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Classical Solids

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Abstract. The range of convex solids studied in ancient Greece includes many interesting geometrical objects besides convex polyhedra, the most familiar examples being the spherical ball, the right circular cylinder and cone, and the double cone.

These objects differ from convex solids in general in the simplicity of the topological and metrical structures of their face-lattices. In this paper, we formulate the notions of *classical solid* and *strictly classical solid* for convex bodies in Euclidean n-space $\mathbf{E}^n$, these being designed to capture the distinctive topological and metrical features of the above solids, respectively. The essential idea is that each component of the space of i-faces should be a manifold that is also a component of an orbit of the symmetry group. For example, in the case of a right regular cone, the symmetry group is $O(2)$, and its orbits on vertices consist of the apex and rim of the circular base, on edges the circle of generators, and on 2-faces the single circular base.

We denote the set of all classical solids by $\mathcal{C}$ and the set of all strictly classical solids by $\mathcal{S}$. We show that $\mathcal{C}$ and $\mathcal{S}$ are both closed under the product $\square$ and the coproduct $\diamond$. Also $\mathcal{C}$ is invariant under duality and $\mathcal{S}$ under polarity.

We show how both $\mathcal{C}$ and $\mathcal{S}$ can be completely determined, using an approach based on Dadok [2], in the spirit of Madden and Robertson [8]. In particular, to illustrate these results, we give a detailed description of all symmetry types of strictly classical n-solids and hence of all face types of classical n-solids for $n \leq 4$. A good example of a classical 4-solid is the convex hull of the Clifford torus.

We observe also that to each face-type in $\mathcal{C}$ there is associated a projective algebraic curve $\phi(B) = 0$ whose defining polynomial describes certain features of

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the face-lattice of B. The polynomial $\phi(B)$ is irreducible if B is $\square$ -prime, with a similar relationship for $\diamond$-primeness.

1. Face-Types

We use the term *n-solid* to mean a compact convex subset B of Euclidean m-space $\mathbf{E}^m$ whose affine hull aff$B$ has dimension n, $(-1 \leq n \leq m)$. The number n is the *dimension* $\dim B$ of B. Thus the empty set $\emptyset$ is the only (-1)-solid.

A *supporting hyperplane* of an m-solid B in $\mathbf{E}^m$ is an affine $(m-1)$-plane H in $\mathbf{E}^m$ such that $H \cap B \neq \emptyset$ and B lies entirely in one of the two closed half-spaces bounded by H. For each such H, the set $H \cap B$ is itself a j-solid, for some $0 \leq j \leq m-1$, and is called a *j-face* of B. Any m-solid B has a unique (-1)-face, namely $\emptyset$ and a unique m-face, namely B itself. Faces other than B and $\emptyset$ are called *proper* faces of B. We write $A < B$ to mean that A is a face of B. Notice that the relation $<$ on solids in general need not to be transitive.

The set FB of all faces of B is a bounded lattice under inclusion, with unit B and zero $\emptyset$, graded by dimension. We call FB the *face lattice* of B. Thus for each $P, Q \in FB$, the meet is $P \cap Q$ and their join is the intersection of all faces of B that contain $P \cup Q$.

In case $B = \emptyset$, FB consists of just one element, namely $\emptyset$ itself. In this case B has no centroid, and FB has a unique topology.

In case $\dim B = 0$, so that $B = \{x\}$ is a singleton, for some $x \in \mathbf{E}^n$, $FB = \{\emptyset, B\}$ and we give FB the discrete topology.

Suppose now that B is an m-solid, for some $m \geq 1$. We may assume, for simplicity, that $B \subset \mathbf{E}^m$, for each i-face A of B with $i \geq 0$, let c_A denote the centroid of A. Now compactify $\mathbf{E}^m$ by adjoining a point ∞ to form a homeomorph of S^m, and define an injective map $\gamma : FB \to S^m$ by putting $\gamma(A) = c_A$ if $\dim A \geq 0$ and $\gamma(\emptyset) = \infty$. We topologise FB by requiring that γ is a topological embedding.

Two solids B and C are said to be *face-equivalent* (or combinatorially equivalent) if there is a lattice isomorphism $\lambda : FB \to FC$ which is also a homeomorphism with respect to the topologies defined above, and preserves dimensions of faces. The map λ is called a *face-equivalence*. The set of all solids face-equivalent to B is called the *face-type* of B, denoted by $[B]$.

2. Symmetry Types

We define the symmetry group $G\emptyset$ of $\emptyset$ to be the identity.

For an n-solid B in $\mathbf{E}^m$ with $n \geq 0$, a self-isometry f of $\mathbf{E}^m$ is a *symmetry* of B iff $f(B) = B$ and $f(X) = X$ for each affine n-plane X parallel to aff B. Thus the set of all symmetries of B is a compact subgroup GB of the Euclidean $E(m)$. In particular, if $\dim B = 0$, then GB is the identity. The action of GB on $\mathbf{E}^m$ induces an action α_B of GB on FB.

