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ALMOST EVERYWHERE STRONG SUMMABILITY OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

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Dedicated to Professor Ferenc Schipp on the occasion of his 75th birthday, to Professor William Wade on the occasion of his 70th birthday and to Professor Péter Simon on the occasion of his 65th birthday.

Abstract. A BMO-estimation of two-dimensional Walsh-Fourier series is proved from which an almost everywhere exponential summability of quadratic partial sums of double Walsh-Fourier series is derived.

1. Introduction

We shall denote the set of all non-negative integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $\mathbb{I} := [0,1)$ by $\mathbb{Q}$. In particular, each element of $\mathbb{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \le p < 2^n$. Set $I_N := [0, 2^{-N})$, $I_N(x) := x \oplus I_N$, where $\oplus$ is the dyadic addition (see [34]).

Let $r_0(x)$ be the function defined by

$$r_0(x) = \begin{cases} 1, \text{ if } x \in [0,1/2) \\ -1, \text{ if } x \in [1/2,1) \end{cases}, \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \qquad n \ge 1.$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e. $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s$ then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

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The Walsh-Dirichlet kernel is defined by

$$D_n(x)=\sum_{k=0}^{n-1} w_k(x),\ n\in\mathbb{N}.$$

Given $x\in\mathbb{I}$, the expansion

$$x=\sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \tag{1}$$

where each $x_k=0$ or 1, will be called a dyadic expansion of x. If $x\in\mathbb{I}\backslash\mathbb{Q}$, then (1) is uniquely determined. For the dyadic expansion $x\in\mathbb{Q}$ we choose the one for which $\lim_{k\to\infty} x_k=0$.

The dyadic sum of $x,y\in\mathbb{I}$ in terms of the dyadic expansion of x and y is defined by

$$x\oplus y=\sum_{k=0}^{\infty}\left|x_k-y_k\right|2^{-(k+1)}.$$

We consider the double system $\{w_n(x)\times w_m(y): n,m\in\mathbb{N}\}$ on the unit square $\mathbb{I}^2=[0,1)\times[0,1)$. The notation $a\lesssim b$ in the whole paper stands for $a\leq c\cdot b$, where c is an absolute constant.

The norm (or quasinorm) of the space $L_p(\mathbb{I}^2)$ is defined by

$$\|f\|_p:=\left(\int_{\mathbb{I}^2}|f(x,y)|^p\,dxdy\right)^{1/p}\qquad(0<p<+\infty).$$

If $f\in L_1(\mathbb{I}^2)$, then

$$\hat{f}(n,m)=\int_{I^2} f(x,y)\,w_n(x)w_m(y)dxdy$$

is the (n,m)-th Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N}(x,y;f)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\hat{f}(m,n)\,w_m(x)w_n(y).$$

Denote

$$S_n^{(1)}(x,y;f):=\sum_{l=0}^{n-1}\widehat{f}(l,y)\,w_l(x),$$

$$S_m^{(2)}(x,y;f):=\sum_{r=0}^{m-1}\widehat{f}(x,r)\,w_r(y),$$

where

$$\widehat{f}(l,y)=\int\limits_{\mathbb{I}} f(x,y)\,w_l(x)\,dx$$

and

$$\widehat{f}(x,r)=\int\limits_{\mathbb{I}} f(x,y)\,w_r(y)\,dy$$

Recall the definition of $BMO[\mathbb{I}]$ space. It is the Banach space of functions $f\in L_1(\mathbb{I})$ with the norm

$$\|f\|_{BMO}:=\sup_I\left(\frac{1}{|I|}\int\limits_I |f-f_I|^2\right)^{1/2}+\left|\int\limits_{\mathbb{I}} f\right|$$

and the supremum is taken over all dyadic intervals $I\subset\mathbb{I}$.

