proved that P_0 intersects each orbit of K and $T_pP_0 \subset \nu_pK(p)$ is valid for any point p of P_0 (see Lemma 6.3 of Chapter V in [10]). By definition this means that P_0 is a section (orthogonally transversal submanifold) of the action ρ. Hence, it can be seen that the principal orbits of ρ are isoparametric submanifolds in N, that is, their normal bundles have trivial holonomy group and they have constant principal curvatures with respect to parallel normal vector fields (see [16]).

The singular orbits of ρ can be found by regarding the diagram of N in $\mathfrak{p}_0$ (see Chapter VII, Proposition 3.2 in [10]). The statement presented below gives the isotropy subgroups of K at the points of P_0.

Proposition 2. *Let w be an element of $\mathfrak{p}_0$ and let $\triangle_w^{\mathbf{Z}}$ denote the set of those roots α in $\triangle^+$ which possess the property $\alpha(w) \in \mathbf{Z}\pi i$. Let K_p denote the isotropy subgroup of K at the point $p = Exp(w)$. Then the Lie algebra of K_p coincides with*

$$\mathfrak{k}_p = \mathfrak{k}_0 + \sum_{\alpha \in \triangle_w^{\mathbf{Z}}} \mathfrak{k}_\alpha. \tag{10}$$

Proof. Let us take the identification $\pi : G|K \to N$ where $\pi(exp(w)\,K) = p$. Since π is an equivariant map with respect to the action of G, it can be seen that K_p can be expressed in the form

$$K_p = \{\ k \in K \mid exp(-w) \cdot k \cdot exp(w) \in K\ \}.$$

Since we have the equalities

$$\begin{aligned} exp(-w)\,exp(tC)\,exp(w) &= exp(Ad_{exp(-w)}(tC)) \\ &= exp(e^{-ad\ w}(tC)) \end{aligned}$$

for any $C \in \mathfrak{k}$ and $t \in \mathbf{R}$, the Lie algebra of K_p coincides with

$$\mathfrak{k}_p = \{\ C \in \mathfrak{k} \ \mid\ e^{-ad\ w}(C) \in \mathfrak{k}\ \}.$$

Using the equalities (4), it can be shown that

$$e^{-ad\ w}(Z_\alpha) = \cos(\alpha(iw))\,Z_\alpha + \sin(\alpha(iw))\,Y_\alpha \tag{11}$$

is true where $Z_\alpha \in \mathfrak{k}_\alpha$ and $Y_\alpha = \mu^{-1}(Z_\alpha) \quad (\alpha \in \triangle^+)$.

Considering a vector C in $\mathfrak{k}$, by Lemma 1 it can uniquely be expressed in the form

$$C = \bar{H} + \sum_{\alpha \in \triangle^+} Z_\alpha$$

where $\bar{H} \in \mathfrak{k}_0$ and $Z_\alpha \in \mathfrak{k}_\alpha$. Using the equality (11) and $e^{-ad\ w}(\bar{H}) = \bar{H}$, it can be seen that the vector $e^{-ad\ w}(C)$ is included in $\mathfrak{k}$ if and only if either $Z_\alpha = 0$ or $\alpha(iw) \in \mathbf{Z}\pi$ holds for any $\alpha \in \triangle^+$. Therefore we obtain that the Lie algebra $\mathfrak{k}_p$ has the expression (10). The proof is completed. □

Considering a point p in P_0, take a vector w in $\mathfrak{p}_0$ such that $Exp(w) = p$ holds. Denote by u the unit vector indicating the direction of w, that is, $w = d \cdot u$ is valid with $d = |w|$. Let $\triangle^+_w$ denote the set of those roots in $\triangle^+$ which are not contained by $\triangle^{\mathbf{Z}}_w$. Let us consider the subspace $\mathfrak{m} = \sum_{\alpha \in \triangle^+_w} \mathfrak{p}_\alpha$ and the normal geodesic $\gamma : [0, d] \to N$ defined by $\gamma(t) = Exp(tu)$, which joins the points $o,\ p$.

