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A GENERALIZATION OF ORLICZ SEQUENCE SPACES BY CESÀRO MEAN OF ORDER ONE

H. DUTTA and F. BAŞAR

Abstract. In this paper, we introduce the Orlicz sequence spaces generated by Cesàro mean of order one associated with a fixed multiplier sequence of non-zero scalars. Furthermore, we emphasize several algebraic and topological properties relevant to these spaces. Finally, we determine the Köthe-Toeplitz dual of the spaces $\ell'_M(C,\Lambda)$ and $h_M(C,\Lambda)$.

1. Preliminaries, Background and Notation

By ω, we denote the space of all complex valued sequences. Any vector subspace of ω which contains ϕ, the set of all finitely non–zero sequences is called a *sequence space*. We write ℓ_∞, c and c_0 for the classical sequence spaces of all bounded, convergent and null sequences which are Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$, where $\mathbb{N} = \{0,1,2,\ldots\}$, the set of natural numbers. A sequence space X with a linear topology is called a $K-$*space* provided each of the maps $p_i\colon X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space X is called an FK*-space* provided X is a complete linear metric space. An FK-space whose topology is normable is called a BK*-space*.

A function $M\colon [0,\infty) \to [0,\infty)$ which is convex with $M(u) \geq 0$ for $u \geq 0$, and $M(u) \to \infty$ as $u \to \infty$, is called as an *Orlicz function*. An Orlicz function M can always be represented in the following integral form

$$M(u) = \int_0^u p(t)\mathrm{d}t,$$

where p, *the kernel of* M, is right differentiable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non–decreasing and $p(t) \to \infty$ as $t \to \infty$ whenever $\frac{M(u)}{u} \uparrow \infty$ as $u \uparrow \infty$.

Consider the kernel p associated with the Orlicz function M and let

$$q(s) = \sup\{t : p(t) \leq s\}.$$

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Then q possesses the same properties as the function p. Suppose now

$$\Phi(x) = \int_0^x q(s)\mathrm{d}s.$$

Then, Φ is an Orlicz function. The functions M and Φ are called *mutually complementary Orlicz functions*.

Now, we give the following well-known results.

Let M and Φ are mutually complementary Orlicz functions. Then, we have:

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), \qquad \text{(Young's inequality)}. \tag{1.1}$$

(ii) For all $u \geq 0$,

$$up(u) = M(u) + \Phi(p(u)). \tag{1.2}$$

(iii) For all $u \geq 0$ and $0 < \lambda < 1$,

$$M(\lambda u) < \lambda M(u). \tag{1.3}$$

An Orlicz function M is said to satisfy the Δ_2-condition for small u or at 0 if for each $k \in \mathbb{N}$, there exist $R_k > 0$ and $u_k > 0$ such that $M(ku) \leq R_k M(u)$ for all $u \in (0, u_k]$. Moreover, an Orlicz function M is said to satisfy the Δ_2-condition if and only if

$$\limsup_{u \to 0^+} \frac{M(2u)}{M(u)} < \infty.$$

Two Orlicz functions M_1 and M_2 are said to be equivalent if there are positive constants α, β and b such that

$$M_1(\alpha u) \leq M_2(u) \leq M_1(\beta u) \quad \text{for all } u \in [0, b]. \tag{1.4}$$

Orlicz used the Orlicz function to introduce the sequence space ℓ_M (see Musielak [**10**]; Lindenstrauss and Tzafriri [**9**]), as follows

$$\ell_M = \left\{ x = (x_k) \in \omega : \sum_k M\left(\frac{|x_k|}{\rho}\right) < \infty \quad \text{for some } \rho > 0 \right\}.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. For relevant terminology and additional knowledge on the Orlicz sequence spaces and related topics, the reader may refer to [**1, 3, 5, 6, 7, 8, 11, 9, 10**] and [**12**].

Throughout the present article, we assume that $\Lambda = (\lambda_k)$ is the sequence of non–zero complex numbers. Then for a sequence space E, the multiplier sequence space $E(\Lambda)$ associated with the multiplier sequence Λ is defined by

$$E(\Lambda) = \{x = (x_k) \in \omega : \Lambda x = (\lambda_k x_k) \in E\}.$$

The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. G. Goes and S. Goes defined the differentiated sequence space dE and integrated sequence space $\int E$ for a given sequence space E, using the multiplier sequences (k^{-1}) and (k) in [**4**], respectively. A multiplier sequence can be used to accelerate the convergence of the sequences in some spaces.

In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction. Sometimes the associated multiplier sequence delays the rate of convergence of a sequence. Thus it also covers a larger class of sequences for study.

Let $C = (c_{nk})$ be the Cesàro matrix of order one defined by

$$c_{nk} := \begin{cases} \dfrac{1}{n+1}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}$$

for all $k, n \in \mathbb{N}$.

Definition 1.1. Let M be any Orlicz function and $\delta(M, x) := \sum_k M(|x_k|)$, where $x = (x_k) \in \omega$. Then, we define the sets $\widetilde{\ell}_M(C, \Lambda)$ and $\widetilde{\ell}_M$ by

$$\widetilde{\ell}_M(C, \Lambda) := \left\{ x = (x_k) \in \omega : \widehat{\delta}_C(M, x) = \sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j x_j\right|}{k+1} \right) < \infty \right\}$$

and

$$\widetilde{\ell}_M := \{x = (x_k) \in \omega : \delta(M, x) < \infty\}.$$

Definition 1.2. Let M and Φ be mutually complementary functions. Then, we define the set $\ell_M(C, \Lambda)$ by

$$\ell_M(C, \Lambda) := \left\{ x = (x_k) \in \omega : \sum_k \left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1} \right) y_k \text{ converges for all } y = (y_k) \in \widetilde{\ell}_\Phi \right\}$$

which is called as Orlicz sequence space associated with the multiplier sequence $\Lambda = (\lambda_k)$ and generated by Cesàro matrix of order one.