Let $\xi:=\{\xi_n : n=0,1,2,\ldots\}$ be an arbitrary sequence of numbers. Taking

$$\delta_k^n:=\left[\frac{k}{2^n},\frac{k+1}{2^n}\right),$$

we define

$$BMO[\xi_n]:=\sup_{0\le n<\infty}\left\|\sum_{k=0}^{2^n-1}\xi_k\mathbb{I}_{\delta_k^n}(t)\right\|_{BMO},$$

where $\mathbb{I}_E$ is the characteristic function of $E\subset\mathbb{I}$.

Set

$$F:=\{J:=[j2^m,(j+1)2^m)\cap\mathbb{N},j,m\in\mathbb{N}\}.$$

Then F is the collection of integer dyadic intervals. The number of elements in $J\in F$ will be denoted by $|J|$. The mean value of the sequence $\xi:=\{\xi_n : n=0,1,2,\ldots\}$ with respect to J is defined by

$$\xi^J:=\frac{1}{|J|}\sum_{l\in J}\xi_l.$$

Then it is easy to see that

$$BMO[\xi_n]=\sup_{J\in F}\left(\frac{1}{|J|}\sum_{k\in J}\left|\xi_k-\xi^J\right|^2\right)^{1/2}.$$

We denote by $L(\log L)^\alpha(\mathbb{I}^2)$ the class of measurable functions f, with

$$\int\limits_{\mathbb{I}^2}|f|\left(\log^+|f|\right)^\alpha<\infty,$$

where $\log^+ u:=\mathbb{I}_{(1,\infty)}(u)\log u$.

Denote by $S_n^T(x,f)$ the partial sums of the trigonometric Fourier series of f and let

$$\sigma_n^T(x,f)=\frac{1}{n+1}\sum_{k=0}^{n}S_k^T(x,f)$$

be the $(C,1)$ means. Fejr [1] proved that $\sigma_n^T(f)$ converges to f uniformly for any 2π-periodic continuous function. Lebesgue in [19] established almost everywhere convergence of $(C,1)$ means if $f\in L_1(\mathbb{T}),\mathbb{T}:=[-\pi,\pi)$. The strong summability problem, i.e. the convergence of the strong means

$$\frac{1}{n+1}\sum_{k=0}^{n}\left|S_k^T(x,f)-f(x)\right|^p,\quad x\in\mathbb{T},\quad p>0, \tag{2}$$

was first considered by Hardy and Littlewood in [16]. They showed that for any $f\in L_r(\mathbb{T})$ $(1<r<\infty)$ the strong means tend to 0 a.e., if $n\to\infty$. The Fourier series of $f\in L_1(\mathbb{T})$ is said to be (H,p)-summable at $x\in T$, if the values (2) converge to 0 as $n\to\infty$. The (H,p)-summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [24] for $p=2$, and later by Zygmund [44] for the general case $1\le p<\infty$. Oskolkov in [26] proved the following: Let $f\in L_1(\mathbb{T})$ and let Φ be a continuous positive convex function on $[0,+\infty)$ with $\Phi(0)=0$ and

$$\ln\Phi(t)=O(t/\ln\ln t)\qquad(t\to\infty). \tag{3}$$

Then for almost all x

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\Phi\left(\left|S_k^T(x,f)-f(x)\right|\right)=0. \tag{4}$$

It was noted in [26] that Totik announced the conjecture that (4) holds almost everywhere for any $f\in L_1(\mathbb{T})$, provided

$$\ln\Phi(t)=O(t)\quad(t\to\infty). \tag{5}$$

In [28] Rodin proved

Theorem R. *Let $f\in L_1(\mathbb{T})$. Then for any $A>0$*

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left(\exp\left(A\left|S_k^T(x,f)-f(x)\right|\right)-1\right)=0$$

for a.e. $x\in\mathbb{T}$.

Karagulyan [17] proved that the following is true.

Theorem K. *Suppose that a continuous increasing function $\Phi:[0,\infty)\to[0,\infty),\Phi(0)=0$, satisfies the condition*

$$\limsup_{t\to+\infty}\frac{\log\Phi(t)}{t}=\infty.$$

Then there exists a function $f \in L_1(\mathbb{T})$ for which the relation

$$\limsup_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi\left(\left|S_k^T(x,f)\right|\right) = \infty$$

holds everywhere on $\mathbb{T}$.