Proposition 3. *The tangent space of the orbit $M = K(p)$ at p coincides with the parallel displacement of $\mathfrak{m}$ along the geodesic γ.*

Proof. Let $\mathfrak{p}^w$ denote the tangent space of $\mathfrak{p}$ at w. Clearly, $\mathfrak{p}^w$ can canonically be identified with $\mathfrak{p}$. Let $\mathfrak{p}^w_\alpha,\ y^w_\alpha$ denote the subspace and the vector in $\mathfrak{p}^w$ which correspond to $\mathfrak{p}_\alpha,\ y_\alpha\ (\alpha \in \triangle^+)$, respectively.

Obviously, w can be expressed in the form $w = \langle a_\alpha, w\rangle a_\alpha + b$ where $b \in \mathfrak{p}_0$ and $\alpha(b) = 0$ hold. Considering the equality (8), for any unit vector y_α we have

$$Ad_{exp(tZ_\alpha)}(w) = b + \langle a_\alpha, w\rangle(\cos(|\alpha|t)\,a_\alpha - \sin(|\alpha|t)\,y_\alpha) \tag{12}$$

where $Z_\alpha = c_\alpha \cdot \mu(y_\alpha)$. The equality (12) implies that the tangent space of the orbit $Ad_K(w)$ at w coincides with $\sum_{\alpha\in\triangle_w} \mathfrak{p}_\alpha^w$ where $\triangle_w = \{\ \alpha\in\triangle^+ \mid \alpha(w)\neq 0\ \}$.

Regarding (9), the tangent linear map $T_w Exp$ maps $T_w Ad_K(w)$ onto $T_p K(p)$. Let $\hat{X}_\alpha$ denote the parallel vector field along γ defined by $\hat{X}_\alpha(\gamma(0)) = y_\alpha$ $(\alpha\in\triangle_w)$. Let us consider the geodesic variation $V(t,\tau) = Exp(t(u+\tau y_\alpha))$ $\quad (t\in[0,d],\ \tau\in(-\varepsilon,\varepsilon))$ of γ. It is well-known that the transversal vector field

$$\hat{Y}_\alpha(\gamma(t)) = t\cdot T_{tu} Exp(y_\alpha^{tu}) \tag{13}$$

of V along γ is a Jacobi vector field such that the initial conditions $\hat{Y}_\alpha(\gamma(0)) = 0$ and $\tilde{\nabla}_u \hat{Y}_\alpha = y_\alpha$ hold (see e.g. [7; pp. 132] and [14; pp. 64]). By the equalities (1) and (4) $R_u(y_\alpha) = \alpha(iu)^2 y_\alpha$ is valid. Therefore using the well-known presentation of Jacobi vector fields along geodesics (see [7; pp. 131]), it can be seen that

$$\hat{Y}_\alpha(\gamma(t)) = \frac{1}{|\alpha(iu)|}\ \sin(|\alpha(iu)|t)\,\hat{X}_\alpha(\gamma(t)) \tag{14}$$

holds. The equalities (13) and (14) immediately imply

$$T_w Exp(y_\alpha^w) = \frac{1}{|\alpha(iw)|}\ \sin(|\alpha(iw)|)\,\hat{X}_\alpha(p). \tag{15}$$

This means that the vector $\hat{X}_\alpha(p)$ $\quad (\alpha\in\triangle_w)$ is contained in T_pM if and only if $\alpha(iw)\notin \mathbf{Z}\pi$, which completes the proof. $\square$

Remark 2. By the relations (6) the subspace $\hat{\mathfrak{q}}^\alpha = \mathfrak{p}_0 + \mathfrak{p}_\alpha + \mathfrak{p}_{2\alpha}$ is a Lie triple system in $\mathfrak{p}$ for any $\alpha\in\triangle^+$. Take the totally geodesic submanifold $\hat{Q}^\alpha = Exp(\hat{\mathfrak{q}}^\alpha)$ in N. Considering the eigenvalues and the eigenspaces of the curvature endomorphisms of $\hat{Q}^\alpha$ at o, by Remark 1 we obtain that $\hat{Q}^\alpha$ is locally isometric with the product symmetric space $\mathbf{R}^{n-1}\times Q_\alpha$, where n denotes the rank of N.