Definition 1.3. The α-dual or Köthe-Toeplitz dual X^α of a sequence space X is defined by

$$X^\alpha := \left\{ a = (a_k) \in \omega : \sum_k |a_k x_k| < \infty \text{ for all } x = (x_k) \in X \right\}.$$

It is known that if $X \subset Y$, then $Y^\alpha \subset X^\alpha$. It is clear that $X \subset X^{\alpha\alpha}$. If $X = X^{\alpha\alpha}$, then X is called as an α space. In particular, an α space is called a Köthe space or a perfect sequence space.

The main purpose of this paper is to introduce the sequence spaces $\ell_M(C, \Lambda)$, $\widetilde{\ell}_M(C, \Lambda)$, $\ell'_M(C, \Lambda)$ and $h_M(C, \Lambda)$, and investigate their certain algebraic and topological properties. Furthermore, it is proved that the spaces $\ell'_M(C, \Lambda)$ and $h_M(C, \Lambda)$ are topologically isomorphic to the spaces $\ell_\infty(C, \Lambda)$ and $c_0(C, \Lambda)$ when $M(u) = 0$ on some interval, respectively. Finally, the α-dual of the spaces $\ell'_M(C, \Lambda)$ and $h_M(C, \Lambda)$ are determined, and therefore the non-perfectness of the space $\ell'_M(C, \Lambda)$ is showed when $M(u) = 0$ on some interval, and some open problems are noted.

2. Main Results

In this section, we emphasize the sequence spaces $\ell_M(C,\Lambda)$, $\widetilde{\ell}_M(C,\Lambda)$, $\ell'_M(C,\Lambda)$ and $h_M(C,\Lambda)$, and give their some algebraic and topological properties.

Proposition 2.1. *For any Orlicz function M, the inclusion $\widetilde{\ell}_M(C,\Lambda)\subset\ell_M(C,\Lambda)$ holds.*

Proof. Let $x=(x_k)\in\widetilde{\ell}_M(C,\Lambda)$. Then, since $\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j x_j\right|}{k+1}\right)<\infty$ we have from (1.1) that

$$\begin{aligned}\left|\sum_k\left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1}\right)y_k\right| &\le \sum_k\left|\left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1}\right)y_k\right|\\ &\le \sum_k M\left(\left|\frac{\sum_{j=0}^k \lambda_j x_j}{k+1}\right|\right)+\sum_k\Phi(|y_k|)<\infty\end{aligned}$$

for every $y=(y_k)\in\widetilde{\ell}_\Phi$. Thus, $x=(x_k)\in\ell_M(C,\Lambda)$. □

Proposition 2.2. *For each $x=(x_k)\in\ell_M(C,\Lambda)$,*

$$\sup\left\{\left|\sum_k\left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1}\right)y_k\right| : \delta(\Phi,y)\le 1\right\}<\infty. \tag{2.1}$$

Proof. Suppose that (2.1) does not hold. Then for each $n\in\mathbb{N}$, there exists y^n with $\delta(\Phi,y^n)\le 1$ such that

$$\left|\sum_k\left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1}\right)y_k^n\right|>2^{n+1}.$$

Without loss of generality, we can assume that $\sum_{j=0}^k \frac{\lambda_j x_j}{k+1},\ \ y^n\ge 0$. Now, we can define a sequence $z=(z_k)$ by $z_k=\sum_n \frac{y_k^n}{2^{n+1}}$ for all $k\in\mathbb{N}$. By the convexity of Φ, we have

$$\begin{aligned}\Phi\left(\sum_{n=0}^{l}\frac{1}{2^{n+1}}y_k^n\right) &\le \frac{1}{2}\left[\Phi(y_k^0)+\Phi\left(y_k^1+\frac{y_k^2}{2}+\cdots+\frac{y_k^l}{2^{l-1}}\right)\right]\\ &\le\cdots\le\sum_{n=0}^{l}\frac{1}{2^{n+1}}\Phi(y_k^n)\end{aligned}$$

for any positive integer l. Hence, using the continuity of Φ, we have

$$\delta(\Phi,z)=\sum_k\Phi(z_k)\le\sum_k\sum_n\frac{1}{2^{n+1}}\Phi(y_k^n)\le\sum_n\frac{1}{2^{n+1}}=1.$$

But for every $l \in \mathbb{N}$, it holds

$$\begin{aligned}\sum_k \left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1} \right) z_k &\geq \sum_k \left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1} \right) \sum_{n=0}^{l} \frac{1}{2^{n+1}} y_k^n \\ &= \sum_{n=0}^{l} \sum_k \left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1} \right) \frac{y_k^n}{2^{n+1}} \geq l.\end{aligned}$$

Hence $\sum_k \left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1} \right) z_k$ diverges and this implies that $x \notin \ell_M(C, \Lambda)$, a contradiction. This leads us to the required result. □

The preceding result encourages us to introduce the following norm $\| \cdot \|_M^C$ on $\ell_M(C, \Lambda)$.

Proposition 2.3. *The following statements hold:*

(i) $\ell_M(C, \Lambda)$ *is a normed linear space under the norm* $\| \cdot \|_M^C$ *defined by*

$$\|x\|_M^C = \sup \left\{ \left| \sum_k \left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1} \right) y_k \right| : \delta(\Phi, y) \leq 1 \right\}. \tag{2.2}$$

(ii) $\ell_M(C, \Lambda)$ *is a Banach space under the norm defined by* (2.2).

(iii) $\ell_M(C, \Lambda)$ *is a BK space under the norm defined by* (2.2).

Proof. (i) It is easy to verify that $\ell_M(C, \Lambda)$ is a linear space with respect to the co-ordinatewise addition and scalar multiplication of sequences. Now we show that $\| \cdot \|_M^C$ is a norm on the space $\ell_M(C, \Lambda)$.

If $x = 0$, then obviously $\|x\|_M^C = 0$. Conversely, assume $\|x\|_M^C = 0$. Then using the definition of the norm given by (2.2), we have

$$\sup \left\{ \left| \sum_k \left(\frac{\sum_{j=0}^k \lambda_j x_j}{k+1} \right) y_k \right| : \delta(\Phi, y) \leq 1 \right\} = 0.$$

This implies that $\left| \sum_k \left[\frac{\sum_{j=0}^k \lambda_j x_j}{k+1} \right] y_k \right| = 0$ for all y such that $\delta(\Phi, y) \leq 1$. Now considering $y = \mathrm{e}^k$ if $\Phi(1) \leq 1$ otherwise considering $y = \mathrm{e}^k / \Phi(1)$ so that $\lambda_k x_k = 0$ for all $k \in \mathbb{N}$, where e^k is a sequence whose only non-zero term is 1 in k^{th} place for each $k \in \mathbb{N}$. Hence we have $x_k = 0$ for all $k \in \mathbb{N}$, since (λ_k) is a sequence of non-zero scalars. Thus, $x = 0$.

