Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 35 (1994), No. 1, 101-108.

Covering the Plane with Congruent Copies of a Convex Disk

Edwin H. Smith*
Department of Mathematics, Auburn University
255 Conway Acres, Auburn, AL 36830, U.S.A.

Abstract. It is shown that there exists a number $d_0 < 8(2\sqrt{3}-3)/3 = 1.237604..$, such that every compact convex set K with an interior point admits a covering of the plane with density smaller than or equal to d_0. This improves on the previous result [9], which showed that a density of $8(2\sqrt{3}-3)/3$ can always be obtained. Since the thinnest covering of the plane with congruent circles is of density $2\pi/\sqrt{27} = 1.20919\ldots$, we strengthen the case for the conjecture that the smallest such number d_0 is $2\pi/\sqrt{27}$.

I. Introduction

A *convex disk* is a compact convex set in the plane having an interior point. We will denote by K an arbitrary convex disk in the plane.

A *packing of the plane with copies of* K is a family $\{K_i\}$ of sets congruent to K whose interiors are mutually disjoint. A *covering of the plane with copies of* K is a family $\{K_i\}$ of sets congruent to K whose union is the plane. A family $\{K_i\}$ which is both a packing and a covering is called a *tiling*. Any convex disk which admits a tiling is called a (convex) *tile*.

Suppose T is a tile and T is contained in (respectively contains) K. If $\{T_i\}$ is a tiling with copies of T, then each T_i is contained in (contains) a copy K_i of K. Then $\{K_i\}$ is a covering (packing) of the plane *associated with the tiling* $\{T_i\}$.

Throughout this paper, the area of any set S in the plane will be denoted by $|S|$. Each covering (packing) is assigned a non-negative number, called the *density* of the covering (packing), which, intuitively, corresponds to the ratio of the sum of the areas of the disks in the covering (packing) to the area of the region which they cover (pack). A definition of density and the detailed investigation concerning its existence and uniqueness can be found in [3, Section III, Chapter 1]. In the case that the covering (packing) $\{K_i\}$ is associated

* This paper is based upon the author's doctoral dissertation, defended at Auburn University November 8, 1994.

0138-4821/93 $ 2.50

with the tiling $\{T_i\}$, the density of $\{K_i\}$ can be computed as $|K|/|T|$. For example, the density of the familiar 'honeycomb' circle packing is $\pi/\sqrt{12}$, and the circle covering by the same honeycomb pattern is of density $2\pi/\sqrt{27}$.

Early in this century, A. Thue (1910, [14]) showed that the density of a packing of congruent circles in the plane never exceeds $\pi/\sqrt{12}$. Later R. Kershner (1939, [4]) proved that the density of a covering of the plane with congruent circles is never less than $2\pi/\sqrt{27}$. It is a conjecture that every convex disk K admits a covering with congruent copies having density less than or equal to $2\pi/\sqrt{27}$.

In this paper we investigate the following problem: find the least number d_0 such that every convex disk admits a covering of the plane with density smaller than or equal to d_0. W. Kuperberg [9] has shown that $d_0 \leq 8(2\sqrt{3}-3)/3$. In this paper we show that $d_0 < 8(2\sqrt{3}-3)/3$.

In a classical theorem [12], E. Sas showed that every convex disk K of area 1 contains an n-sided polygon P_n ($n \geq 3$) with the area of P_n being greater than or equal to that of a maximal area (regular inscribed) n-sided polygon contained in a circle of area 1. If K is centrally symmetric and n is even, then by a theorem of Dowker [1], a centrally symmetric P_n can be found which is also of maximum area among all n-sided polygons in K. Since every centrally symmetric hexagon tiles the plane, it follows immediately that every centrally symmetric convex disk admits a covering of the plane with density smaller than or equal to $2\pi/\sqrt{27}$. In view of the result of Kershner [4] mentioned above, the number $2\pi/\sqrt{27}$ is the least upper bound of covering density for centrally symmetric convex disks.

The analogous problem for packings of convex disks is investigated in [5], where the authors show that every convex disk admits a packing with density greater than or equal to $\sqrt{3}/2 = 0.866025\ldots$. The packing problem for centrally symmetric convex disks has been given more attention due to its role in the Geometry of Numbers. Tammela [13] has shown that any centrally symmetric convex disk admits a packing of the plane with density greater than or equal to $0.89265\ldots$. This bound was obtained through an elaboration of a method of Ennola [2] who suggested this improvement of his own result. However, it is conjectured that the 'smoothed octagon' of Reinhardt [11], for which the packing density equals $0.90241\ldots$, provides the best bound in this context.