For any solids B and C, a *symmetry equivalence* from B to C is a face-equivalence $\lambda : FB \to FC$ for which there is a self-isometry η of $\mathbf{E}^m$ such that

$$\begin{tikzcd} GB \times FB \arrow[r, "\alpha_B"] \arrow[d, "\eta_* \times \lambda"'] & FB \arrow[d, "\lambda"] \\ GC \times FC \arrow[r, "\alpha_C"] & FC \end{tikzcd} \quad ,$$

$\lambda \circ \alpha_B = \alpha_C \circ (\eta_* \times \lambda)$, where η_* denotes the conjugacy of $E(n)$ given by $\eta_*(\phi) = \eta^{-1} \circ \phi \circ \eta$. The set of all solids symmetry equivalent to B is called the *symmetry type* of B, denoted by $\langle B \rangle$. Trivially, $[B]$ is a union of symmetry types, one of which is $\langle B \rangle$.

3. Classical and Strictly Classical Solids

Let B be an n-solid in $\mathbf{E}^m$, $n \geq 0$. The path-components of FB define a partition of FB into mutually exclusive subsets. Another partition of FB, which in general is finer than the first, is given by the path-components of the orbits of GB under the action α_B.

We say that B is *strictly classical* iff these two partitions coincide and are finite. Since GB is a compact subgroup of the Euclidean group $E(m)$, the lattice FB of a strictly classical solid is the disjoint union of finitely many compact connected smooth manifolds without boundary, each embedded in a sphere in $\mathbf{E}^m$ with centre c_B with the trivial exception of the singleton orbit $\{c_B\}$ itself.

A solid B is said to be *classical* iff it is face-equivalent to a strictly classical solid. We say that $\emptyset$ is strictly classical also.

Let $\mathcal{C}$ and $\mathcal{S}$ denote the sets of all classical and strictly classical solids, respectively, and let $\mathcal{P}$ denote the set of all (convex) polytopes, that is to say the set of all solids B for which FB is a finite set. Then $\mathcal{P} \subset \mathcal{S} \subset \mathcal{C}$.

Among 3-solids in $\mathcal{S} \backslash \mathcal{P}$, we mention the spherical ball, the right circular cylinder and the right circular cone. The lattice of a right circular cylinder B, for example, consists of four singletons and three circles, each of which is a component of α_B orbit, as indicated in figure 1.

$\longleftarrow$ one 3-face namely B

$\longleftarrow$ two 2-faces

$\longleftarrow$ one circle of 1-faces

$\longleftarrow$ two circles of 0-faces

$\longleftarrow$ one (-1)-face $\emptyset$

B $\qquad\qquad$ FB

Figure 1. A right circular cylinder and its face-lattice

Before we explore the relation of $\mathcal{C}$ and $\mathcal{S}$ to the operations $\square$ and $\diamond$, and to duality and polarity, or discuss the classification of $\mathcal{C}$ and $\mathcal{S}$, we prove some elementary propositions, and state some problems.
Although the relation $<$ is not transitive for solids in general, it is natural to expect that for a classical solid, a face of a face is a face, as we now show.

Proposition 3.1 *Let B a classical solid, and $A < B$. Then $C < A$ implies $C < B$.*

Proof: It is well known ([5], [9]) that every $x \in C$ is a j-extreme point of B and hence belongs to the closure of the set of j-exposed points of B. That is $x \in Cl(F_j B)$. But for a classical solid B, $F_j B$ is compact and hence $Cl(F_j B) = F_j B$. Thus C is a j-face of B. $\square$

However, when it comes to the "shape" of the faces of a classical (strictly classical) solid, the following questions are still unsettled.

Problem 3.2: Let B an m-solid. Is it true that B is classical if and only if each of its i-faces, $i < m$, is classical?

Problem 3.3: If B is strictly classical, is it true that every proper face of B is strictly classical?

Of course the converse of the above question has a negative answer. For example, an "oblique" circular cylinder has each of its proper faces strictly classical and yet the solid itself is not strictly classical.
We should expect that highly symmetric solids will be strictly classical. Obvious candidates include the completely transitive solids and the regular solids [3]. An m-solid B is said to be *completely transitive* if GB acts transitively on each $F_i B$, $0 \leq i \leq m-1$. A solid B is said to be *regular* if

i) each flag of each face of B is in a maximal flag of B;

ii) GB acts transitively on the set of maximal flags of B.

Clearly, every regular solid is completely transitive, but the converse is not true.