Proposition 4. *Let b be an element of $\mathfrak{p}_0$ such that $\langle a_\alpha, b\rangle = 0$ is valid for a given $\alpha\in\triangle^+$. Then $Q_\alpha^b = Exp(b+\mathfrak{q}_\alpha)$ is a totally geodesic submanifold in N. Furthermore, the map $\varphi: Q_\alpha \to Q_\alpha^b$ defined by the assignment $\varphi(Exp(y)) = Exp(b+y)$ $\quad (y\in\mathfrak{q}_\alpha)$ is an isometry.*

Proof. Let us take a vector y in $\mathfrak{q}_\alpha = \mathbf{R}A_\alpha + \mathfrak{p}_\alpha + \mathfrak{p}_{2\alpha}$. The equality $[y,b] = 0$ implies that $\mathbf{R}b + \mathbf{R}y$ is an Abelian subalgebra in $\mathfrak{g}$. It follows from this that $exp(b+y) = exp(b)exp(y)$ is valid. Therefore the equality

$$exp(-b-y_2)exp(b+y_1) = exp(-y_2)exp(y_1)$$

holds for any $y_1,\ y_2\in\mathfrak{q}_\alpha$. Regarding the identification π, it is clear that $Exp(v_1) = Exp(v_2)$ holds for two vectors $v_1,\ v_2\in\mathfrak{p}$ if and only if the condition $exp(-v_2)exp(v_1)\in K$ is satisfied. The above assertions show that the map φ is well-defined.

Take a real number t such that $Exp(ta_\alpha)$ is not a conjugate point of o in Q_α. Notice that this condition is satisfied if $t \notin \frac{c_\alpha}{2\,|\alpha|}\mathbf{Z}\pi$. Assume that w coincides with one of the vectors $ta_\alpha,\ ta_\alpha + b$. Preserving the notation of the proof of Proposition 3, let $\hat{X}$ be the parallel vector field along γ defined by $\hat{X}(\gamma(0)) = a_\alpha$. It is trivial that $T_w Exp(a_\alpha^w) = \hat{X}(\gamma(d))$ holds. Considering the equality (15), the latter one and the definition of φ, we obtain that the tangent linear map $T_{Exp(ta_\alpha)}\varphi$ is isometric. Lemma 4 and the equality (9) imply that $K^\alpha(Exp(\mathbf{R}a_\alpha)) = Q_\alpha$ is valid. Since the maps $k \circ \varphi$ and $\varphi \circ k$ on Q_α coincide for any $k \in K^\alpha$, it can be seen that $T_q\varphi$ is also isometric for any $q \in Q_\alpha$.

Finally, it is obvious that the reflection of N on the point $Exp(\frac{b}{2})$ carries Q_α into Q_α^b. Hence, Q_α^b is also a totally geodesic submanifold in N. □

In order to get the principal curvatures of the orbits we can use the following statement. The straightforward proof is omitted.

Lemma 6. *Let M be a submanifold in a Riemannian manifold N and T be a totally geodesic submanifold intersecting M in a connected submanifold L $(L = M \cap T)$. Regarding L as a submanifold in T, let $\nu^T(L)$ denote the normal bundle of L in T. Let ξ be a smooth section of the vector bundle $\nu^T(L)$ and let A_ξ^L denote the corresponding shape operator of L in T.*

Assume that $\nu_p^T L \subset \nu_p M$ holds for any point $p \in L$. Then T_pL is an invariant subspace of $A_{\xi(p)}$ for any $p \in L$, furthermore, the restriction of $A_{\xi(p)}$ to T_pL coincides with $A^L_{\xi(p)}$.

Since $exp(\mathfrak{p}_0)$ is an Abelian subgroup in G, it can easily be seen that P_0 is a flat torus.

