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A Classification of Perfect 4-Solids

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Abstract. The notion of regularity of polytopes was generalized to that of perfection in [9] by Robertson. A polytope P is said to be *perfect* if it has maximal symmetry properties in the sense that P cannot be deformed without changing its 'shape' or its symmetry group. In [3], the study of symmetry of convex polytopes was extended by Farran and Robertson to convex solids in general (in other words, compact convex bodies). Thus we have the notion of a regular solid and a perfect solid. Recently, the family of regular solids were classified by Madden and Robertson [5]. However, the more general problem of classifying the perfect solids remains unsolved. In fact, not even the perfect polytopes have been classified. This paper gives a classification of the perfect 4-solids, confirming a conjecture of Rostami [12].

In section 0, we explain the terminology and key ideas. For general background, see Robertson [8]. We introduce the notion of a deformation of a solid (and in particular of a polytope) in section 1. A polytope P is perfect if and only if all deformations of P are similar to P. The properties of the fixed point set of a vertex of a perfect polytope are investigated in section 2. It turns out that this set is one-dimensional. The conditions for which a perfect polytope is regular are determined in section 3. In section 4, the orbit vectors of a perfect polytope P provide useful information about transitivity properties of GP, leading to a classification of perfect 4-polytopes. We conclude with a discussion of the perfect 4-solids in section 5.

The classification in dimensions $n \geq 5$ remains unresolved. In the cases $n \geq 6$, there is a problem in that perfect polytopes are known which are not associated (in terms of their symmetry group) to any regular polytope. Such polytopes are derived from the root systems E_6, E_7 and E_8.

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0. Preliminaries

An *n-solid* (or *solid*) is a compact convex subset B of E^m whose affine hull aff(B) is n-dimensional, for some $m \geq n \geq -1$. The dimension of a solid B is denoted by $\dim(B)$.

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The boundary of B is composed of faces which are solids of lower dimension. A *supporting hyperplane* of an n-solid B in E^n is an affine $(n-1)$-plane Π in E^n such that $(\Pi \cap B)$ is nonempty and B lies entirely in one of the two closed half-spaces bounded by Π. For each such Π, the set $(\Pi \cap B)$ is convex and hence an i-body for some $-1 \leq i \leq n$. The set $(\Pi \cap B)$ is called an *i-face* of B. The faces of B form a lattice FB under inclusion called the *face-lattice* of B, which is graded by dimension. The face-lattice FB is bounded above by B itself and below by the empty set $\emptyset$, to which we assign dimension -1. The solids B and $\emptyset$ are the unique n- and (-1)-faces of B and are called *improper*. All other faces of B are called *proper*. Let A_1, A_2 be faces of B. Then we use the notation $A_1 \lhd A_2$ if A_1 is contained in A_2. A *flag* of B is a non-empty sequence $(A_1, \ldots, A_r)$ of faces A_i, $1 \leq i \leq r$, such that $A_1 \lhd \cdots \lhd A_r$. A flag of B is said to be *maximal* if it is not contained in any other flag of B.

It is customary to refer to the 0-, 1- and $(n-1)$-faces of an n-solid B (if any) as *vertices*, *edges* and *facets* of B respectively. Two i-faces f_1 and f_2, $1 \leq i \leq n-1$, of B are said to be *adjacent* if $(f_1 \cap f_2)$ is an $(i-1)$-face of B. Two vertices v_1 and v_2 are said to be *adjacent* if the line segment $v_1 v_2$ is an edge of B.

If we let $F_j B$ be the set of all j-faces, $0 \leq j \leq n-1$, of an n-solid B, then B is said to be an *n-polytope* (or *polytope*) if $F_0 B$ is finite. It then follows that $F_i B$ is non-empty and finite for all $0 \leq i \leq n-1$. We let $f_r B$ denote the number of r-faces in B. The *face vector* of an n-polytope P is the vector $f(P) = (f_0 P, f_1 P, \ldots, f_{n-1} P)$.

There is a basic equivalence relation between solids defined in terms of their face-lattices. We say that the solids B and C are *face equivalent* if and only if there is a lattice isomorphism λ: $FB \to C$. We denote this by $B \approx C$.

The symmetry group $G = GB$ of an n-solid B in E^n is the set of all rigid transformations of E^n keeping B setwise fixed. We may assume that the centroid of B is O, so $GB \subset \mathrm{O}(n)$. The effective action of GB on B induces an effective action α_B of GB on FB, α_B: $GB \times FB \to FB$. We define an equivalence relation $\simeq$ on B by putting $B \simeq C$ if there exists an isometry β of E^n and a face-lattice isomorphism λ: $FB \to FC$ such that for each $g \in GB$ and each $A \in FB$, $\lambda\big(g(A)\big) = (\beta^{-1} g \beta)\big(\lambda(A)\big)$. We say two solids B and C are *symmetry equivalent* if $B \simeq C$. Each symmetry equivalence class is called a *symmetry type* and the symmetry type to which B belongs is denoted $[B]$. If P is a polyhedron then $[P]$ can be realised as a topological manifold (see [8]), and the dimension of the symmetry type is called the *deficiency* of P, denoted $\operatorname{def} P$.

In [3], a solid B is said to be *regular* if GB is transitive on the set of maximal flags of B. This new definition of regularity for polytopes agrees with the classical concept of regularity. One extension of regularity is that of perfection. This may be defined as follows. A *similarity* of E^n is a map s: $E^n \to E^n$ such that for some $m \in \mathbf{R}$, $m > 0$, $md(x, y) = d\big(s(x), s(y)\big)$ for all $x, y \in E^n$, where d denotes the usual metric on E^n. Two n-solids B, C are said to be *similar*, denoted by $B \sim C$, if there exists a similarity s of E^n such that $B = s(C)$. In such a case, $m = \gamma(B)$ is called the *gauge* (or *scale*) of s.

A solid B is said to be *perfect* if and only if all solids symmetry equivalent to B are similar to B. Among the best known examples of perfect solids are the regular polytopes, of which the most familiar are the five Platonic solids. More generally, all regular solids are perfect (see [3]).

The perfect n-polytopes have been classified up to $n = 3$ by Robertson [8]. Trivially in dimensions -1, 0 and 1, all solids are perfect polytopes. In dimension 2, the perfect polytopes coincide with the regular polygons. Finally in dimension 3, there are four non-regular perfect polyhedra in addition to the regular Platonic solids. These are the cuboctahedron, the icosadodecahedron, the rhombic dodecahedron and the rhombic triacontrahedron. If we add the circular disk and the 3-ball to this list, we have a classification of perfect solids up to dimension 3.

Therefore we are interested in classifying the perfect polytopes and solids in dimensions $n \geq 4$. From [6], it is known that the polar B^* of a perfect solid is also perfect, as is any power $\square^r B = B \square \cdots \square B$ and its polar $(\square^r B)^* \sim \diamond^r B^*$. Besides, a decomposable solid $B \square C$ is perfect only if B and C are isometric and perfect. For further details, also see [3] and [8]. We can therefore restrict our classification to prime perfect solids (that is, those which are $\square$-indecomposable).

Almost all known prime perfect polytopes or their polars are derived from the irreducible root systems of semisimple Lie algebras in the following way. Recall that the irreducible root system of a semisimple Lie algebra $\mathfrak{g}$ is of the form A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4 or G_2. With the exception of E_6, E_7 and E_8, the Weyl group of $\mathfrak{g}$ is the symmetry group of some regular polytope. Let R be an irreducible root system of a semisimple Lie algebra $\mathfrak{g}$. Then associated to R is a Dynkin diagram (or Coxeter graph) Γ and hence a fundamental region D of the Weyl group W of $\mathfrak{g}$. By [11], the convex hull $\mathrm{conv}\big(W(v)\big)$ of the action of W on any vertex $v \neq O$ of D is a perfect polytope P. Each node of Γ corresponds to a vertex $v \neq O$ of D. Thus we denote P by the Coxeter graph where the node corresponding to v is shaded. The process of deriving a polytope from a Coxeter graph or fundamental region is called *Wythoff's construction* (see Coxeter [1]).