Proposition 3.4 *Every completely transitive solid is strictly classical.*

Proof: Let B be completely transitive. If $A_i \in F_i B$, then the orbit of A_i under the action α_B has a finite number of components, each of which is a compact manifold without boundary. (The orbit is in fact a homogeneous space diffeomorphic to GB/G_i, $G_i = G_{A_i}$). The result follows from the fact that $F_i B$ consists of only one orbit. $\square$

Corollary: *Every regular solid is strictly classical.*

In 4. we shall use operations on classical (strictly classical) solids that yield new classical (strictly classical) solids. The proposition below and its corollary give yet another method of obtaining new classical solids from old ones. Recall that if A is an $(m-1)$-solid in $\mathbf{E}^n$, and $v \in \mathbf{E}^n \setminus \operatorname{aff} A$, then the *cone* $C(v,A)$ with *apex* v and *base* A is the convex hull of $A \cup \{v\}$.

Proposition 3.5 *Let A be a strictly classical $(m-1)$-solid in $\mathbf{E}^{m-1}$ having its centroid at the origin. If $v = \lambda e_m$, then the cone B with apex v and base A is strictly classical.*

Proof: The vertex set of FB is F_0A together with the apex v. For $i \geq 1$, each path-component F_iB is either a path-component of A or is diffeomorphic to a path-component of $F_{i-1}A$. □

Corollary: *Let A be a classical $(m-1)$-solid in $\mathbf{E}^{m-1}$ and $v \in \mathbf{E}^m \setminus \operatorname{aff} A$. Then the cone $C(v,A)$ is a classical m-solid.*

The following fact will be used in 9. and 10. below. An *ovaloid* [4] is a solid whose only proper faces are 0-faces.

Proposition 3.6 *Let B be a classical m-solid. If F_0B has a component C of dimension $m-1$, then B is an ovaloid.*

Proof. The boundary ∂B of B is homeomorphic to the $(m-1)$-sphere $\mathbf{S}^{m-1}$. Since each component of F_0B is a submanifold of ∂B and the only $(m-1)$-dimensional submanifold (without boundary) of $\mathbf{S}^{m-1}$ is $\mathbf{S}^{m-1}$ itself, it follows that $F_0B = \partial B$. Thus B is an ovaloid. □

4. Products and Coproducts

The product operation $\square$ and the coproduct operation $\diamond$ on solids are defined in detail in [10]. If A is an r-solid in $\mathbf{E}^m$ and B an s-solid in $\mathbf{E}^n$, then we may take $A \square B$ to be the image of $A \times B$ in $\mathbf{E}^m \times \mathbf{E}^n$, where the latter space is identified with $\mathbf{E}^{m+n}$ with $(x,y) \in \mathbf{E}^m \times \mathbf{E}^n$ mapped to $z \in \mathbf{E}^{m+n}$, where $z_i = x_i \ \ (1 \leq i \leq r)$ and $z_{r+j} = y_j \ \ (1 \leq j \leq s)$. It is easy to show that for all $k \geq 0$, $F_k(A \square B)$ consists of the solids $U \square V$, where $U \in F_pA$, $V \in F_qB$ and $p+q=k$, $p \geq 0$, $q \geq 0$.

Proposition 4.1 *$\mathcal{S}$ and $\mathcal{C}$ are closed under $\square$.*

Proof: Observe that each path-component of $F(A \square B)$ is diffeomorphic to a cartesian product of path-components of FA and FB. Also $G(A \square B)$ contains $GA \times GB$ with the obvious 'product' action on $F(A \square B)$. □

The right circular cylinder in $\mathbf{E}^3$ is isometric to $A \square B$, where A is a 1-solid and B is a circular disc. For a slightly more complex example, consider the convex hull of a Clifford torus in $\mathbf{E}^4$, of the form $A \square B$, where both A and B are circular discs.

The coproduct operation $\diamond$ may be described as follows. Let A and B be solids as above, with $\mathbf{E}^m \times \mathbf{E}^n = \mathbf{E}^{m+n}$. If $A = \emptyset$, take $A \diamond B = B$. Likewise, $A \diamond \emptyset = A$. Now suppose

$r, s \geq 0$ and let A, B have centroids a, b respectively. Then $A \diamond B$ is the convex hull of $A \times \{b\} \cup \{a\} \times B$. In this case, for any $k < r + s$, $F_k(A \diamond B)$ consists of the convex hulls of $(U \times 0) \cup (0 \times V)$, where $U \in F_i A$, $V \in F_j B$ and $i + j = k - 1$.

Proposition 4.2 *$\mathcal{S}$ and $\mathcal{C}$ are closed under $\diamond$.*

Proof: This may be proved directly in the spirit of Prop. 4.1, mutatis mutandis, or may be deduced immediately from Prop. 4.1 and Prop. 5.2 below. $\square$

The double cone formed by two 'opposite' right circular cones with common base and equal height is congruent to $A \diamond B$, where A is a 1-solid and B is a circular disc. Again, to obtain a 4-dimensional example, take the convex hull of $(A \times 0) \cup (0 \times B)$, where both A and B are circular discs.