Lemma 7. *Let p be an arbitrary point in P_0. Let $\gamma_1,\ \gamma_2 : [0,1] \to P_0$ be two piecewise smooth curves which join the points o, p and consist of geodesic segments. Let $E_\alpha^1(p),\ E_\alpha^2(p)$ $(\alpha \in \triangle^+)$ denote the subspaces in T_pN obtained by parallel displacement of $\mathfrak{p}_\alpha$ along $\gamma_1,\ \gamma_2$, respectively. Then $E_\alpha^1(p) = E_\alpha^2(p)$ holds.*

Proof. Since P_0 is a flat torus, each vector v in $\mathfrak{p}_0$ uniquely determines a parallel vector $v(p)$ in T_pP_0. Let us fix a vector v in $\mathfrak{p}_0$ such that the numbers $\beta(iv)^2$ $(\beta \in \triangle^+ \cup \{0\})$ are different. By the equalities (1) and (4) $\mathfrak{p}_\alpha$ is an eigenspace of the curvature endomorphism R_v with the eigenvalue $\alpha(iv)^2$. It is known that the parallel displacements along geodesics preserve the equalities concerning the Riemannian curvature tensor R. This means that $E_\alpha^1(p),\ E_\alpha^2(p)$ $(\alpha \in \triangle^+)$ are also eigenspaces of $R_{v(p)}$ with the same eigenvalue $\alpha(iv)^2$, which completes the proof. □

Let w be a vector in $\mathfrak{p}_0$ and let us consider the orbit $M = K(p)$ where $Exp(w) = p$. Since the flat torus P_0 is a section of ρ, its tangent space T_pP_0 is included in the normal subspace ν_pM. Applying the earlier statements, the shape operators of M at p can explicitly be described as follows.

Let $\gamma : [0,1] \to N$ be the geodesic defined by $\gamma(t) = Exp(tw)$. By Proposition 3 the tangent space T_pM coincides with the parallel displacement of the subspace $\mathfrak{m} = \sum_{\alpha\in\triangle_w^+} \mathfrak{p}_\alpha$ along γ where $\triangle_w^+ = \{\ \alpha \in \triangle^+ \mid \alpha(w) \notin \mathbf{Z}\pi i\ \}$. Denote by $E_\alpha(p)$ that subspace in T_pM which is obtained by the parallel displacement of $\mathfrak{p}_\alpha$ $(\alpha \in \triangle_w^+)$ along γ.

Theorem 1. *Let v be an element of $\mathfrak{p}_0$ and let $v(p)$ denote the corresponding parallel vector in T_pP_0. Then the eigenspaces of the shape operator $A_{v(p)}$ of $M = K(p)$ are $E_\alpha(p)$ $(\alpha \in \triangle_w^+)$ with the corresponding eigenvalues*

$$\lambda_\alpha = -\alpha(iv) \cdot \cot(\alpha(iw)). \tag{16}$$

Proof. Let α be an arbitrary element of $\triangle_w^+$. Let us consider the orthogonal decomposition $w = \langle a_\alpha, w\rangle a_\alpha + b$ where $\alpha(b) = 0$. Using Lemma 4 and the equality (12), we obtain that the orbit $Ad_{K^\alpha}(w)$ coincides with the set $b + S_\alpha$ where S_α denotes the sphere of radius $r = \frac{c_\alpha}{|\alpha|}|\alpha(iw)|$ around 0 in $\mathfrak{q}_\alpha$, that is, $S_\alpha = \{ \ y \in \mathfrak{q}_\alpha \mid |y| = \frac{c_\alpha}{|\alpha|}|\alpha(iw)| \ \}$. Regarding Proposition 4, it is clear that $Exp(Ad_{K^\alpha}(w))$ coincides with the geodesic sphere of radius r around $q = Exp(b)$ in the totally geodesic submanifold Q_α^b.

Take the totally geodesic submanifold $\hat{Q}^\alpha$ in N (see Remark 2). It is an important fact for us that by the equality (9) the relation $K^\alpha(p) = Exp(Ad_{K^\alpha}(w))$ holds. Hence, $K^\alpha(p)$ is included in $M \cap \hat{Q}^\alpha$.