It is easy to show that $\|\alpha x\|_M^C = |\alpha| \|x\|_M^C$ and $\|x + y\|_M^C \leq \|x\|_M^C + \|y\|_M^C$ for all $\alpha \in \mathbb{C}$ and $x, y \in \ell_M(C, \Lambda)$.

(ii) Let (x^s) be any Cauchy sequence in the space $\ell_M(C, \Lambda)$. Then for any $\varepsilon > 0$, there exists a positive integer n_0 such that $\|x^s - x^t\|_M^C < \varepsilon$ for all $s, t \geq n_0$. Using the definition of norm given by (2.2), we get

$$\sup \left\{ \left| \sum_k \left[\frac{\sum_{j=0}^k \lambda_j \left(x_j^s - x_j^t \right)}{k+1} \right] y_k \right| : \delta(\Phi, y) \leq 1 \right\} < \varepsilon$$

for all $s, t \geq n_0$. This implies that

$$\left| \sum_k \left[\frac{\sum_{j=0}^k \lambda_j \left(x_j^s - x_j^t \right)}{k+1} \right] y_k \right| < \varepsilon$$

for all y with $\delta(\Phi, y) \leq 1$ and for all $s, t \geq n_0$. Now considering $y = \mathrm{e}^k$ if $\Phi(1) \leq 1$, otherwise considering $y = \mathrm{e}^k / \Phi(1)$ we have $(\lambda_k x_k^s)$ is a Cauchy sequence in $\mathbb{C}$ for all $k \in \mathbb{N}$. Hence, it is a convergent sequence in $\mathbb{C}$ for all $k \in \mathbb{N}$.

Let $\lim_{s\to\infty} \lambda_k x_k^s = x_k$ for each $k \in \mathbb{N}$. Using the continuity of the modulus, we can derive for all $s \geq n_0$ as $t \to \infty$, that

$$\sup \left\{ \left| \sum_k \left[\frac{\sum_{j=0}^k \lambda_j \left(x_j^s - x_j \right)}{k+1} \right] y_k \right| : \delta(\Phi, y) \leq 1 \right\} < \varepsilon.$$

It follows that $(x^s - x) \in \ell_M(C, \Lambda)$. Since x^s is in the space $\ell_M(C, \Lambda)$ and $\ell_M(C, \Lambda)$ is a linear space, we have $x = (x_k) \in \ell_M(C, \Lambda)$.

(iii) From the above proof, one can easily conclude that $\|x^s\|_M^C \to 0$ implies that $x_k^s \to 0$ for each $s \in \mathbb{N}$ which leads us to the desired result.

Therefore, the proof of the theorem is completed. $\square$

Proposition 2.4. *$\ell_M(C, \Lambda)$ is a normed linear space under the norm $\|\cdot\|_{(M)}^C$ defined by*

$$\|x\|_{(M)}^C = \inf \left\{ \rho > 0 : \sum_k M \left(\frac{\left| \sum_{j=0}^k \lambda_j x_j \right|}{\rho(k+1)} \right) \leq 1 \right\}. \tag{2.3}$$

Proof. Clearly $\|x\|_{(M)}^C = 0$ if $x = 0$. Now, suppose that $\|x\|_{(M)}^C = 0$. Then, we have

$$\inf \left\{ \rho > 0 : \sum_k M \left(\frac{\left| \sum_{j=0}^k \lambda_j x_j \right|}{\rho(k+1)} \right) \leq 1 \right\} = 0.$$

This yields the fact for a given $\varepsilon > 0$ that there exists some $\rho_\varepsilon \in (0, \varepsilon)$ such that

$$\sup_{k\in\mathbb{N}} M \left(\frac{\left| \sum_{j=0}^k \lambda_j x_j \right|}{\rho_\varepsilon(k+1)} \right) \leq 1$$

which implies that

$$M \left(\frac{\left| \sum_{j=0}^k \lambda_j x_j \right|}{\rho_\varepsilon(k+1)} \right) \leq 1$$

for all $k \in \mathbb{N}$. Thus,

$$M \left(\frac{\left| \sum_{j=0}^k \lambda_j x_j \right|}{\varepsilon(k+1)} \right) \leq M \left(\frac{\left| \sum_{j=0}^k \lambda_j x_j \right|}{\rho_\varepsilon(k+1)} \right) \leq 1$$

for all $k \in \mathbb{N}$. Suppose $\frac{\left|\sum_{j=0}^{n_i} \lambda_j x_j\right|}{n_i+1} \neq 0$ for some $n_i \in \mathbb{N}$. Then, $\frac{\left|\sum_{j=0}^{n_i} \lambda_j x_j\right|}{\varepsilon(n_i+1)} \to \infty$ as $\varepsilon \to 0$. It follows that $M\left(\frac{\left|\sum_{j=0}^{n_i} \lambda_j x_j\right|}{\varepsilon(k+1)}\right) \to \infty$ as $\varepsilon \to 0$ for some $n_i \in \mathbb{N}$, which is a contradiction. Therefore, $\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{k+1} = 0$ for all $k \in \mathbb{N}$. It follows that $\lambda_k x_k = 0$ for all $k \in \mathbb{N}$. Hence $x = 0$, since (λ_k) is a sequence of non-zero scalars.