5. Duality and Polarity

Let A and B be any two m-solids. A *duality* from A to B is a homeomorphism $\alpha : FA \to FB$ that is also a lattice anti-isomorphism. Thus for all $S, T \in FA$, $S < T$ if and only if $\alpha(T) < \alpha(S)$. It should be noted that α may map faces of the same dimension in A to faces of different dimension in B (see 6. below).
For any such α, B is said to be *dual* to A, written $A \wr B$. The relation $\wr$ is symmetric.
It is well known [5], [10] that every m-solid B has a dual B^* called the *polar* of B, defined by $B^* = \{y \in \mathbf{E}^m : \forall x \in B, \langle x, y \rangle \leq 1\}$, where 0 is the centroid of B. We put $\emptyset^* = \emptyset$. The anti-isomorphism $\alpha : FB \to FB^*$ is defined as follows. For every face S of B we take $\alpha(S) = \widehat{S} = \{y \in B^* : \langle x, y \rangle = 1,\ x \in S\}$. We may define a map $\delta : \mathcal{G} \to \mathcal{G}$ by $\delta(B) = B^*$, where $\mathcal{G}$ denotes the set of all solids.

Proposition 5.1 *The set of strictly classical solids is invariant under δ.*

Proof: For any m-solid B, $GB = GB^*$, and if σ, σ^* denote the actions of GB on FB and FB^* respectively, then $\sigma^*_\circ(1 \times \alpha) = \alpha \circ \sigma$. The proposition now follows at once. $\square$

Corollary: *Let $A, B \in 0$ be such that $A \wr B$. Then A is classical if and only if B is classical.*

The operations $\square$ and $\diamond$ are linked by polarity as follows.

Proposition 5.2 *Let A and B be solids with common centroid. Then $(A \square B)^* = A^* \diamond B^*$.*

Proof: See [3]. $\square$

This fact, together with proposition 4.1, yields a simple proof of proposition 4.2.

6. Polynomials

Let B be a classical m-solid. We associate to B a polynomial $\phi(B) = \sum_{i,j=0}^{\infty} \mu_{ij} x^i y^j$, where μ_{ij} is the number of i-dimensional components of $F_j B$. For example, the four classical 3-solids shown in Figure 2 have the respective polynomials $x^2 + y^3$, $1 + y^2 + x + xy + y^3$, $2x + xy + 2y^2 + y^3$, and $2 + x + 2xy + y^3$. Notice that B is a polytope if and only if $\phi(B)$ is independent of x, and that B is an n-ovaloid if and only if $\phi(B) = x^{n-1} + y^n$.

ball cone cylinder double cone

Figure 2

Proposition 6.1 *Let A and B be classical solids. Then $\phi(A \square B) = \phi(A)\phi(B)$.*

Proof: For any k with $0 \leq k \leq m+n$, where $m = \dim A$ and $n = \dim B$, $F_k(A \square B)$ is diffeomorphic to

$$\bigcup \{F_i A \times F_j B : i + j = k,\ i \geq 0,\ j \geq 0\}\ .$$

Hence each component of $F_k(A \square B)$ is a product manifold.
Thus the number of i-dimensional components of $F_j(A \square B)$ is

$$\mu_{ij} = \sum \{a_{pr} b_{qs} :\ p + q = i,\ r + s = j\}$$

, where a_{pr} and b_{qs} are the coefficients of $x^p y^r$ and $x^q y^s$ in $\phi(A)$ and $\phi(B)$ respectively.$\square$

The next result shows how $\phi(B)$ behaves under duality. For simplicity, we note that for any classical m-solid B, the coefficient $\mu_{im} = 0$ for $i > 0$ and $\mu_{0m} = 1$. Thus

$$\phi(B) = y^m + \sum_{j<m} \mu_{ij} x^i y^j\ .$$

Proposition 6.2 *Let A and B be classical m-solids with $A \wr B$. Suppose that $\phi(A) = \sum a_{ij}x^iy^j$, $\phi(B) = \sum b_{ij}x^iy^j$. Then $a_{im} = b_{im}$ for $0 \leq i$ and $a_{ij} = b_{i,m-i-j-1}$ for $0 \leq j < m$.*