II. Coverings associated with p-hexagons

A *p-hexagon* is a convex hexagon with a pair of parallel opposite sides of equal length, where 'opposite' means separated by two sides in each, clockwise and counterclockwise direction [6]. Extreme cases are allowed under this definition, in which some sides, even the parallel ones, may be reduced to points. Thus each convex quadrilateral is a p-hexagon, as well as each triangle. It is easy to observe that each p-hexagon tiles the plane (see Figure 1). This property of p-hexagons makes them very useful in obtaining associated packings of high density (see [8] and [9]) and coverings of low density (see [6], [7] and [8]).

The following theorem will produce the main result of this paper as a corollary.

Theorem. *Every convex disk K contains an inscribed p-hexagon H with $|H|/|K| > (2\sqrt{3}+3)/8$.*

Figure 1

We begin with a lemma.

Lemma (W. Kuperberg [9]). *Suppose that $A = (A_x, A_y)$ and $B = (B_x, B_y)$ are points in the first quadrant of the xy-plane with $A_x \geq B_x$ and $A_y \leq B_y$ and let $OXCY$ be a quadrilateral such that O is the origin, X lies on the positive half of the x-axis, Y lies on the positive half of the y-axis, A lies on CX, and B lies on CY, and so that the complement of the quadrilateral in the first quadrant is convex. Then the area of the quadrilateral is greater than or equal to $2A_y(A_x - B_x) + 2B_x(B_y - A_y)$ and this minimum only occurs when A is the midpoint of CX and B is the midpoint of CY.*

Figure 2

Proof. With A and B fixed, the position of C determines the entire quadrilateral (see Figure 2). We consider C a variable and the area of the quadrilateral as its function. The minimum of the function is achieved only when C is placed so that A is the midpoint of CX and B is the midpoint of CY. In this position of C, the area of the quadrilateral equals $2A_y(A_x - B_x) + 2B_x(B_y - A_y)$, which completes the proof.

Proof of the Theorem. The proof which follows is an extension of the proof from [9] which preserves the notation and construction illustrated by Figure 3, the figure itself being redrawn from [9]. Let k_1 and k_2 be a pair of parallel lines of support of K, such that K

lies between them. The direction of the lines can be chosen arbitrarily. Let V_1 and V_2 be points at which k_1 and k_2, respectively, touch K. Let $\overline{W_1W_2}$ be a longest chord of K in the direction of k_1, and denote by m_1 and m_2 the parallel lines of support of K touching K at W_1 and W_2, respectively. The lines k_1, k_2, m_1 and m_2 define a parallelogram containing K. The affine-invariant nature of the Theorem's statement allows us to assume that the parallelogram constructed above is a square of side 4. Denote the vertices of the square by P, Q, R and S as shown on Figure 3. This figure illustrates some additional constructions and notations described as follows. Let $ABCD$ be a parallelogram inscribed in K with sides AB and CD each of length $2\sqrt{3}$ and parallel to k_1. The vertices of $ABCD$ should be arranged cyclically, in the same orientation as those of the square $PQRS$, and so that the side AB lies between $\overline{W_1W_2}$ and k_1, while CD lies between $\overline{W_1W_2}$ and k_2. Let $EFGH$ be a parallelogram inscribed in K with sides EF and GH parallel to k_1 and such that the distance between the lines EF and GH equals $2\sqrt{3}$. The existence of these parallelograms is guaranteed by a continuity argument. Specifically, assuming K is strictly convex, chords of K which are parallel to k_1 change continuously from a length of 0 to a length of 4 and back to zero as they move from k_1 to k_2. In the event that the boundary of K contains a segment which lies in k_1 or k_2, a chord can be grown continuously within the boundary. Notice that the length of EH and FG may be greater than $2\sqrt{3}$. The vertices of $EFGH$ should be arranged cyclically in the same orientation as those of $PQRS$ and so that the side EF lies between $\overline{W_1W_2}$ and k_1 while GH lies between $\overline{W_1W_2}$ and k_2.

Figure 3

Let x denote the length of the sides EF and GH and let y denote the distance between the lines AB and CD. We shall concentrate on two hexagons inscribed in K: the p-hexagon $h_1 = W_1EFW_2GH$ and the p-hexagon $h_2 = V_1BCV_2DA$, whose areas are $|h_1| = \sqrt{3}(x+4)$ and $|h_2| = \sqrt{3}(y+4)$. Let $t = max(x, y)$ and let h denote h_1 or h_2, whichever of these two p-hexagons is of area $\sqrt{3}(t+4)$.