Let $x = (x_k)$ and $y = (y_k)$ be any two elements of $\ell_M(C, \Lambda)$. Then, there exist $\rho_1, \rho_2 > 0$ such that

$$\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{\rho_1(k+1)}\right) \leq 1 \quad \text{and} \quad \sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j y_j\right|}{\rho_2(k+1)}\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by the convexity of M, we have

$$\begin{aligned}\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j (x_j + y_j)\right|}{\rho(k+1)}\right) &\leq \frac{\rho_1}{\rho_1+\rho_2}\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{\rho_1(k+1)}\right) \\ &\quad + \frac{\rho_2}{\rho_1+\rho_2}\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j y_j\right|}{\rho_2(k+1)}\right) \leq 1.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|x+y\|_{(M)}^{C} = &\inf\left\{\rho > 0 : \sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j (x_j + y_j)\right|}{\rho}\right) \leq 1\right\} \\ \leq &\inf\left\{\rho_1 > 0 : \sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{\rho_1}\right) \leq 1\right\} \\ &+ \inf\left\{\rho_2 > 0 : \sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j y_j\right|}{\rho_2}\right) \leq 1\right\}\end{aligned}$$

which gives that $\|x+y\|_{(M)}^{C} \leq \|x\|_{(M)}^{C} + \|y\|_{(M)}^{C}$.

Finally, let α be any scalar and define r by $r = \rho/|\alpha|$. Then,

$$\begin{aligned}\|\alpha x\|_{(M)}^{C} &= \inf\left\{\rho > 0 : \sum_k M\left(\frac{\left|\sum_{j=0}^{k} \alpha\lambda_j x_j\right|}{\rho(k+1)}\right) \leq 1\right\} \\ &= \inf\left\{r|\alpha| > 0 : \sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{r(k+1)}\right) \leq 1\right\} = |\alpha|\|x\|_{(M)}^{C}.\end{aligned}$$

This completes the proof. □

Proposition 2.4 inspires us to define the following sequence space.

Definition 2.5. For any Orlicz function M, we define

$$\ell'_M(C,\Lambda) := \left\{ x = (x_k) \in \omega : \sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{\rho(k+1)} \right) < \infty \text{ for some } \rho > 0 \right\}.$$

Now, we can give the corresponding proposition on the space $\ell'_M(C,\Lambda)$ to the Proposition 2.3.

Proposition 2.6. *The following statements hold:*

(i) $\ell'_M(C,\Lambda)$ *is a normed linear space under the norm* $\|\cdot\|^C_{(M)}$ *defined by* (2.3).
(ii) $\ell'_M(C,\Lambda)$ *is a Banach space under the norm defined by* (2.3).
(iii) $\ell'_M(C,\Lambda)$ *is a BK space under the norm defined by* (2.3).

Proof. (i) Since the proof is similar to the proof of Proposition 2.4, we omit the detail.

(ii) Let (x^s) be any Cauchy sequence in the space $\ell'_M(C,\Lambda)$. Let $\delta > 0$ be fixed and $r > 0$ be given such that $0 < \varepsilon < 1$ and $r\delta \geq 1$. Then, there exists a positive integer n_0 such that $\|x^s - x^t\|^C_{(M)} < \varepsilon/r\delta$ for all $s, t \geq n_0$. Using the definition of the norm given by (2.3), we get

$$\inf \left\{ \rho > 0 : \sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j \left(x_j^s - x_j^t\right)\right|}{\rho(k+1)} \right) \leq 1 \right\} < \frac{\varepsilon}{r\delta}$$

for all $s, t \geq n_0$. This implies that

$$\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j \left(x_j^s - x_j^t\right)\right|}{\|x^s - x^t\|^C_{(M)}(k+1)} \right) \leq 1$$

for all $s, t \geq n_0$. It follows that

$$M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j \left(x_j^s - x_j^t\right)\right|}{\|x^s - x^t\|^C_{(M)}(k+1)} \right) \leq 1$$

for all $s, t \geq n_0$ and for all $k \in \mathbb{N}$. For $r > 0$ with $M(r\delta/2) \geq 1$, we have

$$M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j \left(x_j^s - x_j^t\right)\right|}{\|x^s - x^t\|^C_{(M)}(k+1)} \right) \leq M\left(\frac{r\delta}{2}\right)$$

for all $s, t \geq n_0$ and for all $k \in \mathbb{N}$. Since M is non-decreasing, we have

$$\frac{\left|\sum_{j=0}^{k} \lambda_j \left(x_j^s - x_j^t\right)\right|}{k+1} \leq \frac{r\delta}{2} \cdot \frac{\varepsilon}{r\delta} = \frac{\varepsilon}{2}$$

for all $s, t \geq n_0$ and for all $k \in \mathbb{N}$. Hence, $(\lambda_k x_k^s)$ is a Cauchy sequence in $\mathbb{C}$ for all $k \in \mathbb{N}$ which implies that it is a convergent sequence in $\mathbb{C}$ for all $k \in \mathbb{N}$.

Let $\lim_{s\to\infty}\lambda_k x_k^s = x_k$ for each $k\in\mathbb{N}$. Using the continuity of an Orlicz function and modulus, we can have

$$\inf\left\{\rho>0:\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j(x_j^s-x_j)\right|}{\rho(k+1)}\right)\le 1\right\}<\varepsilon$$

for all $s\ge n_0$, as $j\to\infty$. It follows that $(x^s-x)\in\ell'_M(C,\Lambda)$. Since x^s is in the space $\ell'_M(C,\Lambda)$ and $\ell'_M(C,\Lambda)$ is a linear space, we have $x=(x_k)\in\ell'_M(C,\Lambda)$.

(iii) From the above proof, one can easily conclude that $\|x^s\|_M^C\to 0$ as $s\to\infty$, which implies that $x_k^s\to 0$ as $k\to\infty$ for each $s\in\mathbb{N}$. This leads us to the desired result. □

Proposition 2.7. *The inequality* $\sum_k M\left(\frac{\left|\sum_{j=0}^k\lambda_j x_j\right|}{\|x\|_{(M)}^C(k+1)}\right)\le 1$ *holds for all* $x=(x_k)\in\ell'_M(C,\Lambda)$.

Proof. This is immediate from the definition of the norm $\|\cdot\|_{(M)}^C$ defined by (2.3). □

Definition 2.8. For any Orlicz function M, we define

$$h_M(C,\Lambda):=\left\{x=(x_k)\in\omega:\sum_k M\left(\frac{\left|\sum_{j=0}^k\lambda_j x_j\right|}{\rho(k+1)}\right)<\infty\ \text{ for each }\rho>0\right\}.$$

Clearly $h_M(C,\Lambda)$ is a subspace of $\ell'_M(C,\Lambda)$.

