Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 36 (1995), No. 2, 299-301.

An Equichordal Characterization of Round Spheres

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Abstract. Characterizations of round spheres are plentiful. The aim of this note is to present one based on the notions of horizon and equichordality. This kind of characterization complements research on the relations between the shape of a body and the shapes of its shadow boundaries (see the paper of Kinczes and Kurusa [Beiträge zur Algebra und Geometrie, 36 (1995) 25- 35]).

MSC 1991: 53C40, 53A07

1. Introduction

Let M be a compact, connected, smooth ($= C^\infty$) submanifold of R^{n+1}, $n \geq 2$. We say that M is *equichordal with respect to* O if there exists a diffeomorphism $\delta : M \longrightarrow M$ such that, for every $x \in M$, x, $\delta(x)$ and O are distinct, collinear points and $\|x - \delta(x)\|$ does not depend on x. We shall refer to δ as being an *equichordal diffeomorphism.* It follows that O belongs to the line segment $\overline{x\delta(x)}$, δ is an involution and it is unique with respect to O. The notion of equichordality goes back to M. Fujiwara (see [4,5] for detailed references) and has been extended by the author and S.A. Robertson [2].

From now on we shall assume that M is a hypersurface with non-vanishing Gaussian curvature. Let $N : M \longrightarrow S^n$ be its Gauss map, given by a normal unit vector field. Then N is a diffeomorphism and M is strictly convex, that is, M lies entirely in one of the closed half-spaces determined by each affine tangent hyperplane to M and each affine tangent hyperplane meets M in just one point [6]. Moreover, no straight line in R^{n+1} intersects M in more than two points.

If λ denotes a straight line in R^{n+1} the λ-*horizon* of M, M_λ, is the set of points of M at which the affine tangent hyperplane is parallel to λ. Since M_λ is the inverse image by N of the great $(n-1)$-sphere of S^n contained in the linear hyperplane normal to the direction of λ it is a submanifold of R^{n+1} diffeomorphic to S^{n-1}.

Given straight lines $\lambda_1, \ldots, \lambda_k$, they are *independent* if their directions are spanned by linearly independent vectors. The corresponding horizons $M_{\lambda_1}, \ldots, M_{\lambda_k}$ will also be called *independent.*

0138-4821/93 \$ 2.50

Obviously two horizons M_λ and M_μ coincide if and only if λ and μ are not independent. If they are independent they meet transversally in M.

2. The main theorem

Theorem. *Let M be a convex smooth hypersurface in R^{n+1}, $n \geq 2$, with regular Gauss map. Then M is a round n-sphere with centre O if and only if through every point of M there pass n independent equichordal horizons with respect to O.*

Proof: The proof of the only if part is trivial. As to the if part, O belongs to open line segments determined by points of M hence, according to our assumptions, we conclude that O is inside M. If M_λ and M_μ are independent equichordal horizons with δ_1, δ_2 as equichordal diffeomorphisms and $x \in M_\lambda \cap M_\mu$ we have $\delta_1(x) = \delta_2(x)$ and $\|x_1 - \delta_1(x_1)\| = \|x_2 - \delta_2(x_2)\|$, for $x_1 \in M_\lambda$, $x_2 \in M_\mu$. Consequently if y is the intersection of $\lambda_x \setminus \{x\}$ with M, λ_x being the straight line defined by x and O, and $\delta : M \longrightarrow M$ is given by $\delta(x) = y$ then δ is an equichordal diffeomorphism with respect to O for M.

We next show that $\|x - \delta(x)\| = diameter(M)$, $x \in M$. Let x, y be points in M such that $\|x - y\| = diameter(M)$. Then the affine tangent hyperplanes at x and y are parallel. Choose independent lines $\lambda_1, \ldots, \lambda_n$ giving rise to the independent horizons which pass through x. Then $M_{\lambda_1} \cap \cdots \cap M_{\lambda_n} = \{x, y\}$ and $y = \delta(x)$.

We are now ready to conclude that M is a round n-sphere. First observe that the normal line to M at x is the straight line determined by x and $\delta(x)$ since $\|x - \delta(x)\| = diameter(M)$. Let us take the distance function $d : M \longrightarrow R$ given by $d(x) = \|x - O\|$. Recalling that x is a critical point of d if and only if the normal line to M at x passes through O we see that the critical set of d is the whole M. Hence d is constant and M is a round n-sphere with centre O. $\square$

The number of independent horizons can not be reduced as can be seen with the following example. Take a convex equichordal curve in the plane different from the circle with an axis of reflectional symmetry passing through its centre. If we rotate it around that axis we obtain a surface with one equichordal horizon through every point which is not a round sphere.

For other characterizations concerning the notion of horizon see [1,7].

Acknowledgement: Part of the research reported here was done under " Projecto de Investigacao PBIC/C/CEN/1060/92 ". I am extremely grateful to Professor Bernd Wegner whose comments on a previous version of this note led to several and important improvements.

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Received March 6, 1995, revised version April 19, 1995.