A perfect polytope derived from a regular polytope by Wythoff's construction is called *Wythoffian*. Then up to polarity any other presently-known prime perfect polytope is a q-sided polygon ($q = 5$ or $q \geq 7$), a Wythoffian polyhedron derived from a dodecahedron or a Wythoffian polytope derived from a 120-cell.

There are numerous examples of non-regular non-polytope prime perfect solids as a result of the classification of regular solids (see [5]). The regular solids arise from certain regular polytopes by a process called Kostant's construction. This process assigns every regular solid B to a regular polytope P such that the dimension of P is given by the rank of GB, as follows. Let G be a compact convex semisimple Lie group G with Lie algebra $\mathfrak{g}$ such that the Weyl group W of $\mathfrak{g}$ is the symmetry group of a regular polytope. Let $v \neq O$ be a vertex of the fundamental region D of W. Then $B = \mathrm{conv}\big(G(v)\big)$ is a perfect solid. The solid B is regular in the special cases where the vertex v corresponds to an end node of the Coxeter graph. For more details see [3] and [5].

These results agree with the following conjecture made by Rostami [12].

Rostami's Conjecture 1.1 *A perfect 4-polytope or its polar is either of the form $Q \square Q$ for some regular polygon Q or a Wythoffian polytope derived from a regular 4-polytope.*

A proof of this conjecture is given in [4] and is the basis of this paper. The above results and this conjecture lead to a generalisation of conjecture 1.1 made by Pinto [6].

1. Deformations

A solid B is perfect if and only if every nearby solid that is symmetry equivalent to B is similar to B. We are interested in maps between such solids and B, particularly when B is a polytope. It is useful to introduce some elementary category theory.

Let $\mathcal{S}$ be the space of all solids. Then $\mathcal{S}$ can be regarded as the set of objects in a category $\mathfrak{S}$ of face-maps between solids. The morphisms or face-maps are defined as follows. Let B be an n-solid, $n \geq 0$. First each point x in B is given a sequence of radial coordinates (similar to barycentric coordinates) in terms of the centroids of a flag of B. Let f_1 be the face of B of lowest dimension j_1 such that $x \in f_1$. If $x = c_1$ (the centroid of f_1), then we say x is given by the n coordinates $(0, \dots, 0, 1, 0, \dots, 0)$ where the $(n+1-j)^{th}$ entry is 1. If $x \neq c_1$, there is a unique straight line through points c_1 and x oriented from c_1 towards x. This line meets ∂f_1 in a unique point x_1 at or beyond x. Thus $x = t_1 x_1 + (1 - t_1) c_1$ for some unique $t_1 \in I = [0, 1]$, $t \neq 0$. Let f_2 be the face of B of lowest dimension $j_2 < j_1$ such that $x_1 \in f_2$. If $x_1 = c_2$ (the centroid of f_2), then x_1 is given by coordinates similar to c_1 above but having the only non-zero entry 1 at the $(n-j)^{th}$ position. If $x_1 \neq c_2$, then the above process is repeated on x_1 in terms of x_2 and c_3. After a finite number of steps x_k is the centroid c_k of a face of B. Therefore x is associated with an affine sum of c_i (or x_i), $i = 1, \dots, k$, that is,

$$\begin{aligned}
x &= t_1 x_1 + (1 - t_1) c_1 \\
&= t_1 \big(t_2 x_2 + (1 - t_2) c_2\big) + (1 - t_1) c_1 \\
&= (1 - t_1) c_1 + t_1 (1 - t_2) c_2 + t_1 t_2 x_2 \\
&\ \ \vdots \\
&= (1 - t_1) c_1 + t_1 (1 - t_2) c_2 + \cdots + t_1 (1 - t_2) \cdots (1 - t_{k-1}) c_{k-1} + t_1 t_2 \cdots t_k x_k \\
&= (1 - t_1) c_1 + t_1 (1 - t_2) c_2 + \cdots + t_1 (1 - t_2) \cdots (1 - t_{k-1}) c_{k-1} + t_1 t_2 \cdots t_k c_k.
\end{aligned}$$

Hence $x = \{X_1, \dots, X_n\}$ such that $X_i = t_i$ if $j_i = i$ and $X_i = 0$ otherwise. Then a *face-map* f: $B_1 \to B_2$ between two n-solids B_1, B_2 is a map which takes the centroids of faces of B_1 to centroids of faces of B_2 such that the image of any point x written as an affine sum above in terms of c_i is the affine sum of the images of c_i, that is,

$$\begin{aligned}
f(x) = (1 - t_1) f(c_1) + t_1 (1 - t_2) f(c_2) + \cdots + t_1 (1 - t_2) \cdots (1 - t_{k-1}) f(c_{k-1}) \\
+ t_1 t_2 \cdots t_k f(c_k).
\end{aligned}$$

It follows at once that any face-map is continuous, that the composition of face-maps f: $B_1 \to B_2$ and g: $B_2 \to B_3$ is a face-map $g \circ f$: $B_1 \to B_3$, and that for any nonempty solid B, the identity 1_B is a face-map. It is convenient to regard $\mathcal{S}(\emptyset, B)$, $\mathcal{S}(B, \emptyset)$ and $\mathcal{S}(\emptyset, \emptyset)$ as singletons.

Suppose next that $\mathfrak{M}$ denotes the category of similarities between solids. Then $\mathfrak{M}$ is a subcategory of $\mathfrak{S}$. Thus: every object in $\mathfrak{M}$ is an object in $\mathfrak{S}$ (in fact, both have the same objects, namely $\mathcal{S}$); every similarity between solids is a face-map; and the composition operation is the same in both categories. We can define a notion of symmetry equivalence between solids which is coarser than similarity and finer than face-isomorphism, as follows. Let B_1, B_2 be solids. Then a face-isomorphism f: $B_1 \to B_2$ is said to be a *symmetry equivalence* if and only if there is a group isomorphism f_*: $GB_1 \to GB_2$ such that for all $g \in GB_1$, $f_*(g) \circ f = f \circ g$. If such an f exists, we say that B_1 is symmetry equivalent to B_2 and write $B_1 \simeq B_2$.

A deformation of an n-solid B is now defined in terms of a connected (continuous) path in the space of solids. A *deformation* is a continuous map, δ: $B \times I \to E^n$, such that for all $t \in I$, $\delta(B,t) = B_t$ is an n-solid and f_t: $B \to B_t$, $f_t(x) = \delta(x,t)$, is a symmetry equivalence. We can assume that $GB = GB_t$ for all $t \in I$. If B is an n-polytope P, then since $P = \mathrm{conv}(F_0P)$, the convex hull of F_0P, f_t: $P \to P_t$ may be defined in terms of a set of paths $\{\delta(v,I)\colon v \in F_0P\}$. It follows that each such path $\delta(v,I)$ lies in the fixed point set $\mathrm{fix}_v = \{x \in E^n\colon g(x) = x \ \forall g \in G_v\}$ of v. Since we are interested in the difference between perfect and imperfect polytopes, we consider only deformations involving non-similar polytopes, that is to say, P_t is not similar to P, for some $t \in I$.

If δ is a deformation for some solid B, we call f_t: $B \to \delta(B,t)$ a *D-path* from B. Thus a polytope P is perfect if and only if there are no D-paths from P. The above formulation of the notion of face-map and the definition of symmetry equivalence in terms of categories has been devised by S. A. Robertson.

2. Vertex orbits

Recall ([3]) that an n-polytope P is said to be *i-transitive*, $0 \le i \le n-1$, if GP acts transitively on F_iP. We say that P is $(a_1, a_2, \dots, a_r)$-transitive if P is a_j-transitive for all j, $1 \le j \le r$. If P is i-transitive for all $0 \le i \le n-1$, then we call P *completely transitive*. It is well-known that a polytope P is completely transitive if and only if P is regular. A proof of this can be found in [4], where an alternative inductive definition of regular polytope is given. Trivially, all line segments are regular. An n-polytope P, $n \ge 2$, with congruent regular facets such that all the vertices of P lie on an $(n-1)$-sphere is *regular*. This definition is equivalent to those in [1] and [3].