Here and after we shall write $\|\cdot\|$ instead of $\|\cdot\|_{(M)}^C$ provided it does not lead to any confusion. The topology of $h_M(C,\Lambda)$ is induced by $\|\cdot\|$.

Proposition 2.9. *Let M be an Orlicz function. Then, $(h_M(C,\Lambda),\|\cdot\|)$ is an AK-BK space.*

Proof. First we show that $h_M(C,\Lambda)$ is an AK space. Let $x=(x_k)\in h_M(C,\Lambda)$. Then for each $\varepsilon\in(0,1)$, we can find n_0 such that

$$\sum_{i\ge n_0} M\left(\frac{\left|\sum_{j=0}^k\lambda_j x_j\right|}{\varepsilon(k+1)}\right)\le 1.$$

Define the n^{th} section $x^{(n)}$ of a sequence $x=(x_k)$ by $x^{(n)}=\sum_{k=0}^n x_k\,\mathrm{e}^k$. Hence for $n\ge n_0$, it holds

$$\begin{aligned}\left\|x-x^{(n)}\right\| &= \inf\left\{\rho>0:\sum_{k\ge n_0} M\left(\frac{\left|\sum_{j=0}^k\lambda_j x_j\right|}{\rho(k+1)}\right)\le 1\right\}\\ &\le \inf\left\{\rho>0:\sum_{k\ge n} M\left(\frac{\left|\sum_{j=0}^k\lambda_j x_j\right|}{\rho(k+1)}\right)\le 1\right\}<\varepsilon.\end{aligned}$$

Thus, we can conclude that $h_M(C,\Lambda)$ is an AK space.

Next to show that $h_M(C,\Lambda)$ is a BK space, it is enough to show $h_M(C,\Lambda)$ is a closed subspace of $\ell'_M(C,\Lambda)$. For this, let (x^n) be a sequence in $h_M(C,\Lambda)$ such that $\|x^n - x\| \to 0$ as $n \to \infty$ where $x = (x_k) \in \ell'_M(C,\Lambda)$. To complete the proof we need to show that $x = (x_k) \in h_M(C,\Lambda)$, i.e.,

$$\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j x_j\right|}{\rho(k+1)}\right) < \infty \quad \text{for all} \quad \rho > 0.$$

There is l corresponding to $\rho > 0$ such that $\|x^l - x\| \leq \rho/2$. Then, using the convexity of M, we have by Proposition 2.7 that

$$\begin{aligned}
&\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j x_j\right|}{\rho(k+1)}\right)\\
&\quad= \sum_k M\left(\frac{2\left|\sum_{j=0}^k \lambda_j x_j^l\right| - 2\left(\left|\sum_{j=0}^k \lambda_j x_j^l\right| - \left|\sum_{j=0}^k \lambda_j x_j\right|\right)}{2\rho(k+1)}\right)\\
&\quad\leq \frac{1}{2}\sum_k M\left(\frac{2\left|\sum_{j=0}^k \lambda_j x_j^l\right|}{\rho(k+1)}\right) + \frac{1}{2}\sum_k M\left(\frac{2\left|\sum_{j=0}^k \lambda_j \left(x_j^l - x_j\right)\right|}{\rho(k+1)}\right)\\
&\quad\leq \frac{1}{2}\sum_k M\left(\frac{2\left|\sum_{j=0}^k \lambda_j x_j^l\right|}{\rho(k+1)}\right) + \frac{1}{2}\sum_k M\left(\frac{2\left|\sum_{j=0}^k \lambda_j \left(x_j^l - x_j\right)\right|}{\|x^l - x\|(k+1)}\right)\\
&\quad< \infty.
\end{aligned}$$

Hence, $x = (x_k) \in h_M(C,\Lambda)$ and consequently $h_M(C,\Lambda)$ is a BK space. □

Proposition 2.10. *Let M be an Orlicz function. If M satisfies the Δ_2-condition at 0, then $\ell'_M(C,\Lambda)$ is an AK space.*

Proof. We shall show that $\ell'_M(C,\Lambda) = h_M(C,\Lambda)$ if M satisfies the Δ_2-condition at 0. To do this it is enough to prove that $\ell'_M(C,\Lambda) \subset h_M(C,\Lambda)$. Let $x = (x_k) \in \ell'_M(C,\Lambda)$. Then for some $\rho > 0$,

$$\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j x_j\right|}{\rho(k+1)}\right) < \infty.$$

This implies that

$$\lim_{k\to\infty} M\left(\frac{\left|\sum_{j=0}^k \lambda_j x_j\right|}{\rho(k+1)}\right) = 0. \tag{2.4}$$

Choose an arbitrary $l > 0$. If $\rho \leq l$, then $\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j x_j\right|}{l(k+1)}\right) < \infty$. Now, let $l < \rho$ and put $k = \rho/l$. Since M satisfies Δ_2-condition at 0, there exist $R \equiv R_k > 0$

and $r \equiv r_k > 0$ with $M(kx) \le RM(u)$ for all $x \in (0, r]$. By (2.4), there exists a positive integer n_1 such that

$$M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{\rho(k+1)}\right) < \frac{r}{2} p\left(\frac{r}{2}\right) \quad \text{for all} \quad k \ge n_1.$$

We claim that $\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{\rho(k+1)} \le r$ for all $k \ge n_1$. Otherwise, we can find $d > n_1$ with $\frac{\left|\sum_{j=0}^{d} \lambda_j x_j\right|}{\rho(d+1)} > r$ and thus

$$M\left(\frac{\left|\sum_{j=0}^{d} \lambda_j x_j\right|}{\rho(d+1)}\right) \ge \int_{r/2}^{\left|\sum_{j=0}^{d} \lambda_j x_j\right|/\rho(d+1)} p(t)\mathrm{d}t > \frac{r}{2} p\left(\frac{r}{2}\right),$$

a contradiction. Hence our claim is true. Then, we can find that

$$\sum_{k \ge n_1} M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{l(k+1)}\right) \le R \sum_{k \ge n_1} M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{\rho(k+1)}\right).$$