Proof: To see that B^* is classical it is enough to notice that the map $\theta : FB \to FB^*$ taking each face K of B to the face $\widehat{K} = \{y \in B^* : \langle x, y\rangle = 1, \ x \in K\}$ of B^* is a duality. This means that θ is a homeomorphism and that every F_iB^* is a finite union of components each of which is a compact manifold without boundary. Thus θ preserves the dimension of the components. Knowing that θ takes B to $\emptyset$ and $\emptyset$ to B^*, the proposition will be proved by determining the image of a j-face A, $j < m$, belonging to an i-dimensional component C. If H is a supporting hyperplane with $H \cap B = A$ then H has

$$\dim(\partial B) - \dim A - \dim C = n - i - j - 1 = k - 1$$

degrees of freedom, because H contains A and contains at most one point of C.
But the set of supporting hyperplanes H with $H \cap B = A$ is parametrised by the points $x_0 \in \operatorname{relint}\widehat{A}$ [7]. Thus for each $x_0 \in \operatorname{relint}\widehat{A}$, there is such a supporting hyperplane $H_0 = \{y \in \mathbf{E}^n : \langle y, x_0\rangle = 1\}$. We therefore conclude that $\dim\widehat{A}$ is equal to the number of degrees of freedom of the set of supporting hyperplanes of A. Hence $\dim\widehat{A} = k - 1$. Thus if $\phi(B) = y^n + \sum m_{ij}x^iy^j$ then $\phi(B^*) = y^n + \sum m_{ij}x^iy^{k-1}$, where $k = n - i - j$. □

We may now clarify the remark of 5. concerning the dimensions of the faces of a dual. Consider a right circular cone in $\mathbf{E}^3$. This is a strictly classical 3-solid C with $\phi(C) = 1 + y^2 + y^3 + x + xy$. Likewise, its polar C^* is also a right circular cone so $\phi(C^*) = \phi(C)$. Thus the antiisomorphism $\alpha : FC \to FC^*$ maps the apex of C to the 2-face forming the base of B^*, while the circle of 0-faces forming the rim of the base of C maps to the circle of 1-faces (the 'generators') of B^*. See Figure 3, where the dual of each component K of FC is denoted by K^*.
The failure of α to map each i-face to an $(m - i - 1)$-face is explained by Proposition 6.2, this phenomenon occuring whenever there are components of different dimensions at a given level in the face-lattice.
To obtain an analogue of Proposition 6.1 for the coproduct $\diamond$, we replace $\phi(B)$ with $\psi(B) = \sum v_{ij}x^iy^j$, where v_{ij} is the number of components of $F_{j+1}B$ of dimension i, $0 \leq j + 1 \leq m$. We may then prove

Proposition 6.3 *Let A and B be classical solids. Then $\psi(A \diamond B) = \psi(A)\psi(B)$.*

Proof: Analogous to that of Proposition 6.2. Alternatively, combine Propositions 6.1, 6.2 and 5.2. □

The polynomials $\phi(B)$ and $\psi(B)$ define (complex) algebraic curves, whose properties are related to the structure of B. We observe, for example, that by the above Propositions, the curve $\phi(B) = 0$ is irreducible if B is $\square$-prime (i.e. if B is not isometric to $P \square Q$ for some solids P and Q of positive dimension).

3

2

1

0

-1

Figure 3 Face lattice of a cone

We conclude this section by remarking that if B is a classical solid and C a cone with base B, and if $\phi(B) = \sum \mu_{ij} x^i y^j$, $\psi(C) = \sum \mu'_{ij} x^i y^j$, then for all $i \geq 0$ and $0 \leq j < m+1$, $\mu'_{ij} = \mu_{ij} + \mu_{i,j-1}$.
We may also observe that if B is j-transitive, then $\mu_{ij} \neq 0$ for just one value of i. In particular, if B is completely transitive, then the matrix $M(B)$ whose (i,j)th element is μ_{ij} has just one nonzero element in each column. This is true a fortiori if B is regular. For example, if B is a right regular cylinder in $\mathbf{E}^3$, then B is completely transitive and $M(B) = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

7. Orthogonal Actions on $\mathbf{E}^m$

The classification of strictly classical solids (and hence of classical solids in general) depends on a knowledge of the compact Lie subgroups of the orthogonal group $O(m)$, $m \geq 1$, and how these groups act (by restriction) on $\mathbf{E}^m$. We give a brief resumé of the relevant facts, referring the reader to (for example) [6] for full details.
Suppose, then, that G is a compact subgroup of $O(m)$, acting by restriction of the action of $O(m)$ on $\mathbf{E}^m$, for some $m \geq 1$.
For each $x \in \mathbf{E}^m$, the isotropy subgroup $G_x = \{g \in G : g(x) = x\}$ is a compact subgroup of G, and we define an equivalence relation $\sim$ on $\mathbf{E}^m$ by putting $x \sim y$ iff G_x is conjugate in G to G_y. The equivalence classes of $\sim$ are called *orbit-types* of G. We denote the orbit-type to which x belongs by $[x]$. Then $[x]$ is a union of G-orbits, and in particular $[x] \supset G \cdot x$. Moreover, $\sim$ partitions $\mathbf{E}^m$ into finitely many orbit types.
Let $\langle H \rangle$ denote the conjugacy class of a compact subgroup H of G in G. Then we can

define a partial order $<$ on the set of such conjugacy classes by $\langle H\rangle < \langle K\rangle$ iff H is conjugate in G to a subgroup of K.