Let P be a perfect n-polytope with distinct vertex orbits $G.v_1, \dots, G.v_m$, for some $m \ge 1$. Let $P_i = \mathrm{conv}(G.v_i)$ denote the 0-transitive polytope given by the convex hull of the i^{th} orbit. Then P_i is an n-polytope, since otherwise $\mathrm{aff}(P_i)$ is a proper G-invariant subspace of $\mathrm{aff}(P)$, which is a contradiction by [7]. Moreover, we have the following result.

Proposition 2.1 *If P is perfect then P_i is perfect.*

Proof. Suppose P_i is not perfect for some i. Then there exists a deformation δ of P_i and a D-path δ_t: $P_i \to \delta(P_i, t)$ for some $t \in I$ such that the paths $\{\delta(v,I)\colon v \in F_0P_i\}$ are given by $G.\delta(v_i, I)$. Since $\dim(\mathrm{fix}_{v_i}) \ne 1$, we can assume that O, v_i and $\delta_t(v_i) \in \mathrm{fix}_{v_i}$ are not collinear. Then there exists a deformation δ' of P_i defined by paths of vertices given by $G.\{s\delta_t(v_i) + (1-s)v_i\colon s \in I\}$ such that $s\delta_t(v_i) + (1-s)v_i \in \mathrm{fix}_{v_i}$ for all $s \in I$. Let δ'_s: $P_i \to \delta'(P_i, s)$ be a D-path of δ'. We may consider δ'_s acting on some fundamental region D of the action of GP containing only one vertex v_i of P_i.

Since any point x in P may be given using barycentric coordinates in terms of vertices of P, the deformation δ' induces a deformation of P such that the D-paths on the vertices

$\{G.v_j\colon j \notin G.v_i\}$ are radial projections $v_j \to (1+\epsilon)v_j$ for some ϵ and the D-paths on $\{G.v_i\}$ are given by δ'_s. This is a contradiction to the hypothesis that P is perfect. □

The converse of this proposition is false. We can find numerous counter-examples by taking the imperfect convex hull of the union of two (or more) perfect polytopes which have the same symmetry group. For instance, the convex hull of the union P of a cube P_1 and an octahedron P_2 is perfect only if $P = P_1$ or P_2 or P is a rhombic dodecahedron.

Proposition 2.2 *Let P be a perfect n-polytope. Then* $\dim(\mathrm{fix}_v) = 1$ *and* $\dim(\mathrm{fix}_c) = 1$ *where v is any vertex of P and c is the centroid of any facet of P.*

Proof. Let v be a vertex of a perfect n-polytope P. Suppose that $\dim(\mathrm{fix}_v) \neq 1$. Then $v \in G.v_i$ for some i and P_i is a 0-transitive perfect n-polytope by the above. Therefore P_i cannot be deformed into a symmetry equivalent non-similar polytope. Since $\dim(\mathrm{fix}_v) \neq 1$, we can define a D-path δ_t from P_i by mapping v to $w = v + \epsilon$ such that O, v and w are not collinear. However $P_t = \delta_t(P_i) = \mathrm{conv}(G.w)$ is a symmetry equivalent non-similar polytope. This is a contradiction if P_i is perfect. Therefore $\dim(\mathrm{fix}_v) = 1$. Hence $\dim(\mathrm{fix}_c) = 1$ by polarity. □

In fact it is not difficult to show the following result (see [4]).

Proposition 2.3 *Let P be a $(0, n-1)$-transitive n-polytope. Then P is perfect if and only if* $\dim(\mathrm{fix}_v) = \dim(\mathrm{fix}_c) = 1$*, where v is any vertex of P and c is the centroid of any facet of P.*

3. Regularity for perfect polytopes

Let P be an n-polytope. Then the *orbit vector* of P is the n-vector $\theta(P) = (\theta_0, \dots, \theta_{n-1})$, where θ_i is the number of orbits of i-faces in F_iP, for each $i = 0, \dots, n-1$, under the action of GP. It is possible to express the orbit vector of any polytope in terms of the orbit vectors of the prime polytopes in its □-decomposition.

A polytope P is regular if and only if $\theta(P) = (1, \dots, 1)$. If P_i is the perfect Wythoffian n-polytope derived from a regular n-polytope R by taking the convex hull of the centroids of all i-faces, where $i \leq [\frac{n}{2}]$, then it is also possible to determine the orbit vector of P_i. We note that for such a P_i, $\theta(P_i) = (1, 1, 2, 3, \dots, i, i+1, \dots, i+1, i, i-1, \dots, 4, 3, 2)$. Such a polytope P_i is regular if and only if $i = 0$. (For full details, see [4].) We consider the conditions for which a perfect polytope is regular. Since there are examples of 0-transitive nonregular perfect 4-polytopes, we consider the $(0, 3)$-transitive prime perfect 4-polytopes.

Proposition 3.1 *Let P be a prime perfect $(0, 3)$-transitive 4-polytope. Let f be an i-face with centroid c and f' an $(i+1)$-face with centroid c', such that $f \triangleleft f'$. Then (1) $|cv_j| = |cv_k|$ for all v_j, $v_k \in F_0 f$, and (2) cc' is perpendicular to f.*

Proof. We consider an i-face of P to be given by the intersection of $r \geq (4-i)$ facets of P. We assume P to be □-prime, since otherwise the □-decomposition results in a decomposition of E^4 into orthogonal subspaces. Also if $P = \square^r A$ for some integer r and some polytope A, then P is perfect $(0, 3)$-transitive if and only if A is perfect $(0, 3)$-transitive.

$(i = 3)$. Let f_r be a facet of P with centroid c_r and let v be any vertex of f_r. Then by Proposition 2.2, $\dim(\mathrm{fix}_{c_r}) = \dim(\mathrm{fix}_v) = 1$. Therefore $\mathrm{aff}(f_r)$ and the line Oc_r are orthogonal. In particular, $\angle v_j c_r O = \pi/2$ for all $v_j \in F_0 f_r$. Since P is $(0,3)$-transitive, we have $|Ov_j| = |Ov_k|$ and $|Oc_r| = |Oc_s|$ for all $v_r, v_s \in F_0 P$ and $f_r, f_s \in F_{n-1}P$, respectively. Therefore $|c_r v_j| = |c_r v_k|$ for all $v_j, v_k \in F_0 f_r$.

$(i = 2)$. Let $f = f_1 \cap f_2$ be an 2-face with centroid c, where f_1 and f_2 are facets with centroids c_1 and c_2, respectively. Then for any $x \in f$, $\angle Oc_1 x = \angle Oc_2 x = \pi/2$. Then $|c_1 x| = |c_2 x|$. Therefore cc_1 and cc_2 are perpendicular to f. Let v_1, v_2 be any two vertices of f, then $\angle c_1 c v_1 = \angle c_1 c v_2 = \pi/2$ and $|c_1 v_2| = |c_1 v_2|$. Thus we have $|cv_1| = |cv_2|$.

$(i = 1)$. Let $f = e$ be an edge of P. Then there are two vertices v_1 and v_2 contained in e and the centroid ϵ of e is the midpoint of $v_1 v_2$. Trivially, $|\epsilon v_1| = |\epsilon v_2|$. Suppose e is common to the facets $f_1, \dots, f_r$, $r \geq 3$, with centroids $c_1, \dots, c_r$ respectively. Then e is the intersection of $s \geq 3$ 2-faces $\alpha_1, \dots, \alpha_s$ of P, where α_i is given by the intersection of two facets f_j and f_k, for some j, k. Let α_i have centroid γ_i. Then for any $x \in e$, $\angle O\gamma_i x = \angle O\gamma_j x = \pi/2$ for $i, j = 1, \dots, s$. Therefore e lies in the intersection of the perpendicular bisectors of the points γ_i, γ_j for all $i, j = 1, \dots, s$, $i \neq j$. Therefore e is perpendicular to $\epsilon\gamma_i$ for $i = 1, \dots, s$. $\square$

Let v be any vertex of some polyhedron P. Suppose the number of faces of P containing v is q. Then each face R containing v has two edges which meet at v at an angle ϕ called the *interior angle* of R at v. There are q such angles at v and it is well-known that their sum is less than 2π if v is a vertex of P.