For quadratic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [25] has proved, that if $f \in L\log L(\mathbb{T}^2)$,$\mathbb{T} := [-\pi,\pi)^2$, then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left(S_{kk}^T(x,y,f) - f(x,y)\right) = 0$$

for a.e. $(x,y) \in \mathbb{T}^2$. Zhizhiashvili [43] improved this result showing that class $L\log L(\mathbb{T}^2)$ can be replaced by $L_1(\mathbb{T}^2)$.

From a result of Konyagin [18] it follows that for every $\varepsilon > 0$ there exists a function $f \in L\log^{1-\varepsilon}(\mathbb{T}^2)$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \left|S_{kk}^T(x,y,f) - f(x,y)\right| \neq 0 \text{ for a.e. } (x,y) \in \mathbb{T}^2. \tag{6}$$

These results show that in the one dimensional case we have the same maximal spaces for $(C,1)$ summability and for $(C,1)$ strong summability. That is, in both cases we have $L_1(\mathbb{T})$. But, the situation changes as we step further to the case of two dimensional functions. In other words, the spaces of functions with almost everywhere summable Marcinkiewicz and strong Marcinkiewicz means are different.

In [12] a BMO-estimation of two-dimensional trigonometric Fourier series is proved from which an almost everywhere exponential summability of quadratic partial sums of double Fourier series is derived.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems. For instance, concerning the Walsh system see Schipp [30, 31, 33], Fridli and Schipp [2, 3], Leindler [19, 20, 21, 22, 23], Totik [36, 37, 38], Rodin [27], Weisz [41, 40], Gabisoniya [4].

The problems of summability of cubical partial sums of multiple Fourier series have been investigated by Gogoladze [13, 14, 15], Wang [39], Zhag [42], Glukhov [9], Goginava [10], Gát, Goginava, Tkebuchava [7], Goginava, Gogoladze [11].

For Walsh system Rodin [29] (see also Schipp [32]) proved that the following is true.

Theorem R2 (Rodin)**.** *If $\Phi(t) : [0,\infty) \to [0,\infty)$, $\Phi(0) = 0$, is an increasing continuous function satisfying*

$$\limsup_{t\to\infty} \frac{\log\Phi(t)}{t} < \infty, \tag{7}$$

then the partial sums of Walsh-Fourier series of any function $f \in L^1(\mathbb{I})$ *satisfy the condition*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Phi\left(\left|S_k(x;f)-f(x)\right|\right)=0$$

almost everywhere on $\mathbb{I}$.

In the paper [6] we established, that, as in trigonometric case [17], the bound (7) is sharp for a.e. Φ-summability of Walsh-Fourier series. Moreover, we prove

Theorem GGK1. *If an increasing function* $\Phi(t):[0,\infty)\to[0,\infty)$ *satisfies the condition*

$$\limsup_{t\to\infty}\frac{\log\Phi(t)}{t}=\infty,$$

then there exists a function $f \in L^1(\mathbb{I})$ *such that*

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Phi\left(\left|S_k(x;f)\right|\right)=\infty$$

holds everywhere on $[0,1)$.

Schipp in [32] introduced the following operator

$$V_n(x;f):=\left(\frac{1}{2^n}\int_{\mathbb{I}}\left(\sum_{j=0}^{n-1}2^{j-1}\mathbb{I}_{I_j}(t)\,S_{2^n}f(x\oplus t\oplus e_j)\right)^2dt\right)^{1/2}.$$

Let

$$V(f):=\sup_n V_n(f).$$

The following theorem is proved by Schipp.