Hence,

$$\sum_k M\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{l(k+1)}\right) < \infty \quad \text{for all} \quad l > 0.$$

This completes the proof. □

Proposition 2.11. *Let M_1 and M_2 be two Orlicz functions. If M_1 and M_2 are equivalent, then $\ell'_{M_1}(C,\Lambda) = \ell'_{M_2}(C,\Lambda)$ and the identity map $I: \left(\ell'_{M_1}(C,\Lambda), \|\cdot\|_{M_1}^{C}\right) \to \left(\ell'_{M_2}(C,\Lambda), \|\cdot\|_{M_2}^{C}\right)$ is a topological isomorphism.*

Proof. Let α, β and b be constants from (1.3). Since M_1 and M_2 are equivalent, it is obvious that (1.3) holds. Let us take any $x = (x_k) \in \ell'_{M_2}(C,\Lambda)$. Then,

$$\sum_k M_2\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{\rho(k+1)}\right) < \infty \quad \text{for some} \quad \rho > 0.$$

Hence, for some $l \ge 1$, $\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{l\rho(k+1)} \le b$ for all $k \in \mathbb{N}$. Therefore,

$$\sum_k M_1\left(\frac{\alpha\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{l\rho(k+1)}\right) \le \sum_k M_2\left(\frac{\left|\sum_{j=0}^{k} \lambda_j x_j\right|}{\rho(k+1)}\right) < \infty$$

which shows that the inclusion

$$\ell'_{M_2}(C,\Lambda) \subset \ell'_{M_1}(C,\Lambda) \tag{2.5}$$

holds. One can easily see in the same way that the inclusion

$$\ell'_{M_1}(C,\Lambda) \subset \ell'_{M_2}(C,\Lambda) \tag{2.6}$$

also holds.

By combining the inclusions (2.5) and (2.6), we conclude that $\ell'_{M_1}(C,\Lambda) = \ell'_{M_2}(C,\Lambda)$.

For simplicity in notation, let us write $\|\cdot\|_1$ and $\|\cdot\|_2$ instead of $\|\cdot\|_{M_1}^C$ and $\|\cdot\|_{M_2}^C$, respectively. For $x=(x_k)\in\ell'_{M_2}(C,\Lambda)$, we get

$$\sum_k M_2\left(\frac{\left|\sum_{j=0}^k \lambda_j x_j\right|}{\|x\|_2(k+1)}\right)\le 1.$$

One can find $\mu>1$ with $(b/2)\mu p_2(b/2)\ge 1$, where p_2 is the kernel associated with M_2. Hence,

$$M_2\left(\frac{\left|\sum_{j=0}^k \lambda_j x_j\right|}{\|x\|_2(k+1)}\right)\le \frac{b}{2}\mu p_2\left(\frac{b}{2}\right)\quad\text{for all}\quad k\in\mathbb{N}.$$

This implies that

$$\frac{\left|\sum_{j=0}^k \lambda_j x_j\right|}{\mu\|x\|_2(k+1)}\le b\quad\text{for all}\quad k\in\mathbb{N}.$$

Therefore,

$$\sum_k M_1\left(\frac{\alpha\left|\sum_{j=0}^k \lambda_j x_j\right|}{\mu\|x\|_2(k+1)}\right)<1.$$

Hence, $\|x\|_1\le(\mu/\alpha)\|x\|_2$. Similarly, we can show that $\|x\|_2\le\beta\gamma\|x\|_1$ by choosing γ with $\gamma\beta>1$ such that $\gamma\beta(b/2)p_1(b/2)\ge 1$. Thus, $\alpha\mu^{-1}\|x\|_1\le\|x\|_2\le\beta\gamma\|x\|_1$ which establish that I is a topological isomorphism. □

Proposition 2.12. *Let M be an Orlicz function and p be the corresponding kernel. If $p(x)=0$ for all x in $[0,b]$, where b is some positive number, then the spaces $\ell'_M(C,\Lambda)$ and $h_M(C,\Lambda)$ are topologically isomorphic to the spaces $\ell_\infty(C,\Lambda)$ and $c_0(C,\Lambda)$, respectively; where $\ell_\infty(C,\Lambda)$ and $c_0(C,\Lambda)$ are defined by*

$$\ell_\infty(C,\Lambda)=\left\{x=(x_k)\in\omega:\sup_{k\in\mathbb{N}}\sum_{j=0}^k\frac{|\lambda_j x_j|}{k+1}<\infty\right\}$$

and

$$c_0(C,\Lambda)=\left\{x=(x_k)\in\omega:\lim_{k\to\infty}\sum_{j=0}^k\frac{|\lambda_j x_j|}{k+1}=0\right\}.$$

It is easy to see that the spaces $\ell_\infty(C,\Lambda)$ and $c_0(C,\Lambda)$ are the Banach spaces under the norm $\|x\|_\infty^C=\sup_{k\in\mathbb{N}}\frac{\left|\sum_{j=0}^k\lambda_j x_j\right|}{k+1}$.

Proof. Let $p(x)=0$ for all x in $[0,b]$. If $y\in\ell_\infty(C,\Lambda)$, then we can find $\rho>0$ such that $\frac{\left|\sum_{j=0}^k\lambda_j y_j\right|}{\rho(k+1)}\le b$ for $k\in\mathbb{N}$. Hence, $\sum_k M\left(\frac{\left|\sum_{j=0}^k\lambda_j y_j\right|}{\rho(k+1)}\right)<\infty$. That is to say that $y\in\ell'_M(C,\Lambda)$.