Now each orbit-type $[x]$ is a submanifold of $\mathbf{E}^m$ whose closure is a union of orbit-types $[y]$, where for each such y, $\langle G_x\rangle < \langle G_y\rangle$. We put $[y] < [x]$ iff $[y]$ is contained in the frontier of $[x]$. Then $<$ is a partial order on the set of orbit types, and implies that G_x is conjugate to a proper subgroup of G_y.

There is a unique maximal element $[x]$ with respect to $<$, called the *principal orbit-type* of G. The constituent orbits of $[x]$ are then called *principal orbits*. The principal orbit-type is a dense open subset of $\mathbf{E}^m$.

Suppose then that x is in the principal orbit-type of G. Then $G \cdot x$ is a smooth manifold of $\mathbf{E}^m$, of dimension $d_x = \dim G - \dim G_x$, lying in the sphere with centre 0 and radius $\|x\|$ (unless, of course, $x = 0$, in which case $G \cdot x = \{0\}$). Consider the normal plane to $G \cdot x$ at x. This is a linear subspace N_x of $\mathbf{E}^m$, of dimension $n_x = m - d_x$, intersecting each orbit $G \cdot y$, $y \in \mathbf{E}^m$, orthogonally in a nonempty finite set, and $N_x \cap B = B_x$ is an n_x-solid with centroid 0.

The face-structure of B_x is closely related to that of B, as follows. Consider the quotient spaces $\Lambda A = FB/GB$ and $\Lambda B_x = FB_x/GB_{N_x}$. Both are topological lattices with respect to induced partial orders.

For each i-face A of B_x, $i \geq 0$, consider a supporting hyperplane H_A of B_x in N_x with $H_A \cap B_x = A$. Then there is a (not necessarily unique) supporting hyperplane H'_A of B in $\mathbf{E}^m$ with $H'_A \supset H_A$. Thus $A' = H'_A \cap B$ is a j-face of B for some $j \geq i$.

If $i = 0$ then $j = 0$. For otherwise the orbit of G through the unique point a of $A = \{a\}$ would contain a line-segment, which is impossible since every G-orbit in $\mathbf{E}^m$ lies on a sphere. For $i > 0$, it may happen that $j > i$. For example, let B be a right circular cone and let x be a point in the relative interior of the base other than its centroid. Thus $[x]$ is principal, and B_x is an isosceles triangle. Each vertex of B_x is a vertex of B, as are two of its edges. The third edge is contained in the unique 2-face of B (its base). See Figure 4.

Figure 4

The map of FB_x to FB sending A to A' induces a map $\lambda : \Lambda B_x \to \Lambda B$ since faces in the same G_{N_x}-orbit of FB_x map to the same G-orbit of FB.

Proposition 7.1 *$\lambda : \Lambda B_x \to \Lambda B$ is an isomorphism of topological lattices.*

Proof: This follows immediately from the above construction. Note that ΛB_x and ΛB are both graded by dimension, but λ may increase grades at levels other than $m, 0$ and -1.$\square$

Example: Let B be a right circular cone in $\mathbf{E}^3$ as above. Then ΛB_x and ΛB are finite lattices with just six points, forming hexagonal diagrams as shown in Figure 5.

Figure 5

8. The Classification Problem

We now apply the theory of 7. to the case in which the m-solid B is strictly classical.

Proposition 8.1 *Let B be a strictly classical m-solid in $\mathbf{E}^m$ for $m \geq 2$ with centroid* 0, *and let $x \in B$ lie on a principal orbit of GB. Then B_x is a polytope.*

Proof: This follows at once from Proposition 7.1, since for any strictly classical solid B, FB has finitely many GB-orbits, and hence both ΛB and ΛB_x are finite. But each G-orbit in $\mathbf{E}^m$ meets N_x in a (nonempty) finite set (see 6. above). So FB_x itself is finite. $\square$

We use Theorem 8.1 as the basis for our approach to the classification of strictly classical (and hence of classical) solids as follows.
We saw in 4. that the sets $\mathcal{S}$ and $\mathcal{C}$ of strictly classical and classical solids are closed under both $\square$ and $\diamond$. Thus to classify strictly classical solids it is enough to classify those which are both $\square$-prime and $\diamond$-prime. Let us call a strictly classical solid B *elementary* if and only if $B = \square^n Q$ or $B = \diamond^n Q$ where Q is both $\square$-prime and $\diamond$-prime. So from now on, in this section, we will assume that B is an elementary m-solid in $\mathbf{E}^m$ with $\operatorname{cent} B = 0$. Since the symmetry group G of B is a compact Lie subgroup of $O(m)$, the inclusion $\pi : G \to O(m)$ is a faithful orthogonal representation of G. Because B is elementary, π is irreducible.