Let e be any edge with midpoint ϵ contained in a 3-face f of some 4-polytope P'. Then the subspace of $\mathrm{aff}(f)$ orthogonal to $\mathrm{aff}(e)$ is a 2-space which cuts the boundary of f such that the intersection is locally the boundary of a polygon α with vertex ϵ. The interior angle of α at ϵ is called the *dihedral angle* ϕ of e in f. Then $\phi < \pi$ if e is an edge of f. Likewise the subspace of $\mathrm{aff}(P')$ orthogonal to $\mathrm{aff}(e)$ is a 3-space which cuts $\partial P'$ such that the intersection is locally the boundary of a polyhedron β. Thus the sum of the dihedral angles at any edge e of the 3-faces containing e is less than 2π. For example, suppose P' is the 4-cube (see figure 1). Then α is a square and β is a cube.

By polarity, each 2-face of a prime perfect $(0,3)$-transitive 4-polytope P is a polygon R such that the vertices of R all lie on a circle with centre $\mathrm{cent}(R)$. Suppose R is an m-gon, for some $m \geq 3$. Let $\phi_1, \dots, \phi_m$ be the interior angles of R such that ϕ_1 is the smallest interior angle. It follows that if $0 < \phi_1 < \frac{\pi}{2}$ then $\phi = \frac{\pi}{3}$ and R is an equilateral triangle, and if $\phi_1 = \frac{\pi}{2}$ then R is an equiangular quadrilateral. Let f be any facet of P and let $\mathrm{val}_f(v)$ denote the valency of a vertex v in f. Then it follows that $\mathrm{val}_f(v) = 3, 4$ or 5 (for full details see [4]).

Theorem 3.2 *A prime perfect $(0,3)$-transitive 4-polytope is regular.*

Proof. For any facet f of P, $G(f) = F_3 P$. Suppose c is the centroid of f. Since P is perfect fix_c is an 1-dimensional subspace of E^4. Thus c is the only fixed point in $\mathrm{aff}(f)$ under G_c.

We consider the action of G_c on $\mathrm{aff}(f)$ in terms of orbits $G_c.v_i$ of vertices v_i of f. We show that f is regular using the classification of vertex transitive polyhedra (see [8] and [10]).

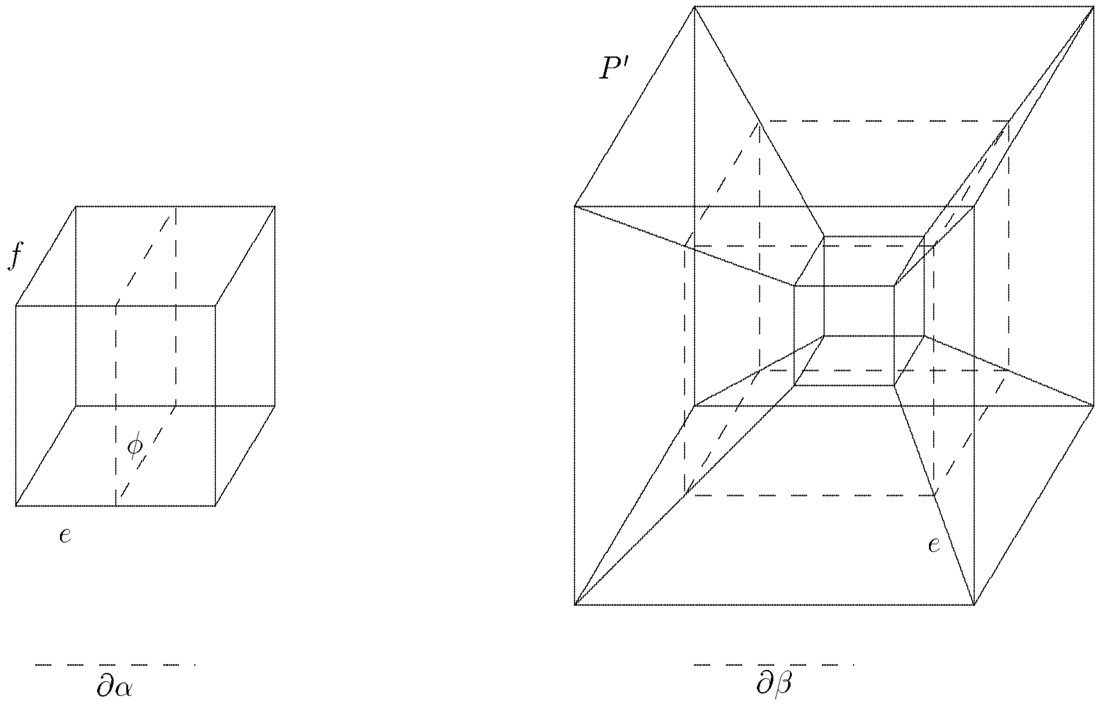

Figure 1

First suppose that $f_i = \mathrm{conv}(G_c.v_i)$ is of dimension 1 or 2 for some vertex v_i. In other words, v_i lies in some G_c-invariant subspace of $\mathrm{aff}(f)$. Since the triangular and quadrilateral faces of f are equiangular, the facet f is of the form $M \diamond I$ or $\mathrm{conv}\big((M \,\square\, I) \cup J\big)$ for some m-gon M and for some suitable intervals $I = [-t_i, t_i]$, $J = [-t_j, t_j]$, such that $t_i < t_j$ and $I \subset J$. (Recall that $M \diamond I$ is a double m-cone and $M \,\square\, I$ is a right m-prism.) If $f = M \diamond I$ for some m-gon M, then the 2-faces of f (and hence P) are equilateral triangles by Propositon 3.1. Since the vertices of f lie on a sphere with centre c and f has regular 2-faces, the facets of P are regular. Hence P is regular since P is 3-transitive. This is a contradiction if G_c is irreducible. Therefore $f = \mathrm{conv}\big((M \,\square\, I) \cup J\big)$ and $m = 3$, 4 or 5 by interior angle considerations. By Proposition 3.1, there exists a 2-sphere S such that $F_0 f \subset S$. Then $F_0 f' \subset S$ for some regular polyhedron $f' = \{3, m\}$ such that v_i is a vertex of f'. This is a contradiction if $m = 3$ or 4, since then $f = f'$. If $m = 5$, then the dihedral angles of f may be calculated. The dihedral angle sum of an edge common to two triangular faces exceeds 2π. Therefore $\dim(f_i) \neq 1, 2$.

Therefore $f_i = \mathrm{conv}(G_c.v_i)$ is a polyhedron, where $v_i \in F_0 f$. There are four possible cases:

1. G_c reducible on $\text{aff}(f)$ and $f = f_i$;
2. G_c reducible on $\text{aff}(f)$;
3. G_c irreducible on $\text{aff}(f)$ and $f = f_i$;

and 4. G_c irreducible on $\text{aff}(f)$.

In each case we consider dihedral angle sums and Euler's relation on the various polyhedra to show that the non-regular polyhedra give rise to contradictions. Let P be a 4-polytope such that $f(P) = (v, e, f, h)$ and Q be a polyhedron such that $f(Q) = (v', e', f')$. Then Euler's relation for P and Q, yields respectively,

$$v - e + f - h = 0 \qquad \text{and} \qquad v' - e' + f' = 2.$$

Case 1.

Suppose that G_c is reducible on $\text{aff}(f)$ and $f = f_i$. Then f is one of the five families of prisms, which are labelled in table 1 (see [8] for more details).

LABEL	DESCRIPTION
C_m	right prism on regular m-gon
D_m	anti-prism on regular m-gon
E_m	skew prism on regular m-gon
F_m	right prism on semi-regular $2m$-gon
G_m	antiprism on semi-regular $2m$-gon

Table 1: The five families of prisms

We first note that the dihedral angle ϕ_{ij} between faces f_i and f_j of C_m is given by

$$\phi_{ij} = \begin{cases} \frac{\pi}{2}, & \text{if } f_i \text{ or } f_j \text{ is not a quadrilateral;} \\ \frac{\pi}{m}(m-2), & \text{if } f_i \text{ and } f_j \text{ are both quadrilaterals.} \end{cases}$$

See, for example, [2]. The dihedral angles of F_m coincide with the dihedral angles of C_{2m}.