Theorem Sch ([32])**.** *Let* $f\in L_1(\mathbb{I})$. *Then*

$$\mu\{|Vf|>\lambda\}\lesssim\frac{\|f\|_1}{\lambda}.$$

Set

$$H_n^pf:=\left(\frac{1}{2^n}\sum_{m=0}^{2^n-1}|S_{mm}f|^p\right)^{1/p}$$

and the maximal strong operator

$$H_*^pf:=\sup_{n\in\mathbb{N}}H_n^pf,\qquad p>0.$$

In [5] we studied the a.e. convergence of strong Marcinkiewicz means of the two-dimensional Walsh-Fourier series. In particular, the following is true.

Theorem GGK2. *Let $f \in L\log L\left(\mathbb{I}^2\right)$ and $p>0$. Then*

$$\mu\left\{H_*^p f>\lambda\right\} \lesssim \frac{1}{\lambda}\left(1+\int\limits_{\mathbb{I}^2}|f|\log^+|f|\right).$$

The weak type $\left(L\log^+ L,1\right)$ inequality and the usual density argument of Marcinkiewicz and Zygmund imply

Theorem GGK3. *Let $f \in L\log L\left(\mathbb{I}^2\right)$ and $p>0$. Then*

$$\left(\frac{1}{n}\sum_{m=0}^{n-1}\left|S_{mm}(x,y,f)-f(x,y)\right|^p\right)^{1/p} \to 0 \text{ for a.e. } (x,y)\in\mathbb{I}^2 \text{ as } n\to\infty.$$

We note that from the theorem of Getsadze [8] it follows that the class $L\log L$ in the last theorem is necessary in the context of strong summability question. That is, it is not possible to give a larger convergence space (of the form $L\log L\phi(L)$ with $\phi(\infty)=0$) than $L\log L$. This means a sharp contrast between the one and two dimensional strong summability.

In [11] the exponential uniform strong approximation of the Marcinkiewicz means of the two-dimensional Walsh-Fourier series was studied. We say that the function ψ belongs to the class Ψ if it increase on $[0,+\infty)$ and

$$\lim_{u\to 0}\psi(u)=\psi(0)=0.$$

Theorem GG ([11]). *a)Let $\varphi\in\Psi$ and let the inequality*

$$\overline{\lim_{u\to\infty}}\frac{\varphi(u)}{\sqrt{u}}<\infty$$

hold. Then for any function $f\in C\left(\mathbb{I}^2\right)$ the equality

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{l=1}^{n}\left(e^{\varphi(|S_{ll}(f)-f|)}-1\right)\right\|_C=0$$

is satisfied.

b) For any function $\varphi\in\Psi$ satisfying the condition

$$\overline{\lim_{u\to\infty}}\frac{\varphi(u)}{\sqrt{u}}=\infty$$

there exists a function $F\in C\left(\mathbb{I}^2\right)$ such that

$$\overline{\lim_{m\to\infty}}\frac{1}{m}\sum_{l=1}^{m}\left(e^{\varphi(|S_{ll}(0,0,F)-F(0,0)|)}-1\right)=+\infty.$$

In this paper we study a BMO-estimation for quadratic partial sums of two-dimensional Walsh-Fourier series from which an almost everywhere exponential summability of quadratic partial sums of double Walsh-Fourier series is derived.

Theorem 1. *If* $f \in L(\log L)^2(\mathbb{I}^2)$, *then*

$$\mu\left\{(x,y)\in I^2 : BMO\left[S_{nn}(x,y;f)\right]>\lambda\right\} \lesssim \frac{1}{\lambda}\left(1+\int\limits_{\mathbb{I}^2}|f|\left(\log|f|\right)^2\right).$$

The following theorem shows that the quadratic sums of two-dimensional Walsh-Fourier series of a function $f \in L(\log L)^2(\mathbb{I}^2)$ are almost everywhere exponentially summable to the function f. It will be obtained from the previous theorem by using the John-Nirenberg theorem (see ([12]).

Theorem 2. *Suppose that* $f \in L(\log L)^2(\mathbb{I}^2)$. *Then for any* $A>0$

$$\lim_{m\to\infty}\frac{1}{m}\sum_{n=1}^{m}\left(\exp\left(A\left|S_{nn}(x,y;f)-f(x,y)\right|\right)-1\right)=0$$

for a.e. $(x,y)\in\mathbb{I}^2$.