On the other hand, let $y \in \ell'_M(C,\Lambda)$. Then for some $\rho > 0$, we have

$$\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j y_j\right|}{\rho(k+1)}\right) < \infty.$$

Therefore, $\frac{\left|\sum_{j=0}^k \lambda_j y_j\right|}{k+1} \leq K < \infty$ for a constant $K > 0$ and for all $k \in \mathbb{N}$ which yields that $y \in \ell_\infty(C,\Lambda)$. Hence, $y \in \ell_\infty(C,\Lambda)$ if and only if $y \in \ell'_M(C,\Lambda)$. We can easily find b such that $M(u_0) \geq 1$. Let $y \in \ell_\infty(C,\Lambda)$ and $\alpha = \|y\|_\infty = \sup_{k\in\mathbb{N}}\left(\left|\frac{\sum_{j=0}^k \lambda_j y_j}{k+1}\right|\right) > 0$. For every $\varepsilon \in (0,\alpha)$, we can determine d with $\left|\frac{\sum_{j=0}^d \lambda_j y_j}{d+1}\right| > \alpha - \varepsilon$ and so

$$\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j y_j\right| b}{\alpha(k+1)}\right) \geq M\left(\frac{\alpha-\varepsilon}{\alpha}b\right).$$

Since M is continuous, $\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j y_j\right| b}{\alpha(k+1)}\right) \geq 1$, and so $\|y\|_\infty \leq b\|y\|$, otherwise $\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j y_j\right|}{\|y\|(k+1)}\right) > 1$ which contradicts Proposition 2.7. Again,

$$\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j y_j\right| b}{\alpha(k+1)}\right) = 0$$

which gives that $\|y\| \leq \|y\|_\infty/b$. That is to say that the identity map $I\colon (\ell'_M(C,\Lambda), \|\cdot\|) \to (\ell_\infty(C,\Lambda), \|\cdot\|)$ is a topological isomorphism.

For the last part, let $y \in h_M(C,\Lambda)$. Then for any $\varepsilon > 0$, $\frac{\left|\sum_{j=0}^k \lambda_j y_j\right|}{k+1} \leq \varepsilon b$ for all sufficiently large k, where b is a positive number such that $p(b) > 0$. Hence, $y \in c_0(C,\Lambda)$. Conversely, let $y \in c_0(C,\Lambda)$. Then, for any $\rho > 0$, $\frac{\left|\sum_{j=0}^k \lambda_j y_j\right|}{\rho(k+1)} < b/2$ for all sufficiently large k. Thus, $\sum_k M\left(\frac{\left|\sum_{j=0}^k \lambda_j y_j\right|}{\rho(k+1)}\right) < \infty$ for all $\rho > 0$ and so $y \in h_M(C,\Lambda)$. Hence, $h_M(C,\Lambda) = c_0(C,\Lambda)$ and this step completes the proof. $\square$

Prior to giving our final two consequences concerning the α-dual of the spaces $\ell'_M(C,\Lambda)$ and $h_M(C,\Lambda)$, we present the following easy lemma without proof.

Lemma 2.13. *For any Orlicz function M, $\Lambda x = (\lambda_k x_k) \in \ell_\infty$ whenever $x = (x_k) \in \ell'_M(C,\Lambda)$.*

Proposition 2.14. *Let M be an Orlicz function and p be the corresponding kernel of M. Define the sets D_1 and D_2 by*

$$D_1 := \left\{a = (a_k) \in \omega : \sum_k \left|\lambda_k^{-1} a_k\right| < \infty\right\}$$

and

$$D_2 := \left\{ b = (b_k) \in \omega : \sup_{k \in \mathbb{N}} |\lambda_k b_k| < \infty \right\}.$$

If $p(x) = 0$ for all x in $[0, d]$, where d is some positive number, then the following statements hold:

(i) Köthe-Toeplitz dual of $\ell'_M(C, \Lambda)$ is the set D_1.
(ii) Köthe-Toeplitz dual of D_1 is the set D_2.

Proof. Since the proof of Part (ii) is similar to that of the proof of Part (i), to avoid the repetition of the similar statements we prove only Part (i).

Let $a = (a_k) \in D_1$ and $x = (x_k) \in \ell'_M(C, \Lambda)$. Then, since

$$\sum_k |a_k x_k| = \sum_k \left| a_k \lambda_k^{-1} \right| |\lambda_k x_k| \leq \sup_{k \in \mathbb{N}} |\lambda_k x_k| \cdot \sum_k \left| a_k \lambda_k^{-1} \right| < \infty,$$

applying Lemma 2.13, we have $a = (a_k) \in \{\ell'_M(C, \Lambda)\}^\alpha$. Hence, the inclusion

$$D_1 \subset \{\ell'_M(C, \Lambda)\}^\alpha \tag{2.7}$$

holds.

Conversely, suppose that $a = (a_k) \in \{\ell'_M(C, \Lambda)\}^\alpha$. Then, $(a_k x_k) \in \ell_1$, the space of all absolutely convergent series, for every $x = (x_k) \in \ell'_M(C, \Lambda)$. So, we can take $x_k = \lambda_k^{-1}$ for all $k \in \mathbb{N}$ because $(x_k) \in \ell'_M(C, \Lambda)$ by Proposition 2.12 whenever $(x_k) \in \ell_\infty(C, \Lambda)$. Therefore, $\sum_k \left| a_k \lambda_k^{-1} \right| = \sum_k |a_k x_k| < \infty$ and we have $a = (a_k) \in D_1$. This leads us to the inclusion

$$\{\ell'_M(C, \Lambda)\}^\alpha \subset D_1. \tag{2.8}$$

By combining the inclusion relations (2.7) and (2.8), we have $\{\ell'_M(C, \Lambda)\}^\alpha = D_1$. □

Proposition 2.14 (ii) shows that $\{\ell'_M(C, \Lambda)\}^{\alpha\alpha} \neq \ell'_M(C, \Lambda)$ which leads us to the consequence that $\ell'_M(C, \Lambda)$ is not perfect under the given conditions.

Proposition 2.15. *Let M be an Orlicz function and p be the corresponding kernel of M and the set D_1 be defined as in the Proposition 2.14. If $p(x) = 0$ for all x in $[0, b]$, where b is a positive number, then the Köthe-Toeplitz dual of $h_M(C, \Lambda)$ is the set D_1.*

Proof. Let $a = (a_k) \in D_1$ and $x = (x_k) \in h_M(C, \Lambda)$. Then, since

$$\sum_k |a_k x_k| = \sum_k \left| a_k \lambda_k^{-1} \right| |\lambda_k x_k| \leq \sup_{k \in \mathbb{N}} |\lambda_k x_k| \cdot \sum_k \left| a_k \lambda_k^{-1} \right| < \infty,$$

we have $a = (a_k) \in \{h_M(C, \Lambda)\}^\alpha$. Hence, the inclusion

$$D_1 \subset \{h_M(C, \Lambda)\}^\alpha \tag{2.9}$$

holds.