Consider $f = C_m$. Suppose $m = 3$, so f has 6 vertices, 9 edges and 5 faces. Any edge is common to only three 2-faces (and hence three facets) as we now show. Suppose e is an edge belonging to some triangular face T of f. In any facet there are no adjacent triangular faces. Hence e is common to at least two quadrilateral faces, two of which belong to facets containing T. If e is contained in another triangular face, then the dihedral angle sum at e is 2π, which is a contradiction. Suppose therefore e is contained in another quadrilateral face f'. Then there are two facets common to f' such that the edge of f' adjacent to e is contained in two triangular faces. This is a contradiction by the above. Hence each edge of P is contained in two quadrilateral faces and one triangular face. Now if $m \geq 4$ then $\phi_{ij} \geq \frac{\pi}{2}$. Therefore each edge of P is common to (at most) three facets and three 2-faces of P. Let e be an edge common to the 2-faces f_1, f_2 and f_3. Since f_i and f_j,

$i \neq j$, are adjacent in some facet, only one of the f_i is an m-gon, $i = 1$, 2 or 3. Therefore each edge is common to two quadrilateral faces and one m-gonal face. For any $m \geq 3$, let $f(P) = (v, e, f, h)$ be the face vector of P. Suppose that the valency of a vertex v_i of P is q and x is the number of facets common with v_i. Then $(m+2)h = 2f$, $mh = e$, $2mh = xv$ and $2e = qv$. Therefore by Euler's relation we have

$$\frac{2mh}{x} - mh + \frac{(m+2)h}{2} - h = 0.$$

Therefore $x = 4$ and likewise $q = 4$ for all $m \geq 3$. Hence P is of the form $\{m\} \square s(\{m\})$, for some similarity s. If s is not the identity then P is not facet transitive. However if s is the identity P is not prime. Both cases are contradictions.

In the case $f = F_m$, any dihedral angle is at least $\frac{\pi}{2}$. Therefore there are three 2-faces and three facets common with each edge of P. A similar argument to that for C_m shows that each edge is contained in one $2m$-gonal face and two quadrilateral faces. Using the above notation we have $(2m+2)h = 2f$, $2mh = e$, $4mh = xv$ and $2e = qv$. By Euler's relation, $x = q = 4$. For each facet f, some vertex v of f is contained in three edges, at least two of which are not equal if f is a semiregular prism. Then whatever the length of the fourth edge of P containing v , P is not facet transitive. This is a contradiction.

The 3- and 4-sided faces of E_m and G_m are not equiangular. Hence $f \neq E_m$ or G_m by proposition 3.1.

Suppose $f = D_m$. Then f has $2m$ vertices, $4m$ edges and $(2m+2)$ faces. If $m = 3$ then f is a regular octahedron and $P = \{3, 4, 3\}$, which is a contradiction if G_c is reducible. Suppose $m \geq 4$. Since the triangular faces of f are regular, f is an Archimedean anti-prism. Thus the dihedral angles of f are

$$\begin{aligned}\theta &= \operatorname{arcsec}\left(-\sqrt{3}\{\operatorname{cosec}\frac{\pi}{m} + \cot\frac{\pi}{m}\}\right)\\ \phi &= \arccos\left(\frac{1}{3}\{1 - 4\cos\frac{\pi}{m}\}\right),\end{aligned}$$

where the dihedral angle between two triangular faces is θ. Since $\theta, \phi \geq \frac{\pi}{2}$, each edge of P is common to 3 facets. Let $f(P) = (v, e, f, h)$. Then $3e = 4mh$, $f = (m+1)h$ and $vx = 2mh$, where x is the number of facets common to v. By Euler's relation $x = 6$. Likewise the valency of v is 8. Therefore a facet f^* of P^* has 6 vertices and 8 faces. Thus $f^* = \{3, 4\}$. It then follows that $P = \{3, 4, 3\}$. This is a contradiction.

Case 2.

Suppose $f = \operatorname{conv}(f_1 \cup \cdots \cup f_r)$ such that $v_i \notin G_c.v_j$ for $i \neq j$, $i, j = 1, \ldots, r$, where f_i is a 'case 1' polyhedron. Then the axis of rotation of each f_i coincides with every other such axis. Hence each f_i is a prism on a regular m-gon or a semiregular m-gon for some fixed m. It follows that no such f exists since the triangular and quadrilateral faces of f are equiangular and the dihedral angles of f are at least those found in each f_i.

Case 3.

Therefore G_c is irreducible on $\operatorname{aff}(f)$ and f is one of the twenty-four vertex transitive non-prism polyhedra, see [8]. These polyhedra have deficiency 0, 1 or 2. A 0-transitive polyhedron f with deficiency 0 or 1 has dihedral angles which are invariant under any deformation δ such that $f \simeq \delta(f)$. If f has deficiency 2 then f has δ-invariant dihedral angles if the 3- or 4-sided 2-faces of $\delta(f)$ are equiangular. The dihedral angles of such polyhedra can be calculated and are given in table 2, see also [2].

	Name	**Dihedral angles**	(approx.)
A	Tetrahedron	$2\arcsin\sqrt{\frac{1}{3}}$	$70°32'$
B	Octahedron	$2\arcsin\sqrt{\frac{2}{3}}$	$109°28'$
C	Cuboctahedron	$\pi - \arcsin\sqrt{\frac{2}{3}}$	$125°26'$
D	Cube	$\frac{\pi}{2}$	$90°$
E	Icosahedron	$2\arcsin(\frac{2}{\sqrt{3}}\sin\frac{3\pi}{10})$	$138°11'$
F	Icosidodecahedron		$142°37'$
G	Dodecahedron		$116°34'$
H	Truncated tetrahedron	$2\arcsin\sqrt{\frac{1}{3}},\ 2\arcsin\sqrt{\frac{2}{3}}$	$70°32',\ 109°28'$
I	—	$\pi - \arcsin\sqrt{\frac{2}{3}}$	$125°26'$
J	†		
K	†		
L	Truncated octahedron	$2\arcsin\sqrt{\frac{2}{3}},\ \pi - \arcsin\sqrt{\frac{2}{3}}$	$109°28',\ 125°26'$
M	Rhombicuboctahedron	$\frac{3\pi}{4},\ \frac{\pi}{2} + \arcsin\sqrt{\frac{2}{3}}$	$135°,\ 144°44'$
N	Truncated cube	$\frac{\pi}{2},\ \pi - \arcsin\sqrt{\frac{2}{3}}$	$90°,\ 125°26'$
O	Truncated icosahedron	$2\arcsin(\frac{2}{\sqrt{3}}\sin\frac{3\pi}{10}),\ 142°37'$	$138°11',\ 142°37'$
P	Rhombiicosidodecahedron		$148°17',\ 159°6'$
Q	Truncated dodecahedron		$116°34',\ 142°37'$

† These polyhedra have irregular triangular faces.

Table 2. The vertex transitive polyhedra

	Name	Dihedral angles	(approx.)
R	†		
S	Snub cube	**	$142°59'$, $153°14'$
T	Snub dodecahedron	**	$152°56'$, $164°11'$
U	‡		
V	Rhombitruncated cuboctahedron	as I and M	each $> \frac{\pi}{3}$
W	Rhombitruncated icosidodecahedron		each $> \frac{\pi}{3}$
X	—	$\frac{\pi}{2} + \arcsin\sqrt{\frac{1}{3}}$, $\pi - 2\arcsin\sqrt{\frac{1}{3}}$	$125°16'$, $109°28'$

** Dihedral angles given when polyhedra are Archimedean.
† These polyhedra have irregular triangular faces.
‡ These polyhedra have trapezium faces.
The polyhedra I and X have not been named.