Definition (Dadok [2]): *Let G be a Lie group and $\pi : G \to O(V)$ be an orthogonal representation for an inner product space V over R. Then π is said to be polar if the normal plane at some (and hence any) point of a principal orbit of the action of G on V intersects every orbit of G orthogonally. Any such normal plane is called a Cartan subspace.*

Proposition 8.2 *Let B be an elementary m-solid in $\mathbf{E}^m$ with* $\operatorname{cent} B = 0$. *If G is the symmetry group of B, then $\pi : G \to O(m)$ is polar.*

Proof: The orbits of π form a transnormal system in the sense of Bolton [1]. See also 7. above. □

If G is connected, a polar representation $\pi : G \to SO(V)$ is said to be a *symmetric space representation* if there exist:

(i) a real simple Lie algebra h with Cartan decomposition $h = K \oplus p$;

(ii) a Lie algebra isomorphism $A : g \to K$; *

(iii) a real vector space isomorphism $L : V \to p$ such that $L \circ \pi(X)(y) = [A(X), y]$ for all $X \in g$, $y \in p$.

Proposition 8.3 (Dadok [2]): *Let $\pi : G \to SO(V)$ be a polar representation of a connected Lie group G. Then there is a connected Lie group $\widetilde{G}$ with symmetrical space representation $\widetilde{\pi} : \widetilde{G} \to SO(V)$ such that the G-orbits and the $\widetilde{G}$-orbits in V coincide.*

The symmetric space H/K associated to $\widetilde{G}$ is of non-compact form and the representation $\widetilde{\pi}$ may be given in terms of the highest weight [2].
Since for an elementary solid B with symmetry group G the representation $\pi : G \to O(m)$ is irreducible and polar, Dadok's classification of irreducible polar representations can be used to determine the set $\mathcal{E}$ of all elementary solids, and hence also the sets $\mathcal{S}$ and $\mathcal{C}$.
Thus for any elementary m-solid B in $\mathbf{E}^m$ with symmetry group G, there is a symmetric-space representation $\widetilde{\pi} : \widetilde{G} \to SO(m)$. Moreover, for any cartan subspace N in $\mathbf{E}^m$, $N \cap B$ is a polytope which is invariant setwise under the action of the Weyl group W on N. The polytope $N \cap B$ may be identified with the polytope B_x described in 7. and in proposition 8.1 (see Theorem 5 of Dadok [2]).
Conversely, for any symmetric space representation $\widetilde{\pi} : \widetilde{G} \to SO(m)$, and any polytope Q in a Cartan subspace, with Q invariant under W, $\widetilde{G} \cdot Q$ is an elementary solid.
This characterisation enables us to describe $\mathcal{S}$ and $\mathcal{C}$ in full detail using Dadok's list [2] as in [8].
In the degenerate case when $\widetilde{G}$ is trivial, the Cartan subspace will be $\mathbf{E}^m$ itself and hence $B = B_x$ is an m-polytope. As an illustration we show how this works in dimensions two, three and four, thus giving a detailed classification of strictly classical and classical solids in these dimensions.

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* where g denotes the Lie algebra of G

9. Dimension two

In this case, the only nontrivial connected subgroup of $O(2)$ is $SO(2)$. The only symmetric space H/K in Dadok's list with K isomorphic to $SO(2)$ is $SO(2,1)/SO(2)$. The principal orbit is one-dimensional and so is the Cartan subspace. The rank of $SO(2,1)/SO(2)$ is one and hence the Weyl group is of order 2 generated by just one reflection. So the only polytope invariant under W is a closed straight line segment whose midpoint is at the origin. Hence B must be the circular disk D^2.
To avoid repetition, we notice that this argument works for any dimension m. That is to say, in $\mathbf{E}^m$, the symmetric space $SO(m,1)/SO(m)$ gives the m-dimensional solid ball D^m. (See also Proposition 3.6). Since a compact connected subgroup of $O(2)$ is either trivial or $SO(2)$, the above discussion gives:

Proposition 9.1 *Every strictly classical 2-solid is either a polygon or a circular disk.*

Corollary 9.2 *Every classical 2-solid is either a polygon or an ovaloid.*

10. Dimension three

The only such symmetric spaces H/K where K is isomorphic to a connected subgroup G of $SO(3)$ are $SO(3,1)/SO(3)$, $SO(2,1)/SO(2)$ and $SL(2,R)/SO(2)$.
The case $SO(3,1)/SO(3)$ gives D^3 as we have remarked earlier. In each one of the other two cases every principal orbit is a circle, and the Cartan subspace N is a 2-plane meeting each principal orbit in exactly two points. Both symmetric spaces have rank one and the Weyl group is generated by one reflection. Hence for every polygon P with an axis of symmetry in N, the rotation of P in 3-space around this axis generates a strictly classical 3-solid B. Such a solid is called a *lantern*.
We may wish to identify different symmetry classes of lanterns. This depends upon the position of the r-gon P relative to the symmetry axis L and the line L' perpendicular to L through the origin as follows.