Conversely, suppose that $a = (a_k) \in \{h_M(C,\Lambda)\}^\alpha \setminus D_1$. Then, there exists a strictly increasing sequence (n_i) of positive integers n_i such that

$$\sum_{k=n_i+1}^{n_{i+1}} |a_k|\,|\lambda_k|^{-1} > i.$$

Define $x = (x_k)$ by

$$x_k := \begin{cases} \lambda_k^{-1} sgn\ a_k/i & , \quad (n_i < k \le n_{i+1}), \\ 0 & , \quad (0 \le k < n_0), \end{cases}$$

for all $k \in \mathbb{N}$. Then, since $x = (x_k) \in c_0(C,\Lambda)$, $x = (x_k) \in h_M(C,\Lambda)$ by Proposition 2.12. Therefore, we have

$$\begin{aligned} \sum_k |a_k x_k| &= \sum_{k=n_0+1}^{n_1} |a_k x_k| + \cdots + \sum_{k=n_i+1}^{n_{i+1}} |a_k x_k| + \cdots \\ &= \sum_{k=n_0+1}^{n_1} \left|a_k \lambda_k^{-1}\right| + \cdots + \frac{1}{i} \sum_{k=n_i+1}^{n_{i+1}} \left|a_k \lambda_k^{-1}\right| + \cdots \\ &> 1 + \cdots + 1 + \cdots = \infty, \end{aligned}$$

which contradicts the hypothesis. Hence, $a = (a_k) \in D_1$. This leads us to the inclusion

$$\{h_M(C,\Lambda)\}^\alpha \subset D_1. \tag{2.10}$$

By combining the inclusion relations (2.9) and (2.10), we obtain the desired result $\{h_M(C,\Lambda)\}^\alpha = D_1$.

This completes the proof. □

3. Conclusion

The difference Orlicz spaces $\ell_M(\Delta,\Lambda)$ and $\widetilde{\ell}_M(\Delta,\Lambda)$ were recently been studied by Dutta [**2**]. Of course, the sequence spaces introduced in this paper can be redefined as a domain of a suitable matrix in the Orlicz sequence space ℓ_M. Indeed, if we define the infinite matrix $C(\lambda) = \{c_{nk}(\lambda)\}$ via the multiplier sequence $\Lambda = (\lambda_k)$ by

$$c_{nk}(\lambda) := \begin{cases} \dfrac{\lambda_k}{n+1}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$, then the sequence spaces $\ell'_M(C,\Lambda)$, $c_0(C,\Lambda)$ and $\ell_\infty(C,\Lambda)$ represent the domain of the matrix $C(\lambda)$ in the sequence spaces ℓ_M, c_0 and ℓ_∞, respectively. Since $c_{nn}(\lambda) \neq 0$ for all $n \in \mathbb{N}$, i.e., $C(\lambda)$ is a triangle, it is obvious that those spaces $\ell'_M(C,\Lambda)$, $c_0(C,\Lambda)$ and $\ell_\infty(C,\Lambda)$ are linearly isomorphic to the spaces ℓ_M, c_0 and ℓ_∞, respectively.

Although some algebraic and topological properties of these new spaces are investigated, the following further suggestions remain open:

(i) What is the relation between the norms $\|\cdot\|_M^C$ and $\|\cdot\|_{(M)}^C$? Are they equivalent?
(ii) What is the relation between the spaces $\ell_M(C,\Lambda)$ and $\ell'_M(C,\Lambda)$? Do they coincide?
(iii) What are the β- and γ-duals of the spaces $\ell'_M(C,\Lambda)$ and $h_M(C,\Lambda)$?
(iv) Under which conditions an infinite matrix transforms the sets $\ell'_M(C,\Lambda)$ or $h_M(C,\Lambda)$ to the sequence spaces ℓ_∞ and c?

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References

1. Chen S., *Geometry of Orlicz spaces.* Dissertationes Math. **356** (1996), 1–204.
2. Dutta H., *On Orlicz difference sequence spaces.* SDU Journal of Science **5** (2010), 119–136.
3. Gribanov Y., *On the theory of ℓ_M spaces.* Uc. Zap. Kazansk un-ta, **117** (1957), 62–65, (in Russian).
4. Goes G.and Goes S., *Sequences of bounded variation and sequences of Fourier coefficients.* Math. Z. **118** (1970), 93–102.
5. Kamthan P. K., *Bases in a certain class of Frechet spaces.* Tamkang J. Math. **7** (1976), 41–49.
6. Kamthan P. K. and Gupta M., *Sequence Spaces and Series.* Marcel Dekker Inc., New York and Basel, 1981.
7. Krasnoselskii M. A. and Rutickii Y. B., *Convex functions and Orlicz spaces.* Groningen, Netherlands, 1961.
8. Köthe G. and Toeplitz O., *Linear Raume mit unendlichvielen koordinaten and Ringe unenlicher Matrizen.* J. Reine Angew. Math. **171** (1934), 193–226.
9. Lindenstrauss J. and Tzafriri L., *On Orlicz sequence spaces.* Israel J. Math. **10** (1971), 379–390.
10. Musielak J., *Orlicz Spaces and Modular Spaces.* Lecture Notes in Mathematics 1083, Springer–Verlag, 1983.
11. Maddox I. J., *Elements of Functional Analysis.* The University Press, 2nd ed. Cambridge, 1988.
12. Tripathy B. C., *A class of difference sequences related to the p-normed space ℓ^p.* Demonstratio Math. **36(4)** (2003), 867–872.

H. Dutta, Department of Mathematics, Gauhati University, Kokrajhar Campus, Assam, India, *e-mail*: `hemen_dutta08@rediffmail.com`

F. Başar, Fatih Üniversitesi, Fen-Edebiyat Fakültesi, Matematik Bölümü, Büyükçekmece Kampüsü 34500-Istanbul, Türkiye, *e-mail*: `fbasar@fatih.edu.tr, feyzibasar@gmail.com`