Case I. $t > 2\sqrt{3}$. Then $|h| > 4\sqrt{3} + 6$, while $|K| \leq 16$. Thus, $|h|/|K| > (2\sqrt{3} + 3)/8$.

Case II. $t \leq 2\sqrt{3}$. In this case, $x \leq 2\sqrt{3}$ which means that the segments EF and GH are not longer than the segments AB and CD. Therefore the aforementioned 12 points are ordered cyclically $W_1AEV_1FBW_2CGV_2HD$ on the boundary of K.

Now we consider the convex dodecagon d defined by the lines of support k_1, k_2, m_1 and m_2 along with the eight lines of support whose points of contact with K are A, E, F, B, C, G, H and D. Since d contains K we have $|K| \leq |d|$. Notice that the complement of d in the square $PQRS$ is the union of four quadrilaterals which we denote q_P, q_Q, q_R and q_S, their subscripts corresponding to the corner in which they lie. For each quadrilateral, introduce a separate coordinate system so that the corresponding corner of the square is the origin, the lines containing their horizontal or vertical side is the x-axis or y-axis respectively, and the quadrilateral lies in the first quadrant. Note that, as in the lemma, the complement of each quadrilateral in its respective first quadrant is convex. In their corresponding coordinate systems the points A, E, F, B, C, G, H and D have their corresponding coordinates $A = (A_x, A_y)$, $B = (B_x, B_y)$, etc., giving us the following equalities:

$$A_x + B_x = 4 - 2\sqrt{3}, \tag{1}$$
$$C_x + D_x = 4 - 2\sqrt{3}, \tag{2}$$
$$E_x + H_y = 4 - 2\sqrt{3}, \tag{3}$$
$$F_y + G_y = 4 - 2\sqrt{3}, \tag{4}$$
$$E_x + F_x = 4 - x, \tag{5}$$
$$H_x + G_x = 4 - x, \tag{6}$$
$$A_y + D_y = 4 - y, \tag{7}$$
$$B_y + C_y = 4 - y. \tag{8}$$

Additionally, since the lines AB, CD, EF, and GH are parallel to k_1 and k_2, we have the equalities:

$$A_y = B_y, C_y = D_y, E_y = F_y, G_y = H_y. \tag{9}$$

Through use of the lemma, we gain the following inequalities:

$$|q_P| \geq 2E_y(E_x - A_x) + 2A_x(A_y - E_y), \tag{10}$$
$$|q_Q| \geq 2F_y(F_x - B_x) + 2B_x(B_y - F_y), \tag{11}$$
$$|q_R| \geq 2G_y(G_x - C_x) + 2C_x(C_y - G_y), \tag{12}$$
$$|q_S| \geq 2H_y(H_x - D_x) + 2D_x(D_y - H_y). \tag{13}$$

Adding inequalities (10)-(13) and using the equalities (1)-(9), we conclude that the area of the union of the four corner quadrilaterals is greater than or equal to $2(4 - 2\sqrt{3})(4\sqrt{3} - (x + y))$, which is in turn greater than or equal to $8(2 - \sqrt{3})(2\sqrt{3} - t)$. Therefore

$$|K| \leq |d| \leq 16 - 8(2 - \sqrt{3})(2\sqrt{3} - t) = 8(2 - \sqrt{3})(4 + t),$$

which implies that

$$|h|/|K| \geq \sqrt{3}/8(2-\sqrt{3}) = (2\sqrt{3}+3)/8,$$

with the equality only occurring when both of the following are satisfied: $K = d$ and each of the points A, E, F, B, C, G, H and D are the midpoints of the respective sides of d which contain them.

Figure 4

Assume now that we have equality above. Let us consider the quadrilaterals q_P and q_Q. A side of these quadrilaterals which is shared with the dodecagon d has A, E, F or B as its midpoint. We denote these segments s_A, s_E, s_F and s_B respectively and denote the sides of K which are contained in m_1 or m_2 as w_1 and w_2 respectively (see Figure 4). Note that w_1 and w_2 may be degenerated to points. Since the midpoints and lower endpoints of s_E and s_F have identical y coordinates, their upper endpoints must also have identical y coordinates. Similarly, the upper endpoints of s_A and s_B have y coordinates which are the same. Likewise, q_S and q_R have common horizontal lines of support. Thus w_1 is the horizontal projection of w_2.