Table 2 (continued)

If each dihedral angle θ of f is such that $\theta \geq \frac{2\pi}{3}$ then the polyhedra comprising of $G(f)$ cannot be 'folded' into a 4-polytope. Thus f is not a polyhedron of the form C, E, F, I, M, O, P, V or W. The polyhedra of the forms J, K and R have irregular triangular faces, since otherwise they would be icosahedra. Likewise $f = U$ has trapezium faces. Hence f is not of any of these forms by proposition 3.1. If f is of the form S or T then the triangular 2-faces of f are equilateral. Therefore f is Archimedean and every dihedral angle is greater than $\frac{2\pi}{3}$. This is impossible by the above. We also note that with the exception of A and H, $\theta \geq \frac{\pi}{2}$. Hence each edge of $f \neq A$, H is common to three 2-faces and three facets of P.

If $f = A$, B, D, or G then P has congruent regular facets. Hence P is regular by proposition 3.1. If $f = A$ then P is the 4-simplex, the 4-cocube or the 600-cell. If $f = B$, D or G then P is the 24-cell, the 4-cube or the 120-cell respectively.

We now check that f is not one of the remaining polyhedra. Let $f(P) = (v, e, f, h)$ denote the face vector of P and $q = \mathrm{val}_P(v)$ be the valency of some vertex v.

(i) $f = H$ The truncated tetrahedron : $f(H) = (12, 18, 8)$
The dihedral angles are $\theta = 2\arcsin\sqrt{\frac{1}{3}}$ and $\phi = 2\arcsin\sqrt{\frac{2}{3}}$. We note that $\theta + \phi = \pi$. Thus any edge is common to three faces and three facets of P. Then $18h = 3e$, $16h = 2f$ and $qv = 2e$. Then by Euler's relation we have

$$\begin{aligned} v - e + f - h &= \frac{36h}{3q} - 6h + 8h - h \\ &= \frac{12h}{q} + h = 0. \end{aligned}$$

Hence $q < 0$ which is a contradiction.

(ii) $f = L$ The truncated octahedron: $f(L) = (24, 36, 14)$
Each edge is common with three 2-faces and three facets. Thus $qv = 2e$, $3e = 36h$ and $2f = 14h$. Then by Euler's relation we have

$$\frac{2e}{q} - 12h + 7h - h = \frac{24h}{q} - 5h = 0.$$

Therefore $q = \frac{24}{5}$ which is impossible.

(iii) $f = N$ The truncated cube: $f(N) = (24, 36, 14)$
Each edge is common to three 2-faces and three facets. Thus $qv = 2e$, $3e = 36h$ and $2f = 14h$. Therefore by Euler's relation we have

$$\frac{24h}{q} - 12h + 7h - h = 0$$

$$\text{and so} \qquad q = 4.$$

Hence v is contained in four facets. Consider the $(0,3)$-transitive prime perfect 4-polytope P^*. A facet f^* of P^* only has four vertices so f^* is a tetrahedron. Therefore f^* is of the form A and hence P^* is regular. Thus $P = \{4,3,3\}$ or $\{3,3,3\}$, which is a contradiction.

(iv) $f = Q$ The truncated dodecahedron: $f(Q) = (60, 90, 32)$
The sum of any three dihedral angles of Q exceeds 2π which is impossible.

(v) $f = X$: $f(X) = (24, 36, 14)$
Each edge is common to three 2-faces and three facets. Thus $qv = 2e$, $3e = 36h$ and $2f = 14$. Thus we get a contradiction as in the case $f = N$.

Thus there are no $(0,3)$-transitive prime perfect 4-polytopes whose facets are vertex transitive with the exception of the regular 4-polytopes.

Case 4.

Suppose that f is not vertex transitive. If $\operatorname{conv}(f_i \cup f_j) \subseteq f$ then the stratifications of the symmetry groups (or subgroups) of f_i and f_j coincide. Therefore we check that f is not the convex hull of two polyhedra with symmetry groups G_i, G_j such that $G_i \subseteq G_j$. Recall from case 3 that $f_i \neq C$, E, F, I, J, K, M, O, P, R, S, T, U, V or W. Also note that $\operatorname{val}_f(v) = 3$ or 4 for any vertex v of f. Thus if $\operatorname{val}_{f_i}(v) = 4$ for some f_i then this would lead to a contradiction in $\operatorname{val}_f(v)$ if new edges were introduced at v upon taking convex hulls.

$f = \operatorname{conv}(A \cup A^*)$
Such a polyhedron has $\operatorname{val}_f(v) = 6$ for some vertex or f is a cube. In the latter case $P = \{4,3,3\}$ which is a contradiction since the facets of P are vertex transitive.

$f = \operatorname{conv}(A \cup B)$
This is possible only if for each vertex v of B, there exists a midpoint ϵ of an edge of A

such that v, ϵ and the centroid of f are collinear. In such a case the resulting polyhedron has a vertex (coinciding with a vertex of A) of valency 6.

$f = \mathrm{conv}(A \cup X)$
Such a polyhedron has irregular triangular faces or trapezium faces which is a contradiction.

$f = \mathrm{conv}(B \cup D)$
Such a polyhedron is either a cuboctahedron or has equilateral triangular faces only, therefore the sphere containing $F_0 f$ also contains the vertices of an octahedron B or cube D. This is a contradiction.

$f = \mathrm{conv}(B \cup L)$
Then B coincides with the octahedron which is truncated to form L. Thus $f = B$, which by case 3 is a contradiction.

$f = \mathrm{conv}(B \cup N)$
Such a polyhedron has a vertex of valency 8, which is a contradiction.

$f = \mathrm{conv}(D \cup L)$
Such a polyhedron has a vertex of valency 6, which is a contradiction.

$f = \mathrm{conv}(D \cup N)$
Since the G_c-stratifications of $\mathrm{aff}(f)$ coincide, this union results in a cube D or a truncated cube N.

$f = \mathrm{conv}(L \cup N)$
The dihedral angle sum inequalities of such a polyhedron give rise to a contradiction.

The case where $f_i = G$ is impossible since the polyhedra with the same symmetry group as G have already been excluded. Thus a $(0,3)$-transitive prime perfect 4-polytope does not have a facet of the form $f = \mathrm{conv}(f_i \cup f_j)$. Since such a facet has a vertex of valency four we conclude that f is vertex transitive since otherwise $\mathrm{val}_f(v) \geq 5$ for some vertex v of f. Thus we have shown that no non-regular $(0,3)$-transitive prime perfect 4-polytope exists.

Let R be a regular 4-polytope. Then it is quite easy to see that if P is a perfect 4-polytope such that $GP = GR$, then P or P^* is Wythoffian since the vertices and centroids of facets of P lie on one-dimensional fixed points sets (the more general statement in n-dimensions is proved in [4]).

4. Orbit vectors

Let P be any n-polytope. For any vertex v of P, let $E_v = \{e \in F_1 P \colon v \lhd e\}$ be the set of all edges of P emanating from v. Let $V_v = \{v_i \in F_0 P \colon v_i \lhd e \text{ for some } e \in E_v\}$ be the set of all vertices of P adjacent to v. Now we suppose that P is perfect and let $P_i = \mathrm{conv}(G.v_i)$ for some $v_i \in F_0 P$. Then each P_i is a perfect 0-transitive polytope.

Proposition 4.1 *Let P be a prime perfect 3-transitive 4-polytope. Let f be any facet of P and v_i be any vertex of P. If P is not 0-transitive, then either $\theta(P) = (2,1,x,1)$, for some $x \in \mathbf{N}$, or $\dim\big(\mathrm{aff}(G.v_i \cap f)\big) < 3$.*

Proof. Let P be as stated, and suppose that P is not 0-transitive. Since P is 3-transitive we may suppose that $v_i \triangleleft f$. Suppose $\mathrm{aff}(G.v_i \cap f)$ is an 3-dimensional space. Then $f' = \mathrm{conv}(G.v_i \cap f)$ is a polyhedron contained within f and $f \neq f'$. Thus the closure of the complement of f' in f is a collection of polyhedra $\alpha_1, \ldots, \alpha_r$ such that $f = f' \cup \alpha_1 \cup \cdots \cup \alpha_r$, for some $r \geq 1$. Let $v \in E_{v_i}$, $v \notin G.v_i$. Thus v and v_i are endpoints of some edge e of f. See figure 2.