(i) If $r = 2s+1$, P has one axis of symmetry, namely L. Then L passes through some vertex of P and bisects the opposite edge orthogonally.

(ii) If $r = 2s$, P has one axis of symmetry, namely L. Then L either passes through two vertices or meets two opposite sides orthogonally.

(iii) If $r = 2s$, P has both L, L' as axes of symmetry.

The above three possibilities determine all sequences of symmetry types. A lantern of *type* T^r_{ij} is obtained when an r-gon P is rotated about an axis L, with i = number of vertices of P on L, and j = number of axes of symmetry of P, where of course $i \in \{0,1,2\}$ and $j \in \{1,2\}$. (See Figures 6 and 7.)

Proposition 10.1 *Every strictly classical 3-solid is a polyhedron, a 3-ball or a lantern of type* T^r_{ij}.

Corollary 10.2 *Every classical solid is a polyhedron, an ovaloid or is face equivalent to a lantern.*

Figure 6

Figure 7

11. Dimension four

Here the only symmetric spaces are $SO(4,1)/SO(4)$, $SO(3,1)/SO(3)$, $SO(2,2)/SO(2) \times SO(2)$ and $SO(2,1)/SO(2)$.

(1) If the symmetric space is $SO(4,1)/SO(4)$ then B is the 4-ball D^4.

(2) If the symmetric space is $SO(3,1)/SO(3)$ then the orbits are 2-spheres in $\mathbf{E}^4$, the Cartan subspace is a 2-plane, the polytope P is a plane polygon with symmetry group (the Weyl group) generated by reflection in a line. As we saw in 10., P may have another line of symmetry L' perpendicular to L. This leads to five sequences of 4-dimensional lanterns T^r_{ij} exactly analogous to those of 10. For reasons that will become clear below, we call these solids *lanterns of second kind.*

(3) If the symmetric space is $SO(2,2)/SO(2) \times SO(2)$ then the principal orbits are 2-dimensional Clifford tori, the Cartan subspace is also of dimension 2, the r-gon P has symmetry group of order 4 generated by reflections in two perpendicular lines, say L and L', which we may take to be the x_1-axis and the x_3-axis. We distinguish here two cases. Firstly, L may pass through two opposite vertices. Then $r = 2s$ for any $s \geq 2$, and there is a sequence of symmetry types S^r_2, $r = 4, 6, 8, \dots$ with $P = P_r$ as indicated in Figure 8. Secondly both L and L' may bisect pairs of opposite edges of P. Then $r \equiv 0 (\operatorname{mod} 4)$ and there is a sequence of symmetry types S^r_0 with P as shown in Figure 9.

Figure 8

In both cases each orbit of $SO(2) \times SO(2)$ is either a circle or a Clifford torus.
If P has the symmetry group of a square, we may again distinguish two cases corresponding to two sequences of symmetry types U^r_0, U^r_2 with polygons P as shown in Figure 10, the group $SO(2) \times SO(2)$ spins L in $\mathbf{E}^2 \times 0$, L' in $0 \times \mathbf{E}^2$ and interchanges these two lines. Notice that U^4_2 contains the 4-solid $D^2 \diamond D^2$, while U^4_0 contains the 4-solid $D^2 \square D^2$.
The orbits of $SO(2) \times SO(2)$ here are again circles or Clifford tori. We refer to S^r_i, U^r_i as *Clifford solids.*

Figure 9

Figure 10

(4) If the symmetry space is $SO(2,1)/SO(2)$ then the principal orbits are circles, the Cartan subspace is 3-dimensional, and the polytope P is a polyhedron invariant under a reflection in a plane E. The group $SO(2)$ rotates P in $\mathbf{E}^4$ about E. We call $B = T_P$ a *lantern of the first kind*. To summarise the classification in dimension four we state the following.

Proposition 11.1 *Let B be a strictly classical 4-solid. Then B is a 4-polytope, a 4-ball D^4, or is symmetry equivalent to a lantern T_P of the first kind, a lantern T_{ij}^r of the second kind, or a Clifford solid S_i^r, U_i^r.*

Corollary 11.2 *Let B be a classical 4-solid. Then B is a 4-polytope, a 4-ovaloid, or is face equivalent to a Clifford solid S_i^r, a lantern T_P of the first kind or a lantern T_{ij}^r of the second kind.*

As can be seen from the discussion in 9., 10., 11., for every symmetric space H/K on Dadok's list, with K isomorphic to a compact connected subgroup $G \subset O(m)$, the Cartan subspace N and the Weyl group W are completely determined. Then for every polytope P in N invariant under W, $G \cdot P = B$ is an elementary solid. The group G is a subgroup of the symmetry group of B. The results of 7., 8. show that all elementary solids are obtained in this way. Knowing that every strictly classical solid can be factorised into its elementary solids, we have determined all strictly classical and hence all classical solids up to face isomorphism.

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