First assume that $2\alpha(w) \notin \mathbf{Z}\pi i$ or $2\alpha \notin \triangle$ holds. In this case, since the orthogonal decomposition

$$T_p\hat{Q}^\alpha = T_pP_0 + E_\alpha(p) + E_{2\alpha}(p)$$

is valid, locally $\hat{Q}^\alpha$ intersects $M = K(p)$ in an $(m_\alpha + m_{2\alpha})$–dimensional submanifold. This implies that $K^\alpha(p)$ coincides with that component of $M \cap \hat{Q}^\alpha$ which contains p. Using Propositions 1 and 4, we obtain that Q_α^b is a totally geodesic submanifold of rank one and has the minimal sectional curvature $\kappa_1 = \frac{|\alpha|^2}{{c_\alpha}^2}$. If $2\alpha \in \triangle$ is valid, then Q_α^b is not of constant curvature and its maximal sectional curvature is equal to $\kappa_2 = 4\frac{|\alpha|^2}{{c_\alpha}^2}$.

It can be seen that in order to get the principal curvatures of M at p we can apply Lemma 6 with $T = \hat{Q}^\alpha$ and $L = K^\alpha(p)$. Obviously, the points q and p are joined by a periodic geodesic $\hat{\gamma}_2$ in Q_α^b which is defined by $\hat{\gamma}_2(t) = Exp(b + ta_\alpha)$. By Proposition 4 $T_o\varphi(\mathfrak{p}_\alpha)$ and $T_o\varphi(\mathfrak{p}_{2\alpha})$ are eigenspaces of the curvature endomorphism $R_{a_\alpha(q)}$ with the eigenvalues κ_1, κ_2. Since $K^\alpha(p)$ is just the geodesic sphere of radius r around q in Q_α^b, Lemma 5 implies that the parallel displacements of these subspaces along $\hat{\gamma}_2$ to p are eigenspaces of the shape operator $A_{a_\alpha(p)}$ with the corresponding eigenvalues

$$\lambda_1 = -\operatorname{sgn}(\langle a_\alpha, w\rangle)\sqrt{\kappa_1}\,\cot(\sqrt{\kappa_1}\cdot r) = \frac{|\alpha|}{c_\alpha}\cot(\alpha(iw)), \qquad \lambda_2 = 2\,\frac{|\alpha|}{c_\alpha}\cot(2\cdot\alpha(iw)).$$

However, by Lemma 7 these eigenspaces are just $E_\alpha(p)$ and $E_{2\alpha}(p)$, respectively, since the tangent linear map $T_o\varphi$ coincides with the parallel displacement of $\mathfrak{q}_\alpha$ along the geodesic $\hat{\gamma}_1 : [0,1] \to N$ where $\hat{\gamma}_1(t) = Exp(tb)$. Concerning the principal curvatures λ_1, λ_2, remark that the equality $\alpha(ia_\alpha) = -\frac{|\alpha|}{c_\alpha}$ is valid.

Clearly, if $u \in \mathfrak{p}_0$ is normal to a_α, then $A_{u(p)}$ vanishes on $E_\alpha(p) + E_{2\alpha}(p)$, since Q_α^b is totally geodesic in N. The above assertions imply that $E_\alpha(p)$ is an eigenspace of $A_{v(p)}$ with the eigenvalue (16) for any $v \in \mathfrak{p}_0$.

Finally, let us consider that special case when $\alpha(w) \in \frac{1}{2}(2\mathbf{Z} + 1)\pi i$ and $2\alpha \in \triangle$ are valid. Then using Lemma 5 and the analogous procedure, it can be seen that $E_\alpha(p)$ is an eigenspace of $A_{v(p)}$ with the eigenvalue 0. Therefore the proof is completed. □

Regarding the orbits of K as submanifolds, we can use the fact that K acts transitively on them. If the orbit $K(p)$ is maximal dimensional, then T_pP_0 coincides with $\nu_pK(p)$. Hence, Theorem 1 implies the following statement.