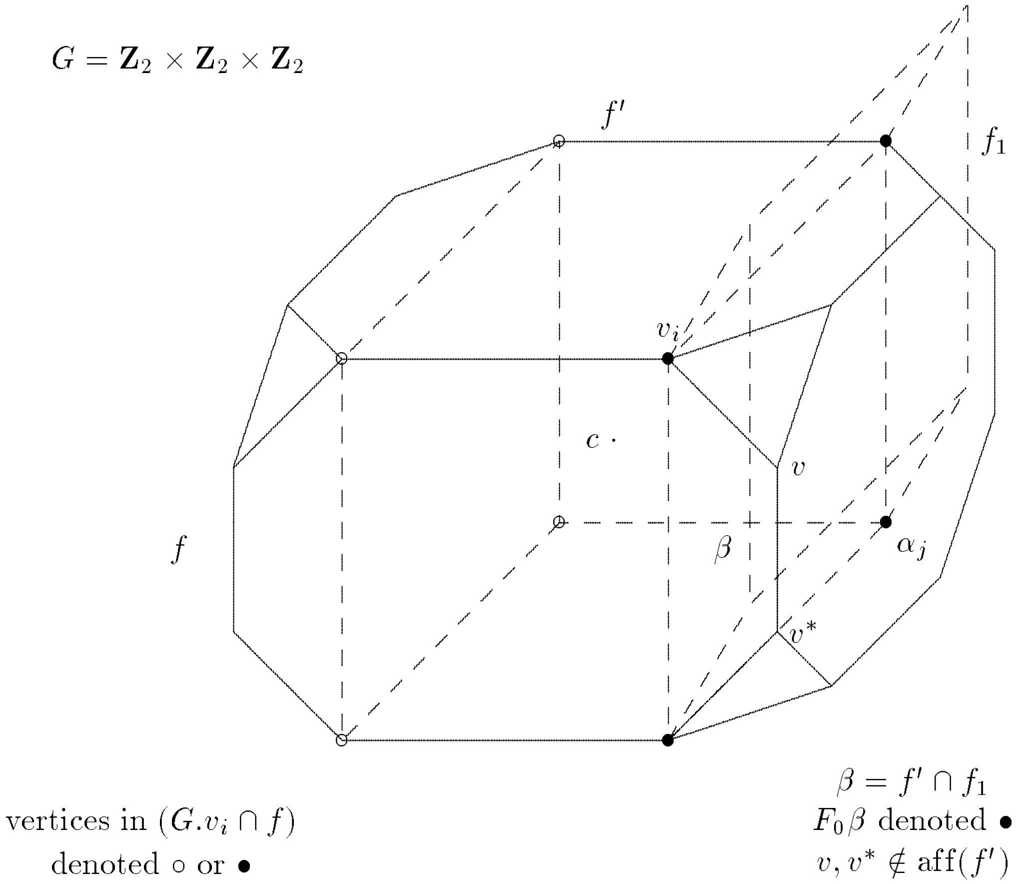

Figure 2

Then if $(f \cap V_v) \not\subseteq G.v_i$, there is a vertex v^* adjacent to v, that is $v^* \in V_v$, such that $v^* \notin G.v_i$. In this case, both v and v^* belong to one of the polyhedra, α_j say. Now α_j and f' share a 2-face β. However β is an 2-face of P_i and hence belongs to a unique facet f_1 of P_i other than f'. Therefore each ray $0v$ and $0v^*$ intersects the interior of f_1 in a distinct point, but these points have 1-dimensional fixed point sets, which is a contradiction. Therefore $f \cap V_v \subset G.v_i$ and hence $V_v \subset G.v_i$ for all $v \notin G.v_i$. Similarly we have $V_{v_i} \subset G.v$ for $v_i \notin G.v$. Thus f contains vertices from two orbits and each edge of P has one endpoint in $G.v_i$. Let $v_i, v \lhd e$ for some edge e. Then $G_v.e$ generates E^n since $\dim(\mathrm{fix}_v) = 1$, and likewise for $G_{v_i}.e$. Thus $\mathrm{conv}(G.e)$ is an n-polytope Q such that $F_0P = F_0Q$. Hence $P = Q$ and $G.e = F_1P$. Since P is 3-transitive we conclude that the orbit vector of P is given by $\theta(P) = (2, 1, x, 1)$. □

Proposition 4.2 *Let P be a prime perfect 3-transitive 4-polytope. Then either* $\mathrm{conv}(G.v)$ *is a prime perfect 4-polytope P^i such that $\theta(P^i) = (1, x, 1, 2)$ or P^* is Wythoffian, where v is a vertex of P or of P^*.*

Proof. Let $P = P^1$ have m_1 distinct vertex orbits given by $G.v_1, \ldots, G.v_{m_1}$. Let $P_i^1 = \mathrm{conv}(G.v_i)$. If the supporting hyperplanes of a facet f of P^1 and of a facet of P_i^1 coincide, in other words, $f' \subset f$ for some facet f' of P_i^1, then $\dim\big(\mathrm{aff}(G.v_i \cap f)\big) = 3$. Therefore either $\theta(P^1) = (2, 1, x, 1)$ or the facets of P_i^1 are only given by $\mathrm{conv}(G.v_i \cap V_v)$ for some vertex $v \notin G.v_i$ of P^1 by Proposition 4.1. Then in the latter case $P^2 = (P_i^1)^*$ is a prime perfect 3-transitive 4-polytope with m_2 distinct vertex orbits, where $m_2 < m_1$ since there are at most $(m_1 - 1)$ vertex orbits to which v can belong. After finitely many repetitions of this procedure, and writing $P^{k+1} = (P_i^k)^*$, we obtain P^j such that $\theta(P^j) = (2, 1, x, 1)$ or $m_j = 1$. If $m_j = 1$ then P^j is a prime perfect $(0, 3)$-transitive 4-polytope and hence regular by Theorem 3.2. It then follows that P or P^* is Wythoffian. □

Let P be a 4-polytope such that $\theta(P) = (x, y, 1, 2)$, for some $x, y \in \mathbf{N}$. Suppose that c is the centroid of any facet f of P. Then it follows that: each 2-face is common to two facets, one from each facet orbit; G_c acts transitively on the 2-faces of f; and $\dim(fix_c) = 1$.

Proposition 4.3 *Let P be any perfect 4-polytope. Then $\theta(P) \neq (1, x, 1, 2)$, for any x.*

Proof. Suppose P is a perfect 4-polytope such that $\theta(P) = (1, x, 1, 2)$, for some $x \in \mathbf{N}$. Let f_1, f_2 be any two adjacent facets of P. Let c_j be the centroid of f_j, $j = 1, 2$ and c_0 any vertex of P. Then $\dim(\mathrm{fix}_{c_i}) = 1$ and G_{c_i} is not reducible on $\mathrm{aff}(f_i)$ by the proof of Proposition 3.1, for $i = 0, 1, 2$, where f_0 is a facet of P^* with centroid on fix_{c_0}. By the above, $f_1 \notin G.f_2$.