2. Proof of Theorem

Let $f\in L_1(\mathbb{I}^2)$. Then the dyadic maximal function is given by

$$Mf(x,y):=\sup_{n\in\mathbb{N}}2^{2n}\int\limits_{I_n(x)\times I_n(y)}|f(s,t)|\,dsdt.$$

For a two-dimensional integrable function f we need to introduce the following hybrid maximal functions

$$M_1f(x,y):=\sup_{n\in\mathbb{N}}2^{n}\int\limits_{I_n(x)}|f(s,y)|\,ds,$$

$$M_2f(x,y):=\sup_{n\in\mathbb{N}}2^{n}\int\limits_{I_n(y)}|f(x,t)|\,dt,$$

(8) $$V_1(x,y,f) :=\sup_{n\in\mathbb{N}}\left(\frac{1}{2^n}\int\limits_{\mathbb{I}}\left(\sum_{j=0}^{n-1}2^{j-1}\mathbb{I}_{I_j}(t)\,S_{2^n}^{(1)}f(x\oplus t\oplus e_j,y)\right)^2dt\right)^{1/2},$$

(9) $$V_2(x,y,f) :=\sup_{n\in\mathbb{N}}\left(\frac{1}{2^n}\int\limits_{\mathbb{I}}\left(\sum_{j=0}^{n-1}2^{j-1}\mathbb{I}_{I_j}(t)\,S_{2^n}^{(2)}f(x,y\oplus t\oplus e_j)\right)^2dt\right)^{1/2}.$$

It is well known [44] that for $f\in L\log^+L$ the following estimation holds

(10) $$\lambda\mu\{Mf>\lambda\}\lesssim 1+\int\limits_{\mathbb{I}^2}|f|\log^+|f|$$

and for $s=1,2$

$$\int\limits_{\mathbb{I}^2} M_s f \lesssim 1+\int\limits_{\mathbb{I}^2} |f|\log^+|f|\,, \tag{11}$$

$$\mu\{: V_s(f)>\lambda\} \lesssim \frac{\|f\|_1}{\lambda}, \qquad f\in L_1\left(\mathbb{I}^2\right). \tag{12}$$

It is proved in [5] that the following estimation holds

$$\begin{aligned} &\left(\frac{1}{2^n}\sum_{m=0}^{2^n-1}|S_{mm}(x,y,f)|^2\right)^{1/2} \\ &\lesssim V_2(x,y,M_1f)+V_1(x,y,M_2f)+Mf(x,y) \\ &\quad +V_2(x,y,A)+V_1(x,y,A)+\|f\|_1\,, \end{aligned} \tag{13}$$

where A is an integrable and nonegative function on $\mathbb{I}^2$ of two variable for which

$$\int\limits_{\mathbb{I}^2} A \lesssim 1+\int\limits_{\mathbb{I}^2} |f|\log^+|f|\,, \qquad f\in L\log L. \tag{14}$$

Proof of Theorem 1. We can write

$$\begin{aligned} &BMO\left[S_{nn}(x,y;f)\right] \\ &=\sup_{m,j}\left(\frac{1}{2^m}\sum_{l=j2^m}^{(j+1)2^m-1}\left|S_{ll}(x,y;f)-\frac{1}{2^m}\sum_{q=j2^m}^{(j+1)2^m-1}S_{qq}(x,y;f)\right|^2\right)^{1/2} \\ &=\sup_{m,j}\left(\frac{1}{2^m}\sum_{l=0}^{2^m-1}\left|S_{l+j2^m,l+j2^m}(x,y;f)\right.\right. \\ &\quad \left.\left.-\frac{1}{2^m}\sum_{q=0}^{2^m-1}S_{q+j2^m,q+j2^m}(x,y;f)\right|^2\right)^{1/2}. \end{aligned} \tag{15}$$