Corollary 1. *If $K(p)$ is a principal orbit, then its shape operators commute.*

Using Theorem 1, the totally geodesic orbits of K can be selected. The following theorem can be obtained from the criterion given by R. Hermann in [11], too (see Remark 3).

Theorem 2. *The submanifold $M = K(p)$ is totally geodesic in N if and only if $\alpha(w) \in \frac{1}{2}\mathbf{Z}\pi i$ holds for each $\alpha \in \triangle^+$.*

Proof. By Theorem 1 it is evident that the given condition is necessary. Conversely, assume that $\alpha(w) \in \frac{1}{2}\mathbf{Z}\pi i$ holds for each $\alpha \in \triangle^+$. Let Exp^M denote the restriction of Exp to the normal bundle $\nu(M)$. By the tangent linear maps of the isometries K acts on $\nu(M)$, too. It is clear that $Exp^M \circ Tk = k \circ Exp^M$ holds for any $k \in K$. Since P_0 intersects all the orbits of ρ, $\cup_{k\in K} k(P_0) = N$ is valid. The restriction of Exp^M to a suitable tubular neighborhood of M in $\nu(M)$ is a diffeomorphism. The above facts imply that $\cup_{k\in K_p} Tk(T_pP_0) = \nu_pM$ holds where K_p denotes the isotropy subgroup of K at p. Hence, if $v(p)$ is an arbitrary vector in ν_pM, then there exists an isometry k in K_p such that $Tk(v(p)) \in T_pP_0$ is valid. Since $A_{v(p)}$ and $A_{Tk(v(p))}$ have the same eigenvalues, by Theorem 1 we obtain that $A_{v(p)}$ vanishes on T_pM, which completes the proof. $\square$

Remark 3. In the paper [11] R. Hermann has studied totally geodesic orbits of polar isometric actions on Riemannian symmetric spaces. Without using root space decompositions, he has given a criterion for an orbit to be totally geodesic. By Theorem 2.2 in [11] the orbit $K(h(o))$ is totally geodesic for an element $h \in exp(\mathfrak{p}_0)$ if and only if $Ad_{h^4} = id$ holds. However, Lemma 3 implies that $h = exp(w)$ satisfies the above condition if and only if $\alpha(w) \in \frac{1}{2}\mathbf{Z}\pi i$ is valid for any root α in $\triangle$.

Regarding Theorem 2, we can state the following assertion.

Corollary 2. *If N is irreducible and not of constant curvature, then all the totally geodesic orbits of K are singular.*

Remark 4. Evidently, we call a point p antipodal to o if there is a closed geodesic (circle) in N which contains o and p as opposite points. It is known that if p is an antipodal point to o, then the orbit $K(p)$ is totally geodesic and called a polar of o (see [4]). It is natural to raise the question: in which symmetric spaces are there totally geodesic orbits of K which are not polars of o.

Assume that the Lie group G is simply connected. Let us take the lattice

$$\Lambda_1 = \{\ w \in \mathfrak{p}_0 \mid \alpha(w) \in \frac{1}{2}\mathbf{Z}\pi i \quad \textit{for any } \alpha \in \triangle^+\ \}$$

presenting the totally geodesic orbits. Furthermore, let Λ_2 denote the lattice in $\mathfrak{p}_0$ generated by the vectors $\frac{\pi}{|\alpha|^2}A_\alpha$ $(\alpha\in\triangle^+)$ which is a sublattice of Λ_1. Using Theorem 8.5 of Chapter VII in [10], we obtain that $Exp(w)$ is an antipodal point to o or coincides with o if and only if $w\in\Lambda_2$. In this case there are totally geodesic orbits consisting of points not antipodal to o if and only if $\Lambda_2\neq\Lambda_1$.

Let N be a simply connected symmetric space of type II. This means that N coincides with such a compact Lie group the Lie algebra of which is a compact real form in a simple complex Lie algebra. By the above results the isotropy subgroup has totally geodesic orbits which are not polars of o in the following "symmetric spaces" (see [12; pp. 67-69]): $SU(n)$, $Spin(n)$, $Sp(n)$, E_6, E_7.