Now we choose parallel lines of support of K $(= d)$ whose angle of inclination differs only slightly from that of k_1 and k_2. We denote these as k_1^* and k_2^* as illustrated in figure 5. In the case in which w_1 and w_2 are segments not degenerated to points, denote by p the line which is parallel to k_1^* and passes through the lower endpoint of w_1. Through an affine transformation f from the plane onto the plane, we take the parallelogram defined by the lines m_1, m_2, k_1^* and k_2^* onto a square of side 4 (Figure 6) and repeat the above construction of an inscribed p-hexagon and a dodecagon containing $f(K)$ which we denote as h^* and d^* respectively. We now observe (Figure 6) that $f(w_1)$ is not the horizontal projection of $f(w_2)$ and we therefore will not satisfy both of the conditions for the above extreme case in which $|h^*|/|f(K)| = (2\sqrt{3}+3)/8$. In the event that w_1 and w_2 are reduced to points, a longest chord in the direction of k_1^* can be arranged to have either w_1 or w_2 as one of its endpoints and the other endpoint will lie in the interior of a side of the dodecagon K. This will also not satisfy the conditions of the extreme case since one of the parallel

Figure 5

Figure 6

lines of support for this longest chord will contain only a point of K while the other will contain a segment in K. Thus we have $|h^*|/|f(K)| > (2\sqrt{3}+3)/8$, which completes our proof.

Corollary. *There exists a number* $d_0 < 8(2\sqrt{3}-3)/3$ *such that for every convex disk* K*, there exists a covering of the plane with congruent copies of* K*, whose density is smaller than or equal to* d_0*.*

Proof. Let C be the set of all affine equivalence classes of convex disks. A. M. Macbeath [10] proved that C (and in fact, the n-dimensional analog of C) is a compact metric space. The metrization is accomplished by defining the distance between classes $[K]$ and $[L]$ to be $d([K],[L]) = log\ p(K,L) + log\ p(L,K)$ where $p(K,L) = inf\{|f(L)|/|K|\}$ and $f(L)$ ranges through all convex disks which are affinely equivalent to L and contain K with $|K| = 1$. Define the function h from C into the non-negative real numbers by assigning to each convex disk K, the ratio between the area of a maximum area p-hexagon contained in K and the area of K. Since the affine image of a p-hexagon is a p-hexagon and ratios of area are preserved under an affine transformation, h is independent of the affine representation chosen for K as it is also continuous over C (see [7] Lemma 3.2 and [8] Lemma 2.2). Therefore h attains its minimum in the compact space C and by the Theorem, this minimum must be smaller than $8(2\sqrt{3}-3)/3$.

References

[1] C. H. Dowker, On minimum circumscribed polygons, *Bull. Amer. Math. Soc.* 50, 120-122 (1944).

[2] V. Ennola, On the lattice constant of a symmetric convex domain, *J. London Math. Soc.* 36, 135-138 (1961).

[3] L. Fejes Toth, *Lagerungen in der Ebene, auf der Kugel und im Raum* (Spinger, Berlin, 1972).

[4] R. Kershner, The number of circles covering a set, *American J. Mathematics* 61, 665-671 (1939).

[5] G. Kuperberg and W. Kuperberg, Double-lattice packings of convex bodies in the plane, *Discrete and Computational Geometry* 5, 389-397 (1990).

[6] W. Kuperberg, Packing convex bodies in the plane with density greater than 3/4, *Geometriae Dedicata* 13, 149-155 (1982).

[7] W. Kuperberg, On packing the plane with congruent copies of a convex body, *Intuitive Geom.* 48, Janos Bolyai Math. Soc., 317-329 (1985).

[8] W. Kuperberg, An inequality linking packing and covering densities of convex bodies, *Geometriae Dedicata* 23, 59-66 (1987).

[9] W. Kuperberg, Covering the plane with congruent copies of a convex body, *Bull. London Math. Soc.* 21, 82-86 (1989).

[10] A. M. Macbeath, A compactness theorem for affine equivalence classes of convex regions, *Canadian J. Math.* 3, 54-61 (1951).

[11] K. Reinhardt, Uber die dichteste gitterformige Lagerung kongruenter Bereiche in der Ebene und eine besondere Art konvexer Kurven, *Abh. math Sem. hansischer Univ.* 10, 216-230 (1934).

[12] E. Sas, Uber eine Extremumeigenschaft der Ellipsen,*Composito Math.* 6, 468-470 (1939).

[13] P. Tammela, An estimate of the critical determinant of a two-dimensional convex symmetric domain (Russian), *Izv. Vyss. Ucebn. Zaved. Mat.* 12, 103-107, (1970).

[14] A. Thue, Uber die dichteste Zusammenstellung von kongruenten Kreisen in der Ebene, *Christinia Vid. Selsk. Skr.* 1, 3-9 (1910).

Received 11.10.93