Since $(0\le l<2^m)$

$$\begin{aligned} S_{l+j2^m,l+j2^m}(x,y;f) &= S_{j2^m,j2^m}(x,y;f)+S_{j2^m,l}\left(x,y;fw_{j2^m}(x)\right)w_{j2^m}(y) \\ &\quad +S_{l,j2^m}\left(x,y;fw_{j2^m}(y)\right)w_{j2^m}(x) \\ &\quad +S_{l,l}\left(x,y;fw_{j2^m}(x)\otimes w_{j2^m}(y)\right)w_{j2^m}(x)\,w_{j2^m}(y) \end{aligned}$$

from (15) we obtain

$$\begin{aligned} &BMO\left[S_{nn}(x,y;f)\right] \\ &\le \sup_{m,j}\left(\frac{1}{2^m}\sum_{l=0}^{2^m-1}\left|S_{l,l}\left(x,y;fw_{j2^m}(x)\otimes w_{j2^m}(y)\right)\right.\right. \end{aligned} \tag{16}$$

$$
\left.\left.-\frac{1}{2^m}\sum_{q=0}^{2^m-1}S_{q,q}\left(x,y;fw_{j2^m}\left(x\right)\otimes w_{j2^m}\left(y\right)\right)\right|^2\right)^{1/2}
$$

$$
+\sup_{m,j}\left(\frac{1}{2^m}\sum_{l=0}^{2^m-1}\left|S_{l,j2^m}\left(x,y;fw_{j2^m}\left(y\right)\right)\right.\right.
$$

$$
\left.\left.-\frac{1}{2^m}\sum_{q=0}^{2^m-1}S_{q,j2^m}\left(x,y;fw_{j2^m}\left(y\right)\right)\right|^2\right)^{1/2}
$$

$$
+\sup_{m,j}\left(\frac{1}{2^m}\sum_{l=0}^{2^m-1}\left|S_{j2^m,l}\left(x,y;fw_{j2^m}\left(x\right)\right)\right.\right.
$$

$$
\left.\left.-\frac{1}{2^m}\sum_{q=0}^{2^m-1}S_{j2^m,q}\left(x,y;fw_{j2^m}\left(x\right)\right)\right|^2\right)^{1/2}
$$

$$
\begin{aligned}
&\leq 2\sup_{m,j}\left(\frac{1}{2^m}\sum_{l=0}^{2^m-1}\left|S_{l,l}\left(x,y;fw_{j2^m}\left(x\right)\otimes w_{j2^m}\left(y\right)\right)\right|^2\right)^{1/2}\\
&\quad+2\sup_{m,j}\left(\frac{1}{2^m}\sum_{l=0}^{2^m-1}\left|S_{l,j2^m}\left(x,y;fw_{j2^m}\left(y\right)\right)\right|^2\right)^{1/2}\\
&\quad+2\sup_{m,j}\left(\frac{1}{2^m}\sum_{l=0}^{2^m-1}\left|S_{j2^m,l}\left(x,y;fw_{j2^m}\left(x\right)\right)\right|^2\right)^{1/2}\\
&:=T_1+T_2+T_3.
\end{aligned}
$$

From (13) we have

$$
\begin{aligned}
T_1 &\lesssim V_2\left(x,y,M_1f\right)+V_1\left(x,y,M_2f\right)+Mf\left(x,y\right)\\
&\quad+V_2\left(x,y,A\right)+V_1\left(x,y,A\right)+\left\|f\right\|_1.
\end{aligned}
\tag{17}
$$

Since

$$
S_{l,j2^m}\left(x,y;fw_{j2^m}\right)=S_l^{(1)}\left(x,y;S_{j2^m}^{(2)}w_{j2^m}\right)
$$

for T_2 we can write

$$
T_2\lesssim\sup_{m,j}\left(\frac{1}{2^m}\sum_{l=0}^{2^m-1}\left|S_l^{(1)}\left(x,y;S_{j2^m}^{(2)}\left(fw_{j2^m}\right)\right)\right|^2\right)^{1/2}.
\tag{18}
$$