A connected totally geodesic submanifold M in N is said to be reflective if there exists an involutive isometry in N such that M is a component of its fixed point set. Reflective submanifolds in irreducible symmetric spaces have been discussed in several papers by D. S. P. Leung.

Theorem 3. *Assume that N is simply connected. If $K(p)$ is a totally geodesic orbit, then it is a reflective submanifold in N.*

Proof. Take a totally geodesic orbit $K(p)$ $(p\in P_0)$ and a vector $w\in\mathfrak{p}_0$ such that $Exp(w)=p$ holds. By Theorem 2 the numbers $2\frac{\alpha(iw)}{\pi}$ $(\alpha\in\triangle)$ are integers, therefore by their parities the set $\triangle^+$ can be divided into two disjoint subsets which are $\triangle^+_w$ and $\triangle^{\mathbf{Z}}_w$ (see Propositions 2 and 3). Using the relations (6), it can be seen that the orthogonal subspaces

$$\mathfrak{p}^1_w=\sum_{\alpha\in\triangle^+_w}\mathfrak{p}_\alpha,\qquad\qquad \mathfrak{p}^2_w=\mathfrak{p}_0+\sum_{\alpha\in\triangle^{\mathbf{Z}}_w}\mathfrak{p}_\alpha$$

(the upper index indicates the parity) are Lie triple systems in $\mathfrak{p}$ which satisfy the relations

$$[\,[\mathfrak{p}^j_w,\mathfrak{p}^l_w]\,\mathfrak{p}^j_w\,]\subset\mathfrak{p}^l_w,\qquad\qquad [\,[\mathfrak{p}^j_w,\mathfrak{p}^l_w]\,\mathfrak{p}^l_w\,]\subset\mathfrak{p}^j_w$$

for $j,\ l\in\{1,2\}$. Take the corresponding totally geodesic submanifolds $P^j=Exp(\mathfrak{p}^j_w)$, $(j=1,2)$. By Theorem 1.1 of the paper [15] P^1 and P^2 are reflective submanifolds in N. Denote by $\tau_1,\ \tau_2$ the corresponding involutive isometries.

The earlier statements of this paper imply that the normal subspace $\nu_pK(p)$ coincides with T_pP^2. It follows from this that $K(p)$ is a component of the fixed point set of the involutive isometry $r_p\circ\tau_2$ where r_p denotes the reflection of N on p.

If we omit the assumption that N is simply connected, then there exists a local reflection on $K(p)$, since the submanifold $Exp(\nu_qK(p))$ is totally geodesic for any $q\in K(p)$ (see Theorems 1 and 3 in [5]). Finally, notice that the reflection of N on $Exp(\frac{w}{2})$ carries P^1 and $K(p)$ into each other. $\square$

The so-called curvature-adapted submanifolds have been studied in the papers [1] and [2]. The following result has already been proved for principal orbits of isotropy subgroups in symmetric spaces of the noncompact type (see [21]).

Corollary 3. *All the orbits of the isometric action ρ are curvature-adapted submanifolds in N.*

Proof. Take an orbit $K(p)$ $(p \in P_0)$ and a normal vector $v(p) \in T_p P_0$. Considering the proof of Lemma 7 and Theorem 1, we obtain that $A_{v(p)}$ and $R_{v(p)}$ commute on $T_p K(p)$. This implies that if $K(p)$ is a principal orbit, then it is a curvature-adapted submanifold. On the other hand, if $K(p)$ is a singular one, then we can use the procedure applied in the proof of Theorem 2. □

Remark 5. It is clear that excepting Theorems 2 and 3, analogous statements can be deduced in symmetric spaces of the noncompact type, too. Since in such a symmetric space the exponential map at a point is a diffeomorphism, we can take global normal coordinate systems. Hence, orbits of isotropy subgroups can easily be characterized. Clearly, the hyperbolic function *coth* has to be used in the analogue of Theorem 1. In these symmetric spaces totally geodesic foliations on isoparametric orbits have been discussed in the paper [21].

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