If $\mathrm{aff}(f_1)$ and $\mathrm{aff}(f_2)$ are perpendicular, then all facets adjacent to one another are perpendicular. In this case, any vertex v is given by the intersection of n mutually perpendicular hyperplanes. Now v is contained in some facet in $G.f_1$ and some facet in $G.f_2$. Since $\dim(\mathrm{fix}_v) = 1$, it follows that G_v is reducible on the hyperplane orthogonal to fix_v. This is a contradiction by the above. Therefore $\mathrm{aff}(f_1)$ and $\mathrm{aff}(Oc_2)$ intersect in some point x. Now G_{c_j} is transitive on the facets adjacent to f_j for $j = 1, 2$. Since $x \in \mathrm{fix}_{c_2}$, $\dim(\mathrm{fix}_x) = 1$ and G_x is transitive on the facets adjacent to f_2, and hence on their affine hulls. Let $F_0Q = \{y \in E^n\colon y = g.x \text{ for some } g \in G\}$ and let $Q = \mathrm{conv}(F_0Q)$. Then $y \in F_0Q$ is such that $y \in \mathrm{fix}_{g(c_2)}$ and G_y is transitive on the affine hulls of all facets adjacent to $g(f_2)$, for some $g \in G$. If $h = \mathrm{aff}(f_1)$, then $G_y(h) = \{H \subset E^n\colon H = g(h) \text{ for some } g \in G_y, y \in F_0Q\}$ is the set of all hyperplanes supporting facets in $G.f_1$. Now G_{c_1} is transitive on the set F_{f_1} of facets adjacent to f_1, and hence on the fixed point sets of centroids of facets in F_{f_i}. Therefore the affine hull of $(G_{c_1}.x)$ is the hyperplane h and the convex hull is a facet f'

of Q. Clearly G_{c_1} is transitive on the points of $G_{c_1}.x$. Therefore the centroids of f_1 and f' coincide and $h = \mathrm{aff}(f_1) = \mathrm{aff}(f')$. Thus Q is a $(0,3)$-transitive 4-polytope. Since the fixed point set of the centroid of any facet or any vertex is 1-dimensional, it follows that Q is perfect. By Theorem 3.2, Q is regular. Obviously, if we considered the supporting hyperplanes of facets in $G.f_2$, we should get the polar of Q as our polytope. Let G be the symmetry group of P and $G(Q)$ be the symmetry group of Q. Then $G(Q) \subseteq G$.

Now P^* has 2 vertex orbits $G.v_1, G.v_2$. Let $(P^*)_i$ be the convex hull of the i^{th} vertex orbit of P^*, $i = 1, 2$. Consider the perfect 4-polytopes $(P^*)_1$ and $(P^*)_2$. Then $\theta\big((P^*)_i\big) = (1, x_i, 1, 2)$ or $\theta\big((P^*)_i\big) = (1, x_i, y_i, 1)$, for some $x_i, y_i \in \mathbf{N}$. In the latter case $Q_i = (P^*)_i$ is regular. In the former case, $(P^*)_i$ is $(0,2)$-transitive and has two facet orbits. $(P^*)_1$ and $(P^*)_2$ have vertex-set given by the orbits of c_1 and c_2, respectively, and have facet-set given by the union of orbits of c_0 with c_2 and c_0 with c_1, respectively. Let $Q = Q_0$. For $(P^*)_1$ and $(P^*)_2$, we can define regular polytopes Q_1 and Q_2, respectively, in a fashion similar to the above, such that Q_i has vertex orbit $G.c_j$, where $i \neq j$ and $i, j = 0, 1, 2$. Now $\{f_0 Q_i, f_{n-1} Q_i\}$ is independent of $i = 0, 1, 2$. Hence for each $i, j = 0, 1, 2$, Q_i is similar to Q_j or to Q_j^*, since any two regular 4-polytopes R and R' are similar if and only if $f_0 R = f_0 R'$. If Q_0 and Q_1, say, have the same vertex set $G.c_2$, then the fixed point sets of c_0 and c_1 coincide. This is a contradiction to the face-structure of P if P is perfect. □

Classification Theorem 4.4 *Let P be a prime perfect 4-polytope, then P or P^* is Wythoffian.*

Proof. Suppose P is a prime perfect 4-polytope. Let $P_i = \mathrm{conv}(G.v_i)$. Then P_i is a 0-transitive perfect 4-polytope. Therefore P_i is Wythoffian or $\theta(P) = (1, x, 1, 2)$, in which case P is not perfect by Proposition 4.3. Therefore $GP_i = GR$ for some regular 4-polytope R. Hence P or P^* is Wythoffian. □

Now the non-prime perfect 4-polytopes are of the form $\square^4 I = \square_4$, $\diamond^4 I = \diamond_4$, $\square^2 Q$ and $\diamond^2 Q$ for some q-gon Q, $q \geq 3$, where I is a line segment[1]. Therefore Theorem 4.4 confirms Rostami's conjecture.

5. Perfect 4-solids

With the classification of perfect 4-polytopes complete, we turn our attention to the perfect 4-solids. Let G be the symmetry group of a perfect non-polytope 4-solid. Then $G \subseteq \mathrm{O}(4)$ and $\dim(G) \geq 1$. There are three known perfect 4-solids which are not polytopes, the 4-ball D^4, the product of the disk with itself $D^2 \square D^2$, and its polar $D^2 \diamond D^2$. By considering the compact subgroups of $\mathrm{O}(4)$, we show that $G = \mathrm{O}(2) \wr Z_2$ or $G = \mathrm{O}(4)$. In other words, G is the symmetry group of $D^2 \square D^2$ or D^4.

First suppose that G is not the direct product of compact subgroups of $\mathrm{O}(4)$ or a wreath product in $\mathrm{O}(4)$. If $\dim(G) < 4$ then there exists a linear subspace C, $\dim(C) = 4 - \dim(G)$, held pointwise fixed by G. Thus $\mathrm{fix}_B = C$ which contradicts [6]. If $\dim(G) \geq 4$,

[1] Recall $Q^* \sim Q$ for all q-gons Q, $q \geq 3$.

then $G = \mathrm{O}(4)$ or $G = \mathrm{SO}(4)$ (which has the same orbits as $\mathrm{O}(4)$). The only 4-solid with such symmetry is D^4.

Now suppose G is the direct product $J \times K$ of the compact subgroups J, K of $\mathrm{O}(4)$. Then the inclusion π: $G \to \mathrm{O}(4)$ is a reducible representation. This is a contradiction to [7]. In any case, there is a decomposition of B, for instance, $B = B_1 \square B_2$, where $B_1 \neq B_2$, such that $G = GB_1 \times GB_2$ which is a contradiction.

Suppose now that $G = GK \wr S_r$, $r > 1$, where K is a $\square$-prime i-solid, $i < 4$. Then $B = \square^r K$ and $ri = 4$. Thus $(i, r) = (1, 4)$ or $(2, 2)$. If $i = 1$, then B is a polytope and $\dim(G) = 0$, which is a contradiction. Hence $G = GK \wr S_2 = GK \wr Z_2$ and K is a perfect 2-solid. In other words $G = \mathrm{O}(2) \wr Z_2$ and $K = D^2$.

We now check that there are no perfect solids whose symmetry group is $\mathrm{O}(4)$ or $\mathrm{O}(2) \wr Z_2$ other than those mentioned above. The fundamental region of the action of $\mathrm{O}(4)$ on S^3 is a radial line. Hence D^4 is the only solid with symmetry group $\mathrm{O}(4)$. Now consider the action of $G = \mathrm{O}(2) \wr Z_2$ on S^3. The fundamental region of this action may be given as follows. Let E^4 be identified with the orthogonal product $E^2 \times E^2$. Then $C_1 = E^2 \times 0$ and $C_2 = 0 \times E^2$ are two non-intersecting linked great circles of S^3. Let v_i be any point on C_i, $i = 1, 2$. Then there exists a great circle C of S^3 through v_i, $i = 1, 2$, which intersects C_1 and C_2 each in one further point. Then C is divided into four equal arcs. If one of these arcs is subdivided equally into two, then the result is an arc A subtended by an angle of $\frac{\pi}{4}$ at O. Suppose that the endpoints of A are x and y, such that $x \in C_i$, $i = 1$ or 2. Then the convex hull of A and O is a fundamental region D of the action of G such that $\mathrm{conv}(G.x) = D^2 \diamond D^2$ and $\mathrm{conv}(G.y) = D^2 \square D^2$. We may assume that a point v on A is given by $e^{i\theta}$, $0 \leq \theta \leq \frac{\pi}{4}$, where $v = x$ if $\theta = 0$ and $v = y$ if $\theta = \frac{\pi}{4}$. Then there is a one-parameter family of face equivalent 4-solids given by the convex hull of the action of G on a point v on A. These 4-solids are face equivalent since $\dim(\mathrm{fix}_v) = 4$ and v may be mapped along a D-path by a deformation in a similar fashion to that in section 1. The solids in this family are not perfect as v may be deformed by changing θ. Similarly a solid given by the convex hull of the action of G on a number of points of D is not perfect. Thus the only perfect 4-solids are those already mentioned.

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