Schipp proved the following estimation (see [32])

$$
\left(\frac{1}{2^m}\sum_{l=0}^{2^m-1}\left|S_l\left(x;f\right)\right|^2\right)^{1/2}\lesssim V\left(x,f\right).
\tag{19}
$$

Combining (18) and (19) we get

$$T_2 \lesssim \sup_{m,j} V_1 \left(x, y; \left|S^{(2)}_{j2^m}(f)\right|\right) \lesssim V_1 \left(x, y; S^{(2)}_{*}(f)\right),$$

where

$$S^{(2)}_{*}(x, y; f) := \sup_{n} \left|S^{(2)}_{n}(x, y; f)\right|.$$

Let $f \in L(\log L)^2(\mathbb{I}^2)$. Then $f(x, \cdot) \in L(\log L)^2(\mathbb{I})$ for a.e. $x \in \mathbb{I}$,and from the well-known theorem ([35]) $S^{(2)}_{*}(x, \cdot; f) \in L_1(\mathbb{I})$ for a.e. $x \in \mathbb{I}$. Moreover,

$$\int_{\mathbb{I}} \left|S^{(2)}_{*}(x, y; f)\right| dy \lesssim \left(\int_{\mathbb{I}} |f(x, y)| \left(\log^{+} |f(x, y)|\right)^2 dy + 1\right) \tag{20}$$

for a.e. $x \in \mathbb{I}$.

Setting

$$\Omega := \left\{(x, y) \in \mathbb{I}^2 : V_1(x, y, f) > \lambda\right\}.$$

we can use Fubini's Theorem and Theorem 1 to write

$$\begin{aligned} |\Omega| &= \int_{\mathbb{I}^2} 1_{\Omega}(x, y)\, dxdy \\ &= \int_{\mathbb{I}} \left(\int_{\mathbb{I}} 1_{\Omega}(x, y)\, dx\right) dy \\ &\lesssim \frac{1}{\lambda} \int_{\mathbb{I}} \left(\int_{\mathbb{I}} |f(x, y)|\, dx\right) dy. \end{aligned} \tag{21}$$

Consequently, from (20) we obtain

$$\begin{aligned} &\left|\left\{(x, y) \in \mathbb{I}^2 : V_1\left(x, y; S^{(2)}_{*}(f)\right) > \lambda\right\}\right| \\ &\lesssim \frac{1}{\lambda} \int_{\mathbb{I}} \left(\int_{\mathbb{I}} \left|S^{(2)}_{*}(x, y; f)\right| dx\right) dy \\ &\lesssim \frac{1}{\lambda} \int_{\mathbb{I}} \left(\int_{\mathbb{I}} |f(x, y)| \left(\log^{+} |f(x, y)|\right)^2 dy + 1\right) dx \\ &= \frac{c}{\lambda} \left(\int_{\mathbb{I}^2} \left(|f(x, y)| \left(\log^{+} |f(x, y)|\right)^2 + 1\right) dxdy\right)^{1/2}. \end{aligned} \tag{22}$$

Analogously, we can prove that

$$|\{T_3>\lambda\}| \lesssim \frac{1}{\lambda}\left(\int\limits_{\mathbb{I}^2}\left(|f(x,y)|\left(\log^+|f(x,y)|\right)^2+1\right)dxdy\right). \tag{23}$$

From (10), (11), (21), (12), (13), (14) and Theorem D we conclude that

$$\begin{aligned}&|\{T_1>\lambda\}|\\ &\lesssim \frac{1}{\lambda}\left(\|M_1f\|_1+\|M_2f\|_1+\|A\|_1+\|f\|_1\right)\\ &\lesssim \frac{1}{\lambda}\left(1+\int\limits_{\mathbb{I}^2}|f|\log^+|f|\right).\end{aligned}$$

Combining (16), (17), (22) and (23) we conclude the proof of Theorem 1. □

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